Dr. Martin Bays (baysm@uni-muenster.de)
Prof. Dr. Martin Hils (hils@uni-muenster.de)
Prof. Dr. Dr. Katrin Tent (tent@uni-muenster.de)

Summer term 2021

Seminar *Model theory of pseudofinite structures*

Time: Wed 10:00 – 11:30

Seminar Plan

The seminar will introduce to the model theory of pseudofinite structures. The first half of the seminar will be devoted to pseudofinite fields, the second half to pseudofinite groups, both very active research areas.

In 1968, James Ax determined the theory of all finite fields and showed that it is decidable ([1]). To understand this theory, it is crucial to study its infinite models, the so-called pseudofinite fields. Pseudofinite fields allow for a very elegant axiomatization. They are just those perfect fields which have a unique extension of degree n, for every natural number n, and which are pseudo-algebraically closed. Starting with the foundational work [1] of Ax, we will study various aspects of the model theory of pseudofinite fields and in particular see the construction of the Chatzidakis-van den Dries-Macintyre measure (introduced in [3]) for definable sets in pseudofinite fields, which is an important tool in many applications. We will mainly follow the course notes [2] by Chatzidakis.
We will then study pseudofinite groups from various perspectives, starting with a general treatment of asymptotic classes and measurable structures (see [4]) which are defined through the existence of a measure on definable sets mimicking the Chatzidakis-van den Dries-Macintyre measure. This will be followed by an investigation of measurable groups (see [6]) and continued by further works on pseudofinite groups (e.g., from [5, 8, 9]), depending on the interest of the participants.

The material treated in the seminar may serve as a basis for a Bachelor or a Master thesis.

List of talks

1. Some basic results from field theory and algebraic geometry (14 April)
2. The elementary theory of all pseudofinite fields (21 April)
3. The embedding lemma for pseudofinite fields and the completions of Psf (28 April)
4. Characterization of pseudofinite fields and decidability of Psf (5 May)
5. The measure of Chatzidakis-van den Dries-Macintyre I (12 May)
6. The measure of Chatzidakis-van den Dries-Macintyre II (19 May)
7. Asymptotic classes and measurable structures (2 June)
8. Measurable groups of low dimension I (9 June)
9. Measurable groups of low dimension II (16 June)
10. Up to four further talks on pseudofinite groups, to be determined (23 June, 30 June, 7 July, 14 July)

We decided to skip the initially planned first talk on infinite Galois theory and suggest that those not familiar with the relevant material read this by themselves.
In the seminar, we will need some basic results about finite fields ([2, Section 2]), profinite groups and infinite Galois theory ([2, 3.1–3.4]), and in particular some results about pro-cyclic groups and quasi-finite groups ([2, 3.5–3.8]) as well as the correspondence between supernatural numbers, algebraic extensions of a quasi-finite field and closed subgroups of $\widehat{\mathbb{Z}}$ ([1, first paragraph of Section 1]). A detailed exposition of infinite Galois theory may be found in [10, Chapter 7], the full statement of the infinite Galois correspondence being [10, Theorem 7.12].

Detailed description of the talks

Talk 1 (**Some basic results from field theory and algebraic geometry**)
In this second introductory talk, some results from field theory and basic algebraic geometry should be given, mostly without proofs. Themes covered from field theory should include the tensor product of commutative K-algebras, linear disjointness and algebraic independence in fields, and regular field extensions ([2, 4.5–4.7], see, e.g., [7, Section III.1] for a more detailed treetment).
In the second half of the talk, the material from [2, 4.1–4.4] should be presented, adding more details at places (e.g., following [7, Chapter II & III]). In particular, the following material should be briefly covered: algebraic sets, (affine) algebraic varieties and Hilbert's Nullstellensatz; the Zariski topology and irreducible components; the coordinate ring, rational function field, dimension and generic point of an affine algebraic variety

Talk 2 (**The elementary theory of all pseudofinite fields**)
One should first treat [2, 5.1 – 5.5], citing the existence of the relevant uniform bounds for polynomial ideals [2, Thm 5.2(1) and (4)] without proof; then one should show that the class of pseudofinite fields is elementary ([2, Lemma 6.2]), followed by a proof of [2, Thm 6.4] and [1, Corollary on top of p. 253], citing the Lang-Weil estimates ([2, 6.5]) without proof. Finally, [2, Lemma 6.7] should be proven, as a preparation for the following talk.

Talk 3 (**The embedding lemma for pseudofinite fields and the completions of Psf**)
The first half of this talk should be devoted to the proof of the key embedding lemma ([2, 6.8]), covering [2, 6.8 and 6.10]. One should then infer a description of the completions of Psf, covering [2, 6.11 - 6.14] (leaving out the remark in [2, (6.13)]).

[The proof of the embedding lemma is quite algebraic and uses in particular a lot of Galois theory.]

Talk 4 (**Characterization of pseudofinite fields and decidability of Psf**)
In this talk, one should first treat quantifier reduction and model completeness ([2, 6.15–6.17, leaving out 6.13']), and then establish the characterization of the pseudofinite fields as the infinite models of the theory T_f of all finite fields ([2, 6.18 and 6.19]). In the proof, one may use the consequence of the Chebotarev density theorem [2, bottom of page 31] as a black box.
One should then sketch the proof the decidability of the following theories as in [2, 6.20]: Psf, Psf_0, T_f, T_{prime} (the common theory of all fields of the form $\mathbb{F}_p$ for p prime). For this, one may use that the constants in the Lang-Weil estimates and occurring in the bounds on polynomial ideals are effective.

The following two talks should be prepared in close coordination.

Talk 5 (**The measure of Chatzidakis-van den Dries-Macintyre I**)
In this talk, one should state the main result of [3] without proof (this is [2, Thm 7.1]) and then discuss some applications, covering [2, 7.2, 7.3 and 7.5], and maybe also [2, 7.6], if there is enough time.

Talk 6 (**The measure of Chatzidakis-van den Dries-Macintyre II**)
In this talk, the construction of a measure on definable sets in Psf, by Chatzidakis-van den Dries-Macintyre, should be given. One should start with a sketch of the construction of the F-absolute kernel V^* of a variety

V defined over F, paying attention to uniformity in parameters. For this, [3, §1] should be covered, stating [3, Lemma 1.12(ii)] without proof. One should then present the proof of the main result of [3], following [3, 3.3–3.8]. When dealing with an arbitrary (not necessarily quantifier-free) formula ϕ ([3, p. 123]), one should use [2, Thm 6.17] from Talk 4 instead of [3, 2.7].

Talk 7 (**Asymptotic classes and measurable structures**)
The notion of an asymptotic class of finite structures is modeled along the class of finite fields. Asymptotic classes give rise measurable ultraproducts with supersimple theory [4] Sec 1-3.

Talk 8 (**Measurable groups of low dimension I**)
In this talk and in the next one we will see the similarities to groups of low Morley Rank and Zilber's field interpretation theorem. Elwes and Ryten show that groups of asymptotic dimension one are essentially abelian, and in asymptotic dimension two essentially soluble. If time permits, the second talk could also cover measurable group actions [6].

Talk 9 (**Measurable groups of low dimension II**)
See the description of the previous talk.

References

[1] J. Ax, *The Elementary Theory of Finite Fields*, Annals of Math. **88**(2) (1968), 239–271.

[2] Z. Chatzidakis, *Notes on the model theory of finite and pseudo-finite fields*, Course notes, 2005. (The manuscript may be downloaded at `http://www.math.ens.fr/~zchatzid/`).

[3] Z. Chatzidakis, L. van den Dries and A. Macintyre, *Definable sets over finite fields*, J. Reine Angew. Math. **427** (1992), 107–135.

[4] R. Elwes and D. Macpherson, *A survey of asymptotic classes and measurable structures.* Model theory with applications to algebra and analysis. Vol. 2, 125–159, London Math. Soc. Lecture Note Ser., **350**, Cambridge Univ. Press, Cambridge, 2008.

[5] R. Elwes, E. Jaligot, D. Macpherson and M. Ryten, *Groups in supersimple and pseudofinite theories*, Proc. Lond. Math. Soc. (3) **103** (2011), 1049–1082.

[6] R. Elwes and M. Ryten, *Measurable groups of low dimension*, Math. Log. Q. **54** (2008), 374–386.

[7] S. Lang, *Introduction to algebraic geometry*, Third printing, with corrections. Addison-Wesley, Reading, Mass., 1972.

[8] D. Macpherson, *Model theory of finite and pseudofinite groups*, Arch. Math. Logic **57** (2018), 159–184.

[9] A. Ould Houcine and F. Point, *Alternatives for pseudofinite groups*, J. Group Theory **16** (2013), 461–495.

[10] J.S. Milne, *Fields and Galois Theory*, Course notes, 2015. (The manuscript may be downloaded at `http://www.jmilne.org/math/CourseNotes/FT.pdf`).

[11] K. Tent and M. Ziegler, *A Course in Model Theory*, Lecture Notes in Logic, CUP, 2012.