The Group Configuration Theorem

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An exposition of the group configuration theorem for stable theories,
following Chapter 5 of [Pillay-GST].

Introduction and Preliminaries

Work in monster model $\mathbb{M}$ of a stable theory T.

Notation:
$a, b, c, d, e, w, x, y, z, \alpha, \beta, \gamma$ etc will take values in $\mathbb{M}^{\mathrm{eq}}$,
and A,B,C etc in $\mathbb{P}(\mathbb{M}^{\mathrm{eq}})$.

AB means $A \cup B$;
ab means $(a, b) \in \mathbb{M}^{\mathrm{eq}}$;
when appropriate, a means $\{a\}$; e.g. Ab is short for $A \cup \{b\}$.

Group configuration, first statement:
Let G be a connected $\bigwedge$-definable group $/\emptyset$.
Let $a, b, x \in G$ be an independent triple of generics.
Let $b' := c' * a'$, $x' := a' * y'$, and $z' := b' * y'$ (so $z' = c' * a' * y' = c' * x'$).
Then we have

```
      a
     / \
    b   x
   / `z' \
  / .' `. \
 /.'     `.\
c'         `y
```

satisfying:

(*)
- any non-collinear triple is independent
  (i.e. each element is independent from the other two);
- if (d, e, f) is collinear then $\mathrm{acl}^{\mathrm{eq}}(de) = \mathrm{acl}^{\mathrm{eq}}(ef) = \mathrm{acl}^{\mathrm{eq}}(df)$.

The group configuration theorem provides a converse statement:
if a tuple (a, b, c, x, y, z) satisfies (*),
then possibly after base change
(i.e. adding parameters independent from abcxyz to the language),
there is a connected $\bigwedge$-definable group $G/\emptyset$,
and there are (a', b', c', x', y', z') defined as above from G,
such that each unprimed element is interalgebraic with the corresponding primed element.

Remark:
e.g. if each element realises a strongly minimal type,
(*) says that the Morley rank of a non-collinear triple is 3, and that of a collinear triple is 2.

Stability theory preliminaries:
We have an independence notion, non-forking,
satisfying (even after adding parameters) for all A,B,C:

- Transitivity and Monotonicity:

$$A \mathop{\smile\!\!\!\!\!\!\mid} BC \Leftrightarrow (A \mathop{\smile\!\!\!\!\!\!\mid} B \text{ and } A \mathop{\smile\!\!\!\!\!\!\mid}_B C)$$

- Symmetry:

$$A \mathop{\smile\!\!\!\!\!\!\mid} C \Leftrightarrow C \mathop{\smile\!\!\!\!\!\!\mid} A$$

- Reflexivity:

$$A \mathop{\smile\!\!\!\!\!\!\mid} A \Leftrightarrow A \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$$

- "algebraic $\Rightarrow$ nonforking":

$$A \mathop{\smile\!\!\!\!\!\!\mid} C \Leftrightarrow A \mathop{\smile\!\!\!\!\!\!\mid} \mathrm{acl}^{\mathrm{eq}}(C)$$

$p \in S(A)$ is stationary iff for any B,

$$x \models p \text{ and } x \mathop{\smile\!\!\!\!\!\!\mid}_A B$$

defines a well-defined type $p|_B \in S(AB)$,
the non-forking extension of p to AB.

$p \in S(A)$ is stationary iff it has a unique extension to $\mathrm{acl}^{\mathrm{eq}}(A)$.

So $\mathrm{stp}(a/B) := \mathrm{tp}(a/\,\mathrm{acl}^{\mathrm{eq}}(B))$ is stationary.

Stationary types $p \in S(A)$ are definable,
i.e. T eliminates Hrushovski quantifiers:
for any formula/$\emptyset$ $\phi(x, y)$, there is a formula/A

$$\psi(y) \equiv: d_p x.\phi(x, y)$$

(read "$d_p x$." as "for generic x,"),
such that $\models \psi(b)$ iff for a $\models p$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$,

$$\models \phi(a, b),$$

i.e. iff $\phi(x, b) \in p|_b$.

Note that for $A \subseteq B$,

$$d_{p|_B} x.\phi(x, y) \equiv d_p x.\phi(x, y).$$

The canonical base of a stationary type $p \in S(A)$, $A = \mathrm{dcl}^{\mathrm{eq}}(A)$, is the least $\mathrm{dcl}^{\mathrm{eq}}$ closed set $\mathrm{Cb}(p) \subseteq A$ such that the restriction of p to $\mathrm{Cb}(p)$ is stationary and p is its non-forking extension,
i.e. such that all $d_p x.\ \phi(x, y)$ are defined over $\mathrm{Cb}(p)$.

Let $\mathrm{aCb}(a/B) := \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(\mathrm{stp}(a/B)))$.
Then $a \mathop{\smile\!\!\!\!\!\!\mid}_C B \Leftrightarrow \mathrm{aCb}(a/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(C)$.

Example - ACF:
If K is a perfect subfield and $p \in S(K)$,
p is the generic type of an irreducible variety V over K;
p stationary $\Leftrightarrow$ V absolutely irreducible;
$\mathrm{Cb}(p) =$ (perfect closure of) field of definition of V.

A $\bigwedge$-definable group is a $\bigwedge$-definable set G together with a relatively definable group operation
(meaning that its graph is the restriction to G^3 of a definable set).

If G acts transitively on a $\bigwedge$-definable set S, with the action relatively definable, we call (G, S) a $\bigwedge$-definable homogeneous space.

S is connected iff there is a stationary type s extending S such that if $g \in G$ and $b \models s|_g$, then $g * b \models s|_g$
(i.e. $\mathrm{Stab}(s) = G$).
s is then called the generic type of G.

If (G, S) is definable of finite Morley rank, S is connected iff $\mathrm{MD}(S) = 1$, and $a \in S$ is generic iff $\mathrm{RM}(a) = \mathrm{RM}(S)$.

Generics and connectedness are defined for G by considering the left (equivalently: right) action of G on itself.

Germs and Hrushovski-Weil

Definition:
Let p and q be stationary types $/\emptyset$.

A generic map $p \to q$ is the germ $\widetilde{f_b}$ of a definable partial function f_b
(i.e. $f_b(x) = y$ is given by a formula$/\emptyset$ $\phi_f(x, y, b)$),
such that if $a \models p|_b$ then $f_b(a) \models q|_b$,
where equality of germs is defined by $\widetilde{f_b} = \widetilde{g_c}$ iff for $a \models p|_{bc}$, $f_b(a) = g_c(a)$;
i.e. $\models d_p x . f_b(x) = g_c(x)$.

Example:
In ACF, the generic maps $p \to q$ are precisely the dominant rational maps $\mathrm{locus}(p) \to \mathrm{locus}(q)$.

Lemma:

(i) "Equality of germs" is indeed an equivalence relation.

(ii) For any B, if $a \models p|_{Bb}$ then $f_b(a) \models q|_{Bb}$.

(iii) Composition of germs is well-defined,
yielding a category structure with objects the stationary types.

(iv) A germ $\widetilde{f_b}$ is invertible (i.e. an isomorphism) iff f_b is injective on $p|_b$.

Proof:

(i) Symmetry and reflexivity are clear.
Suppose $f_b = g_c$ on $p|_{bc}$ and $g_c = h_d$ on $p|_{cd}$.
Then clearly $f_b = h_d$ on $p|_{bcd}$.
But $\phi(x) :\equiv f_b(x) = h_d(x)$ is defined over bd,
so already $\phi(x) \in p|_{bd}(x)$.
So $f_b = h_d$ on $p|_{bd}$.

(ii) WLOG, $p, q \in S(\emptyset)$.
$a \mathop{\smile\!\!\!\!\!\!\mid}\, Bb \Rightarrow a \mathop{\smile\!\!\!\!\!\!\mid}_b B \Rightarrow f_b(a) \mathop{\smile\!\!\!\!\!\!\mid}_b B$;
but $f_b(a) \mathop{\smile\!\!\!\!\!\!\mid}\, b$,
so $f_b(a) \mathop{\smile\!\!\!\!\!\!\mid}\, Bb$.

(iii) Suppose $\widetilde{f_b} : p \to q$ and $\widetilde{g_c} : q \to r$.
By (ii), if $a \models p|_{bc}$ then $g_c(f_b(a)) \models r|_{bc}$,
so $(\widetilde{g_c \circ f_b})$ is a germ : $p \to r$.

(iv) Clear.

□

Notation:

- $\mathrm{Hom}(p, q) :=$ set of germs $p \to q$;
- $\mathrm{Iso}(p, q) :=$ set of invertible germs $p \to q$;
- $\mathrm{Aut}(p) :=$ group of invertible germs $p \to p$.

Definition:
A family of generic maps $p \to q$ based on s is a family of germs $\{\widetilde{f_b} \mid b \models s\}$ of a definable family of partial functions f_z.
The family is canonical if for b,$b' \models s$,

$$\widetilde{f_b} = \widetilde{f_{b'}} \Leftrightarrow b = b'.$$

Note that by definability of types, any family can be made canonical by quotienting s by the definable equivalence relation of equality of germs.

Remark:
In ACF, algebraic families of dominant rational maps $V \to W$ can be made canonical by parametrising them using the Hilbert scheme of $V \times W$.

Remark:
If $\widetilde{f_z}$ is a family of generic maps $p \to q$ based on s,
let $b \models s$ and $x \models p|_b$, and let $y = \widetilde{f_b}(x)$;
then

$$x \perp b;\ y \perp b;\ y \in \mathrm{dcl}^{\mathrm{eq}}(bx). \qquad (+)$$

Conversely, if (b, x, y) satisfy $(+)$,
let $f_b(x) = y$ be a formula witnessing $y \in \mathrm{dcl}^{\mathrm{eq}}(bx)$;
then $\widetilde{f_z}$ is a canonical family of generic maps $\mathrm{stp}(x) \to \mathrm{stp}(y)$ based on $\mathrm{stp}(b)$.

Lemma:
In the correspondence of the previous remark,

(i) $\widetilde{f_z}$ is invertible iff $x \in \mathrm{dcl}^{\mathrm{eq}}(by)$ i.e. iff x and y are interdefinable over b;

(ii) $\widetilde{f_z}$ is canonical iff $\mathrm{Cb}(\mathrm{stp}(xy/b)) = \mathrm{dcl}^{\mathrm{eq}}(b)$

Proof:

(i) Clear

(ii) Suppose that $\widetilde{f_z}$ is canonical, and obtain (b, x, y) as above.
$\mathrm{tp}(xy/b)$ is stationary since $\mathrm{tp}(x/b)$ is,
so $C := \mathrm{Cb}(\mathrm{stp}(xy/b)) \subseteq \mathrm{dcl}^{\mathrm{eq}}(b)$.

If $C \neq \mathrm{dcl}^{\mathrm{eq}}(b)$,
say $b' \neq b$ with $b' \equiv_C b$ and $b' \perp_C bx$;
then $xy \perp_C b'$, so $xyb \equiv xyb'$,
so since $f_b(x) = y$, also $f_{b'}(x) = y$,
but $x \models p|_{bb'}$ so this contradicts canonicity of the family $\widetilde{f_z}$.

For the converse, let $f_b(x) = y$ be a formula witnessing $y \in \mathrm{dcl}^{\mathrm{eq}}(bx)$.

Suppose $\widetilde{f_b}$ is not canonical. By definability of types,
some $\widetilde{g_c}$ is canonical with $c \in \mathrm{dcl}^{\mathrm{eq}}(b)$ but $b \notin \mathrm{dcl}^{\mathrm{eq}}(c)$.
Then $\mathrm{Cb}(\mathrm{stp}(xy/b) \subseteq \mathrm{dcl}^{\mathrm{eq}}(c)$ since

$$x \perp b \Rightarrow x \underset{c}{\perp} b \Rightarrow xy \underset{c}{\perp} b,$$

and $\mathrm{tp}(xy/c)$ is stationary since $\mathrm{tp}(x/c)$ is and $y \in \mathrm{dcl}^{\mathrm{eq}}(xc)$.
This contradicts $\mathrm{dcl}^{\mathrm{eq}}(b) = \mathrm{Cb}(\mathrm{stp}(xy/b))$.

□

Remark:
Suppose (b, x, y) "lie on a line" in the sense of the group configuation statement above,
i.e. $\mathrm{acl}^{\mathrm{eq}}(bx) = \mathrm{acl}^{\mathrm{eq}}(xy) = \mathrm{acl}^{\mathrm{eq}}(yb)$.
So x is interalgebraic with y over b.
Since $b \in \mathrm{acl}^{\mathrm{eq}}(ac)$, b is interalgebraic with $\mathrm{Cb}(\mathrm{stp}(ac/b))$; indeed:
let $D = \mathrm{aCb}(ac/b)$; then $ac \perp_D \mathrm{acl}^{\mathrm{eq}}(b)$,

so $\mathrm{acl}^{\mathrm{eq}}(b) \downfree_D \mathrm{acl}^{\mathrm{eq}}(b)$,
so $\mathrm{acl}^{\mathrm{eq}}(b) = D$.

So (b, x, y) is "nearly" a triple corresponding to a canonical family of invertible germs.

Lemma HW:
Suppose $\widetilde{f}_z$ is a canonical family of generic bijections $p \to p$ based on s.
Let $G_0 := \{\widetilde{f}_b \mid b \models s\} \subseteq Aut(p)$.
Suppose that G_0 is closed under inverse,
and suppose that for b_1 and b_2 independent realisations of s,

$$\widetilde{f}_{b_1} \circ \widetilde{f}_{b_2} = \widetilde{f}_{b_3}$$

with $b_3 \models s|_{b_i}$ for $i = 1, 2$.

Then, identifying $\widetilde{f}_b$ with b,
the group $G \leq Aut(p)$ generated by G_0 is connected $\bigwedge$-definable, with s its generic type.

Remark:
This is essentially the Hrushovski-Weil "group chunk" theorem.
There, one starts with a generically associative binary operation $*$, and then applies the above statement to the germs of $x \mapsto a * x$ to obtain a group structure extending $*$.

Proof:
First, we show that any element of G is a composition of two generators.

It suffices to see that any composition of three generators

$$\widetilde{f}_{b_1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$$

is the composition of two. But indeed, let $b' \models s|_{b_1, b_2, b_3}$.
Then

$$\widetilde{f}_{b_1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3} = \widetilde{f}_{b_1} \circ \widetilde{f}_{b'} \circ \widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$$

Now $b' \downfree b_2$, so say $\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} = \widetilde{f}_{b''}$ with $b'' \models s$ independent from b' and from b_2;
then $b'' \downfree b_3$ since:
$b' \downfree_{b_2} b_3$,
so $b'' \downfree_{b_2} b_3$ (since $b'' \in \mathrm{dcl}^{\mathrm{eq}}(b'', b_2)$),
so since $b'' \downfree b_2$,
$b'' \downfree b_2 b_3$, and in particular $b'' \downfree b_3$.
Also $b' \downfree b_1$.
So $\widetilde{f}_{b_1} \circ \widetilde{f}_{b'}$ and $\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$ each "realise s".

Now G is defined as pairs of realisations of s, modulo generic equality of their compositions, and the group operation is defined by composition.

Finally, to see that G is connected with generic type s:
if $g \models s$ and $b \models s|_g$, then $g * b \models s|_g$,
and then by induction the same holds for any $g \in G$. □

In the context of the group configuration, we work with definable families of bijections between two types, rather than from a type to itself. The following key lemma gives conditions for this to give rise to a group.

Lemma A:
Suppose $\widetilde{f}_z$ is a canonical family of generic bijections $p \to q$ based on r.
Let b_1 and b_2 be independent realisations of r and say

$$\widetilde{f}_{b_1}^{-1} \circ \widetilde{f}_{b_2} = \widetilde{g}_c$$

with $\widetilde{g}_w$ a canonical family of generic bijections $p \to p$ based on $s = \mathrm{stp}(c)$,
and suppose

$$c \downfree b_i \text{ for } i = 1, 2. \qquad (+)$$

Then $\widetilde{g}_w$ satisfies the assumptions of Lemma HW.

Remark:
In the finite Morley rank setting (e.g. ACF),

$$\oplus \Leftrightarrow \mathrm{RM}(c) = \mathrm{RM}(s).$$

Proof:
Let $c' \models s|_c$.
Let $b \models r|_{c,c'}$.
Then by (+), $bc \equiv b_2c$, so say $b'_1bc \equiv b_1b_2c$;
similarly, $bc' \equiv b_1c$, so say $bb'_2c' \equiv b_1b_2c$.
Then

$$\widetilde{g}_c \circ \widetilde{g}_{c'} = \widetilde{f}_{b'_1}^{-1} \circ \widetilde{f}_b \circ \widetilde{f}_b^{-1} \circ \widetilde{f}_{b'_2} = \widetilde{f}_{b'_1}^{-1} \circ \widetilde{f}_{b'_2}.$$

```
                 *       *       *         q
                / \    .'|`.    / \
               / c'\.'   |   `./ c \
              /    .'\   |b  /`.    \
             /   .'   \  |  /   `.   \
            / .'       \ | /      `.  \
           /.' b_2'     \|/    b_1' `.\
          *'             *           `*  p
```

Now (b, b'_1, b'_2) is an independent triple,
since $b'_i \perp b$ by choice of b'_i, since $b_1 \perp b_2$,
and $b'_1 \perp_b b'_2$, since $c \perp_b c'$, since $c \perp c'$ and $b \perp cc'$.

So b'_1 and b'_2 are independent realisations of r,
so say $\widetilde{f}_{b'_1}^{-1} \circ \widetilde{f}_{b'_2} = \widetilde{g}_{c''}$.

Then by (+), $c'' \perp b'_1$, and hence $c'' \perp b'_1b$
(since $b \perp b'_1b'_2$, so $b \perp c''b'_1$),
so $c'' \perp c$.
Similarly $c'' \perp c'$.
□

The Group Configuration Theorem

Now we turn to applying this lemma to prove the group configuration theorem.
In fact, the proof we will give naturally proves a more general, less symmetric, statement than that above.

Group Configuration Theorem:

```
        a
       / \
      b   x
     / `z' \
    / .' `. \
   /.'     `.\
  c'         `y
```

```
     x
    / \
 a /   \ c
  /     \
 y-------z
     b
```

Suppose (a, b, c, x, y, z) satisfy:

(*)
- any non-collinear triple is independent,
- $\mathrm{acl}^{\mathrm{eq}}(ab) = \mathrm{acl}^{\mathrm{eq}}(bc) = \mathrm{acl}^{\mathrm{eq}}(ac)$,
- x is interalgebraic with y over a, and a is interalgebraic with $\mathrm{Cb}(\mathrm{stp}(xy/a))$; similarly for bzy and czx.

Then, possibly after base change,
there is a $\bigwedge$-definable homogeneous space (G, S),
and an independent triple (a', b', x') with a', b' generics of G and x' generic in S,
such that with $b' := c' * a'$, $x' := a' * y'$, $z' := b' * y'$ (so $z' = c' * a' * y' = c' * x'$),
each unprimed element is interalgebraic with the corresponding primed element.

Example:
In ACF, we can restate as follows:
(b, z, y) fits into a group configuration (i.e. extends to (a, b, c, x, y, z) satisfying (*))
iff it is a generic point of a "pseudo-action",
i.e. iff there is an algebraic group G acting birationally on a variety S,
and there are generically finite-to-finite algebraic correspondences $f : G' \leftrightarrow G$, $g_1 : S'_1 \leftrightarrow S$ and $g_2 : S'_2 \leftrightarrow S$,
such that (b, z, y) is a generic point of the image under (f, g_1, g_2) of the graph $\Gamma_* \subseteq G \times S \times S$ of the action.
(c.f. 6.2 in [HrushovskiZilber-ZariskiGeometries].)

Example:
if $\mathrm{RM}(a) = \mathrm{RM}(b) = \mathrm{RM}(c) = 2$, and $\mathrm{RM}(x) = \mathrm{RM}(y) = \mathrm{RM}(z) = 1$,
and $\mathrm{RM}(abc) = \mathrm{RM}(ab) = \mathrm{RM}(bc) = \mathrm{RM}(ac) = 4$, $\mathrm{RM}(axy) = \mathrm{RM}(bzy) = \mathrm{RM}(czx) = 3$,
and there are no further dependencies,
then the conditions of the group configuration theorem are satisfied,
and we obtain a rank two group acting on a rank one set,
and with some further work one obtains a definable field,
such that the action is essentially $(a, b)x \mapsto ax + b$.

This is sometimes called the "field configuration", and appears in many proofs,
e.g. Hrushovski's proof that unimodularity implies local modularity,
and hence that ω-categorical stable theories are 1-based;
the proof of the Zilber dichotomy for Zariski structures;
and also e.g. Hasson-Kowalski's work on trichotomy for strongly minimal reducts of RCF.

Very rough sketch of proof:

(I) "reduce $\mathrm{acl}^{\mathrm{eq}}$ to $\mathrm{dcl}^{\mathrm{eq}}$" to show we may assume (b, z, y) to define a canonical family of germs of canonical bijections as in Lemma A;

(II) prove the independence assumption of Lemma A;

(III) connect resulting group action to original group configuration.

Proof of Group Configuration Theorem:
In the proof, we may at any time

- add independent parameters to the language, or
- replace any point of the configuration with an interalgebraic point of $\mathbb{M}^{\mathrm{eq}}$.

Performing these operations preserves (*), and the conclusion allows them.

By adding further algebraic parameters whenever necessary, we will assume throughout that $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$, so types over $\emptyset$ are stationary.

(I) First, we see that it follows from (*) that if we let $\widetilde{z} \in \mathbb{M}^{\mathrm{eq}}$ be the set $\widetilde{z} = \{z_1, ..., z_d\}$ of conjugates z_i of z over $ybxc$, then $\widetilde{z}$ is interalgebraic with z. (Here, $\{z_1, ..., z_d\}$ is the image of $(z_1, ..., z_d)$ under quotienting by the action by permutations of the symmetric group S_d.)

For this, we require that the conjugates are interalgebraic, $\mathrm{acl}^{\mathrm{eq}}(z_i) = \mathrm{acl}^{\mathrm{eq}}(z_j)$.
Indeed, then $\mathrm{acl}^{\mathrm{eq}}(\widetilde{z}) \subseteq \mathrm{acl}^{\mathrm{eq}}(z_1, ..., z_n) = \mathrm{acl}^{\mathrm{eq}}(z)$;
and $z \in \mathrm{acl}^{\mathrm{eq}}(\widetilde{z})$, since it satisfies the algebraic formula $z \in \widetilde{z}$.

But indeed:
$z \in \mathrm{acl}^{\mathrm{eq}}(xc) \cap \mathrm{acl}^{\mathrm{eq}}(yb) =: B$ by (*), but meanwhile $xc \perp_z yb$ so $B \perp_z B$ so $B \subseteq \mathrm{acl}^{\mathrm{eq}}(z)$.
So $\mathrm{acl}^{\mathrm{eq}}(z) = B$,
and by the same argument $\mathrm{acl}^{\mathrm{eq}}(z_i) = B$ for each z_i.

Now let $a' \models \mathrm{tp}(a)|_{abcxyz}$.
Say $a'x'c' \equiv_{ybz} axc$. So $a'bc'x'yz$ is also a group configuration.
So as above, the set $\widetilde{z}$ of conjugates of z over $ybx'c'$ is interalgebraic with z.
Note that $\widetilde{z} \in \mathrm{dcl}^{\mathrm{eq}}(ybx'c')$.

So add a' as a parameter,
and replace y with yx', b with bc', and z with $\widetilde{z}$.
Then since $x' \in \mathrm{acl}^{\mathrm{eq}}(ya')$ and $c' \in \mathrm{acl}^{\mathrm{eq}}(ba')$,
the resulting group configuration is interalgebraic with the original,
and now satisfies

$$z \in \mathrm{dcl}^{\mathrm{eq}}(by).$$

Repeating this procedure by adding an independent copy of b and enlarging a and y and replacing x with an $\widetilde{x}$,
we can also ensure that

$$x \in \mathrm{dcl}^{\mathrm{eq}}(ay).$$

Finally, we repeat once more: add an independent copy c' of c,
let $a'x'c' \equiv_{ybz} axc$,
let $\widetilde{y}$ be the set of conjugates of y over $ba'zx'$.

Now since $x' \in \mathrm{dcl}^{\mathrm{eq}}(a'y)$ and $z \in \mathrm{dcl}^{\mathrm{eq}}(by)$,
we have $zx' \in \mathrm{dcl}^{\mathrm{eq}}(ba'y)$ and so $zx' \in \mathrm{dcl}^{\mathrm{eq}}(ba'\widetilde{y})$.
So after replacing b with ba', z with zx', and y with $\widetilde{y}$,
so in the previous two cases $y \in \mathrm{dcl}^{\mathrm{eq}}(bz)$,
and now also $z \in \mathrm{dcl}^{\mathrm{eq}}(by)$.

Finally, replace b with $\mathrm{Cb}(yz/b)$, with which it is interalgebraic by (*).

(II) (b, y, z) now corresponds to a canonical family $\widetilde{f}_w$ of germs of bijections $\mathrm{tp}(y) \to \mathrm{tp}(z)$ over $r := \mathrm{tp}(b)$.

To apply lemma A to obtain a group,
we must show that if $b' \models r|_b$ and

$$\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_b = \widetilde{g}_d$$

with $\widetilde{g}_u$ canonical,
then $d \mathop{\smile\!\!\!\!\!\!|} b$ and $d \mathop{\smile\!\!\!\!\!\!|} b'$.

We may assume $b' \mathop{\smile\!\!\!\!\!\!|} abcxyz$.
Say $b'y'a' \equiv_{xcz} bya$.

So by canonicity, $\mathrm{dcl}^{\mathrm{eq}}(d) = \mathrm{Cb}(\mathrm{stp}(yy'/bb'))$.

Now $y \mathop{\smile\!\!\!\!\!\!|} abc$, and $b' \mathop{\smile\!\!\!\!\!\!|} yabc$, so $y \mathop{\smile\!\!\!\!\!\!|} abcb'$, and since $a' \in \mathrm{acl}^{\mathrm{eq}}(cb')$, we have $y \mathop{\smile\!\!\!\!\!\!|} aa'bb'$.
Since also $y' \in \mathrm{acl}^{\mathrm{eq}}(yaa')$, we have

$$yy' \mathop{\smile\!\!\!\!\!\!|}_{aa'} bb'.$$

Similarly,

$$yy' \mathop{\smile\!\!\!\!\!\!|}_{bb'} aa'.$$

So $\mathrm{aCb}(yy'/bb') = \mathrm{aCb}(yy'/aa'bb') = \mathrm{aCb}(yy'/aa')$,
so $d \in \mathrm{acl}^{\mathrm{eq}}(aa')$.

Claim: $b \mathop{\smile\!\!\!\!\!\!|} aa'$.
Proof: $abc \mathop{\smile\!\!\!\!\!\!|} b'$, so $ab \mathop{\smile\!\!\!\!\!\!|}_c b'$, so $ab \mathop{\smile\!\!\!\!\!\!|}_c a'$ since $a' \in \mathrm{acl}^{\mathrm{eq}}(cb')$.
But $a' \mathop{\smile\!\!\!\!\!\!|} c$, so $ab \mathop{\smile\!\!\!\!\!\!|} a'$, so $b \mathop{\smile\!\!\!\!\!\!|}_a a'$.
Now $a \mathop{\smile\!\!\!\!\!\!|} b$, so $b \mathop{\smile\!\!\!\!\!\!|} aa'$.

So $b \mathop{\smile\!\!\!\!\!\!|} d$, and similarly $b' \mathop{\smile\!\!\!\!\!\!|} d$, as required.

(III) By (II) and Lemma A,
we obtain a connected $\bigwedge$-definable group G, with a generic action of its generic type s on $p := \mathrm{tp}(y)$,
i.e. $g * a$ is defined for $g \models s$ and $a \models p$ with $g \mathop{\smile\!\!\!\!\!\!|} s$.
To get a $\bigwedge$-definable homogeneous space,
define S to be $(G \times p)/E$ where $(g, a)E(g', a')$ iff $d_s h.(h * g) * a = (h * g') * a'$,
with the action of G:
$h * (g, a)/E := (h * g, a)/E$.

Finally, we must show that the original group configuration is interalgebraic with that of (G, S). This will involve adding further parameters.

First, let $b' \models \mathrm{tp}(b)|_{abcxyz}$.
Say $y'b' \equiv_{xzac} yb$.
Say $g \models s$ codes $\widetilde{f_{b'}}^{-1} \circ \widetilde{f_b}$, so $y' = g * y$.
Then g is interdefinable with b over b'.
So add b' to the language,
and replace b with $g \models s$ and z with $g * y \models p$.

Now let $c' \models \mathrm{tp}(c)|_{abcxyz}$,
and say $b'z'c' \equiv_{axy} bzc$.
Add c' to the language,
and replace a by $b' \models s$ and x by $z' = b' * y \models p$.

Let $h := b * a^{-1}$.
$x = a * y$ and $z = b * y$, so $z = h * x$.
So $\mathrm{aCb}(xz/ab) = \mathrm{acl}^{\mathrm{eq}}(h)$; but also x and z are interalgebraic over c,
so $\mathrm{aCb}(xz/ab) = \mathrm{aCb}(xz/c) = \mathrm{acl}^{\mathrm{eq}}(c)$.
So replace c with h, and we are done.

$\square$