Geometric Stability Theory

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1 Introduction

These are lecture notes from a course given in the Münster Model Theory Month "spring school" of 2016.

The course covers some of the foundational results of geometric stability theory. We focus on the geometry of minimal sets. The main aim is an account of Hrushovski's result that unimodular (in particular, locally finite or pseudofinite) minimal sets are locally modular; along the way, we discuss the Zilber Trichotomy and the Group and Field Configurations.

We assume the basics of stability theory (forking calculus, U-rank, canonical bases, stable groups and homogeneous spaces), as can be found e.g. in [Pal17].

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This version includes some corrections to the published text. Thanks to Zhengqing He and Tomás Ibarlucía for pointing out some errors, and to Itay Kaplan for finding many more. Probably most egregious, the statement of the type-definable case of Fact 6.5 was incorrect in the published version.

Your name too could appear here! If you find any errors or omissions, please mail me at `mbays@sdf.org`.

1.1 References

The original results of Zilber are collected in [Zil93]. The results of Hrushovski we discuss are mostly in [Hru86], [Hru87], and [Hru92]. Our presentation is based in large part on Pillay's book [Pil96].

2 Preliminaries

We work in a monster model $\mathbb{M}$ of a stable theory T, so $\mathbb{M}$ is κ-saturated and strongly κ-homogeneous for some suitably large κ. "Small" means of cardinality less than κ.

Notation.

- $a, b, c, d, e, \alpha, \beta, \gamma$ etc denote elements of $\mathbb{M}^{\mathrm{eq}}$, $\overline{a}, \overline{b}, \overline{c}$ etc denote possibly long tuples from $\mathbb{M}^{\mathrm{eq}}$, and A,B,C denote small subsets of $\mathbb{M}^{\mathrm{eq}}$.
- AB means $A \cup B$; ab means $(a, b) \in \mathbb{M}^{\mathrm{eq}}$; when appropriate, a means $\{a\}$; e.g. Ab is short for $A \cup \{b\}$.
- A $\bigwedge$-definable set (over A) is a subset of $\mathbb{M}^{\mathrm{eq}}$ defined by a partial type consisting of formulae whose parameters come from a small set (namely A).
- If p is stationary and $A := \mathrm{Cb}(p)$, then $p{\restriction}_A$ has a unique global non-forking extension $\mathfrak{p} \in S(\mathbb{M})$. For $A \subseteq B \subseteq \mathbb{M}^{\mathrm{eq}}$, we write $p{\restriction}_B$ for $\mathfrak{p}{\restriction}_B$. So $a \vDash p{\restriction}_B$ iff $a \vDash p$ and $a \mathop{\smile\!\!\!\!\!\mid}_A B$.
- For α an ordinal, $p^{(\alpha)}$ is the type of a Morley sequence in p of length α (so $\overline{a}b \vDash p^{(\alpha^+)}$ iff $\overline{a} \vDash p^{(\alpha)}$ and $b \vDash p{\restriction}_{\overline{a}}$).
- Define $\mathrm{Cb}(a/B) := \mathrm{Cb}(\mathrm{stp}(a/B))$ and $\overline{\mathrm{Cb}}(a/B) := \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(a/B))$. So $a \mathop{\smile\!\!\!\!\!\mid}_C B \Leftrightarrow \overline{\mathrm{Cb}}(a/BC) \subseteq \mathrm{acl}^{\mathrm{eq}}(C)$.
- We often disregard the distinction between an element of $\mathbb{M}^{\mathrm{eq}}$ and its $\mathrm{dcl}^{\mathrm{eq}}$-closure. In particular, $c = \mathrm{Cb}(p)$ means $\mathrm{dcl}^{\mathrm{eq}}(c) = \mathrm{Cb}(p)$, and we then allow ourselves to write $\mathrm{Cb}(p) \in \mathbb{M}^{\mathrm{eq}}$.
- We use exponential notation b^σ for the action of an automorphism (so automorphisms always act on the right).

Lemma 2.1. *U-rank is additive for finite ranks: $U(ab/C) = U(a/bC) + U(b/C)$ whenever all terms are finite.*

Proof. This is immediate from the Lascar inequalities [Pal17, Theorem 5.7] (and in fact one only needs that one of the sides of the equality is finite). ☐

3 Minimal sets and the trichotomy

3.1 Minimal sets and pregeometries

Definition. A <u>minimal set</u> is a $\bigwedge$-definable set D, defined by a partial type which has a unique unrealised global completion $\mathfrak{p} \in S(\mathbb{M})$. Call $\mathfrak{p}$ the global <u>generic type</u> of D.

A <u>strongly minimal set</u> is a definable set which is minimal.

Let D be a minimal set, and $\mathfrak{p}$ its global generic type. Adding parameters to the language if necessary, we will assume D is defined over $\emptyset$.

Exercise.

- If $a \in D$ and $C \subseteq \mathbb{M}^{\mathrm{eq}}$, then $a \vDash \mathfrak{p}{\restriction_C}$ iff $a \notin \mathrm{acl}^{\mathrm{eq}}(C)$.
- A $\bigwedge$-definable set is minimal iff every relatively definable subset is finite or cofinite.
- A complete type is minimal iff it is stationary and of U-rank 1.

Remark. Historically the focus was on strongly minimal formulae, due to their role in uncountable categoricity (Baldwin-Lachlan), but minimality is the natural generality for the theory we develop here. Examples of minimal $\bigwedge$-definable sets include the maximal perfect subfield $K^{p^\infty} = \bigcap_n K^{p^n}$ of a separably closed field, and the maximal divisible subgroup $\bigcap_n n({}^*\mathbb{Z})$ of ${}^*\mathbb{Z} \vDash \mathrm{Th}(\mathbb{Z}; +)$.

One can also work in the greater generality of (strongly) regular types, and much of what we will cover in this course goes through for them. See in particular [Hru87].

Definition. A pregeometry is a set X with a map $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that

- cl is a closure operator: $A \subseteq \mathrm{cl}(A)$, $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
- Exchange: $b \in \mathrm{cl}(Ac) \setminus \mathrm{cl}(A) \Rightarrow c \in \mathrm{cl}(Ab)$.
- Finite character: $\mathrm{cl}(A) = \bigcup_{A_0 \subseteq_{\mathrm{fin}} A} \mathrm{cl}(A_0)$;

Often, it is more useful to think of a pregeometry in terms of its lattice of closed sets. An equivalent definition of a pregeometry is as a set X with a set of closed subsets $\mathcal{C} \subseteq \mathcal{P}(X)$ such that

- $\mathcal{C}$ is closed under intersections: if $\mathcal{B} \subseteq \mathcal{C}$, then $\bigcap \mathcal{B} \in \mathcal{C}$.
- Exchange: If $C \in \mathcal{C}$ and $a \in X \setminus C$, there is an immediate closed extension C' of C containing a, i.e. there exists $C' \in \mathcal{C}$ such that $C' \supseteq Ca$ and there does not exist $C'' \in \mathcal{C}$ with $C \subsetneq C'' \subsetneq C'$.
- Finite character: if $\mathcal{B} \subseteq \mathcal{C}$, then $\bigvee \mathcal{B} = \bigcup_{\mathcal{B}_0 \subseteq_{\mathrm{fin}} \mathcal{B}} (\bigvee \mathcal{B}_0)$, where $\bigvee \mathcal{B} := \bigcap \{C \in \mathcal{C} \mid \bigcup \mathcal{B} \subseteq C\}$ is the smallest closed set containing all $B \in \mathcal{B}$,

So setting $\bigwedge \mathcal{B} := \bigcap \mathcal{B}$ and $A \leq B \Leftrightarrow A \subseteq B$, $\mathcal{C}$ forms a complete lattice.

We can pass between the two definitions as follows: given a closure operator cl, define $\mathcal{C} := \mathrm{im}(\mathrm{cl})$; conversely, given a set $\mathcal{C}$ of closed subsets, define $\mathrm{cl}(A) := \bigcap \{C \in \mathcal{C} \mid A \subseteq C\}$.

For $A, B \subseteq X$, $\dim(A/B)$ is the cardinality of a basis for A over B, a minimal subset $A' \subseteq A$ for which $\mathrm{cl}(A'B) = \mathrm{cl}(AB)$. When it is finite, $\dim(A/B)$ is equivalently the length r of any chain of immediate extensions of closed sets $\mathrm{cl}(B) = C_0 \lneq C_1 \lneq \ldots \lneq C_r = \mathrm{cl}(AB)$.

The localisation of X at A is $X_A := (X, \mathrm{cl}_A)$ where $\mathrm{cl}_A(B) = \mathrm{cl}(AB)$; equivalently, the closed sets of X_A are the closed sets of X which contain A.

X is homogeneous if for any closed $C \subseteq X$ and any $a, b \in X \setminus C$, there is an automorphism over C of the pregeometry (a bijection preserving cl, or equivalently $\mathcal{C}$, and fixing C pointwise) which sends a to b.

A geometry is a pregeometry (X, cl) such that $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(a) = \{a\}$ for any $a \in X$. For a pregeometry (X, cl), the associated geometry is the set of dimension 1 closed sets of X with the obvious closure, cl' defined by $A \in \mathrm{cl}'(B) \Leftrightarrow A \subseteq \mathrm{cl}(\bigcup B)$.

Remark. Finite pregeometries are also known as matroids.

Lemma 3.1. (D, acl_D) *is a homogeneous pregeometry, where* $\mathrm{acl}_D(A) := \mathrm{acl}^{\mathrm{eq}}(A) \cap D$.

Proof. Everything is immediate from the definition of acl_D except homogeneity and exchange. Exchange corresponds to symmetry of forking: for $b, c \in D$ and $A \subseteq D$,

$$\begin{aligned} b \in \mathrm{acl}_D(Ac) \setminus \mathrm{acl}_D(A) &\Leftrightarrow b \vDash \mathfrak{p}\restriction_A \text{ and } b \nvDash \mathfrak{p}\restriction_{Ac} \\ &\Leftrightarrow b \not\mathop{\smile\!\!\!\!\!\!\mid}_A c \\ &\Leftrightarrow c \not\mathop{\smile\!\!\!\!\!\!\mid}_A b \\ &\Leftrightarrow c \vDash \mathfrak{p}\restriction_A \text{ and } c \nvDash \mathfrak{p}\restriction_{Ab} \\ &\Leftrightarrow c \in \mathrm{acl}_D(Ab) \setminus \mathrm{acl}_D(A). \end{aligned}$$

For homogeneity, if $C \subseteq D$ is acl_D-closed, and $a_0, b_0 \in D \setminus C$, then they can be extended to acl_D-bases $(a_i)_{i<\lambda}, (b_i)_{i<\lambda}$ for D over C. Then by inductively considering isolating algebraic formulae, $a_i \mapsto b_i$ can be extended to an elementary map $D \twoheadrightarrow D$ fixing C pointwise, which is in particular an automorphism of the pregeometry over C. (Note that this is not just a matter of strong homogeneity of $\mathbb{M}^{\mathrm{eq}}$, since C need not be small.) $\square$

Remark. For $a \in D^{<\omega}$ and $B \subseteq D$, $U(a/B) = \dim(a/B)$. Indeed, this is immediate for singletons, and then follows for tuples by additivity.

Remark. Adding $A \subseteq D$ to the language corresponds to localising the pregeometry at A.

3.2 Triviality and modularity

Definition. A pregeometry is trivial (or degenerate) if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$; equivalently, $\bigvee = \bigcup$ in the lattice of closed sets.

Example.

- The theory of equality on an infinite set (i.e. with trivial language), is strongly minimal with trivial pregeometry; all subsets are closed, $\mathrm{acl}_D(B) = B$.
- An action of a group G on an infinite set D without fixed points, in the language $(D; ((g*))_{g \in G})$, is strongly minimal with trivial pregeometry; closed sets are unions of G-orbits; $\mathrm{acl}_D(B) = GB$.

Definition. A pregeometry (X, cl) is modular if for any finite-dimensional closed $A, B \leq X$,

$$\dim(A \vee B) = \dim(A) + \dim(B) - \dim(A \wedge B), \tag{1}$$

i.e. $\dim(A/B) = \dim(A/A \wedge B)$.

Remark. The pregeometry of a minimal set D is modular iff $A \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} B$ for any acl_D-closed $A, B \subseteq D$.

Lemma 3.2. *Suppose* (X, cl) *is such that (1) holds when* $\dim(A) = 2$. *Then* (X, cl) *is modular.*

Proof. Suppose (1) holds when $\dim(A) = 2$.

We first show that (1) holds when $\dim(A/A \wedge B) = 2$. Say a_1, a_2 is a basis for A over $A \wedge B$; then setting $A_0 := \mathrm{cl}(a_1, a_2)$, $A_0 \wedge B = A_0 \wedge (A \wedge B) = \mathrm{cl}(\emptyset)$, and $A_0 \vee B \supseteq A_0 \vee (A \wedge B) = A$ so $A_0 \vee B = A \vee B$. So indeed, $\dim(A/B) = \dim(A \vee B/B) = \dim(A_0 \vee B/B) = \dim(A_0/A_0 \wedge B) = \dim(A_0) = \dim(A/A \wedge B)$.

Now let A and B be arbitrary, and say $n = \dim(A/B) = \dim(A \vee B) - \dim(B)$. Decompose $B \leq A \vee B$ as an immediate chain of closed subsets

$$B = B_0 \lneq B_1 \lneq \ldots \lneq B_n = A \vee B.$$

Let $A_i := A \wedge B_i$. Clearly $\dim(A/A \wedge B) \geq \dim(A/B) = n$, so it suffices to see that $\dim(A_{i+1}/A_i) \leq 1$. Else, we have $A_i \leq A' \leq A_{i+1}$ with $\dim(A'/A_i) = 2$ and A' closed; but then $A' \wedge B_i = A_i$ and $A' \vee B_i \subseteq B_{i+1}$, contradicting the modularity established above. □

Exercise. A lattice is called <u>modular</u> if for any A, B, C with $A \leq C$,

$$A \vee (B \wedge C) = (A \vee B) \wedge C.$$

Show that (X, cl) is modular iff the lattice of finite-dimensional closed subsets is modular.

Example 3.3 (Projective Geometries). Let V be a vector space over a division ring K with closed sets the vector subspaces. For V infinite dimensional, this is the pregeometry of the strongly minimal structure $(V; +, (k*)_{k \in K})$. It is a standard result of linear algebra that this is modular.

The corresponding geometry is obtained by projectivising - deleting 0 and quotienting by the action of multiplication by scalars.

Fact 3.4. *Any non-trivial modular homogeneous geometry of dimension* ≥ 4 *is isomorphic to a projective geometry over a division ring.*

The proof of this fact goes as follows. By classical theorems of geometric algebra [Sei62, Chapter V], given an incidence relation between a set of points and a set of lines such that

(i) any pair of points $a \neq b$ lies on a unique common line (a, b),

(ii) any pair of lines intersect in at most one point,

(iii) any line has at least three points,

(iv) if (a, b) and (c, d) intersect then (a, c) and (b, d) intersect (this is the Veblan-Young axiom, sometimes referred to as Pasch's axiom),

(v) Not all points are on a common plane, where a *plane* consists of the points on lines (a_0, b) where a_0 is a fixed point and b varies through the points on a fixed line not containing a_0,

one can find a Desarguesian projective plane, and from this construct a division ring, and conclude that the original incidence relation is isomorphic to a projective space over the division ring. We obtain such an incidence relation from our geometry by taking the points resp. lines to be the closed sets of dimension 1 resp. 2. Axioms (i) and (ii) are immediate, (iv) follows easily from modularity, and (v) from the dimension assumption. For (iii), non-triviality yields a relation $a_1 \in \mathrm{cl}(A, a_2)$ for some finite A with $a_i \notin \mathrm{cl}(A)$; then by modularity, there is $e \in \mathrm{cl}(a_1 a_2) \cap \mathrm{cl}(A)$, so the line (a_1, a_2) has at least three points. By homogeneity, so has every line.

In the model theoretic context, essentially this construction actually arises definably. If D is a non-trivial modular minimal type, it is non-orthogonal to (i.e. in generically defined finite-to-finite relatively definable correspondence with) a minimal abelian group; its ring of definable finite-to-finite quasi-endomorphisms is a division ring, and the geometry of this, and hence of D, is then projective geometry over that division ring. See [Pil96, Remark 5.1.9] for details.

3.3 1-basedness and local modularity

Remark. Localisations of modular pregeometries are also modular. However, the converse is false, as the following example shows.

Example 3.5 (Affine Geometry)*.* Let V be an infinite dimensional vector space over a division ring K, with closed sets the affine spaces, i.e. cosets of subspaces. For $\mathrm{char}(K) \neq 2$, this is the pregeometry of the strongly minimal structure

$$(V; (\{z = \lambda x + (1-\lambda)y\} \mid \lambda \in K)).$$

Affine geometries are not quite modular: parallel lines within a common plane are dependent but have trivial intersection.

Localising at 0 yields the projective geometry of Example 3.3.

In this subsection, we establish a dichotomy between minimal sets which have modular localisations and those which don't. This dichotomy becomes clearest when we include imaginaries in our considerations.

Definition. $D^{\mathrm{eq}} := \mathrm{dcl}^{\mathrm{eq}}(D) \subseteq \mathbb{M}^{\mathrm{eq}}$.

Definition. D is <u>1-based</u> if for any $a \in D^{\mathrm{eq}}$ and $B \subseteq D^{\mathrm{eq}}$, $\mathrm{Cb}(a/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(a)$.

Remark. The correct definition of 1-basedness in more general situations allows B to be an arbitrary subset of $\mathbb{M}^{\mathrm{eq}}$. But in our case this is equivalent to the above definition, since $\mathrm{Cb}(a/B) = \mathrm{Cb}(a/\overline{a})$ where $\overline{a} \subseteq D^{\mathrm{eq}}$ is a Morley sequence in $\mathrm{tp}(a/B)$.

Lemma 3.6. *D is 1-based iff for any $\mathrm{acl}^{\mathrm{eq}}$-closed subsets $A, B \subseteq D^{\mathrm{eq}}$,*

$$A \underset{A \cap B}{\perp} B.$$

Proof. Immediate from finite character of forking and the properties of canonical bases. □

Lemma 3.7. *1-basedness is invariant under adding parameters to the language.*

Proof. That a 1-based set remains 1-based on adding parameters follows from the definition and the remark after it.

For the converse, suppose D is 1-based after adding C. Let $A, B \subseteq D^{\mathrm{eq}}$ be $\mathrm{acl}^{\mathrm{eq}}$ closed, and let $I := A \cap B$; we must show $A \perp_I B$.

By taking a realisation of $\mathrm{tp}(AB)$ independent from C, we may assume $AB \perp C$.

Claim 3.8. $I' := \mathrm{acl}^{\mathrm{eq}}(AC) \cap \mathrm{acl}^{\mathrm{eq}}(BC) = \mathrm{acl}^{\mathrm{eq}}(IC)$

Proof. $AC \mathop{\smile\!\!\!\!\!|}_A AB$ and $BC \mathop{\smile\!\!\!\!\!|}_B AB$, so by considering $\mathrm{Cb}(I'/AB)$, we have $I' \mathop{\smile\!\!\!\!\!|}_I AB$. So since $I \subseteq IC \subseteq I'$, also $I' \mathop{\smile\!\!\!\!\!|}_{IC} ABC$. But $I' \subseteq \mathrm{acl}^{\mathrm{eq}}(ABC)$, so $I' = \mathrm{acl}^{\mathrm{eq}}(IC)$. □

Applying Lemma 3.6 in the expanded language, we deduce $A \mathop{\smile\!\!\!\!\!|}_{IC} B$. Meanwhile $AB \mathop{\smile\!\!\!\!\!|} C$ and $I \subseteq AB$, so $AB \mathop{\smile\!\!\!\!\!|}_I IC$, and so $A \mathop{\smile\!\!\!\!\!|}_I IC$. So by transitivity, $A \mathop{\smile\!\!\!\!\!|}_I B$. □

Definition. A $\bigwedge$-definable set X has elimination of imaginaries (EI), resp. geometric EI (gEI), resp. weak EI (wEI), if for any $c \in X^{\mathrm{eq}}$, there exists $b \in X^{<\omega}$ such that $\mathrm{dcl}^{\mathrm{eq}}(b) = \mathrm{dcl}^{\mathrm{eq}}(c)$, resp. $\mathrm{acl}^{\mathrm{eq}}(b) = \mathrm{acl}^{\mathrm{eq}}(c)$, resp. $\mathrm{dcl}^{\mathrm{eq}}(c) \subseteq \mathrm{dcl}^{\mathrm{eq}}(b) \subseteq \mathrm{acl}^{\mathrm{eq}}(c)$.

Remark. EI $\Rightarrow$ wEI $\Rightarrow$ gEI.

Lemma 3.9. *Suppose D has gEI. Then D is modular iff D is 1-based.*

Proof. By gEI and finite character of $\mathrm{acl}^{\mathrm{eq}}$, for $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq D^{\mathrm{eq}}$ we have $A = \mathrm{acl}^{\mathrm{eq}}(A \cap D)$. So for $A, B \subseteq D^{\mathrm{eq}}$ $\mathrm{acl}^{\mathrm{eq}}$-closed,

$$A \mathop{\smile\!\!\!\!\!|}_{A\cap B} B \Leftrightarrow (A \cap D) \mathop{\smile\!\!\!\!\!|}_{A\cap B\cap D} (B \cap D).$$

We conclude on noting that for $A \subseteq D$, $A = \mathrm{acl}_D(A) \Rightarrow A = \mathrm{acl}^{\mathrm{eq}}(A) \cap D$. □

Lemma 3.10. *Let $\overline{a} \vDash \mathfrak{p}{\restriction}_{\emptyset}^{(\omega)}$. Then after expanding the language by parameters for $\overline{a}$, D has wEI.*

Proof. Working in the original language, we will prove wEI in the expanded language by showing that if $c \in \mathrm{dcl}^{\mathrm{eq}}(D, \overline{a}) = D^{\mathrm{eq}}$, then there exists $b \in D^{<\omega}$ such that $c \in \mathrm{dcl}^{\mathrm{eq}}(b\overline{a})$ and $b \in \mathrm{acl}^{\mathrm{eq}}(c\overline{a})$.

So say $c = f(b')$ with $b' \in D^n$ and f a partial function defined over $\emptyset$. It suffices to show that $f^{-1}(c) \cap \mathrm{acl}^{\mathrm{eq}}(c\overline{a}) \neq \emptyset$. In fact we will prove by induction on n the stronger statement that if $\emptyset \neq X \subseteq D^n$ is $\bigwedge$-definable over $c \in D^{\mathrm{eq}}$, then $X \cap \mathrm{acl}^{\mathrm{eq}}(c\overline{a}) \neq \emptyset$.

First suppose $n = 1$. Then either X is finite, in which case any point is algebraic over c, or else $\mathfrak{p}{\restriction}_c(x) \vDash x \in X$, in which case we are done as $a_i \vDash \mathfrak{p}{\restriction}_c$ for some i, since $c \in D^{\mathrm{eq}}$.

(Indeed, say $U(c) = k$, and suppose $U(a_i/c) = 0$ for $i = 0, \ldots, k$. Then $U(c/a_{<i}a_i) = U(ca_i/a_{<i}) - U(a_i/a_{<i}) = U(a_i/ca_{<i}) + U(c/a_{<i}) - U(a_i/a_{<i}) = U(c/a_{<i}) - 1$, so we have a contradiction.)

If $n > 1$, consider a co-ordinate projection $\pi : X \to D$. By compactness and saturation, $\pi(X)$ is also $\bigwedge$-definable over c. By the $n = 1$ case, say $b \in \pi(X) \cap \mathrm{acl}^{\mathrm{eq}}(c)$; then the fibre $\pi^{-1}(b)$, considered as a $\bigwedge$-definable subset of D^{n-1}, is over bc, so by induction has a point algebraic over bc and hence over c. □

It follows from Lemmas 3.7, 3.9, and 3.10 that D is 1-based iff some localisation is modular. But we now prove a sharper result.

Definition. D is linear if whenever $a, b \in D$ and $C \subseteq D^{\mathrm{eq}}$ and $U(ab/C) = 1$, then $U(\mathrm{Cb}(ab/C)) \leq 1$.

(We can think of this as D having "no 2-dimensional family of plane curves".)

Example 3.11. An algebraically closed field is not linear, since e.g. there is a two-dimensional family of straight lines in the plane. We can see this in terms of the definition of linearity as follows. Let a, b, c, d be generic such that $b = ca + d$. So (a, b) is a generic point of the algebraic variety $V_{cd} = \{(x, y) \mid y = cx + d\}$, whose field of definition is generated by c, d. Then $U(ab/cd) = 1$, but $\mathrm{Cb}(ab/cd) = \mathrm{dcl}^{\mathrm{eq}}(cd)$ and $U(cd) = 2$.

Definition. A pregeometry (X, cl) is locally modular if the localisation at any $e \in X \setminus \mathrm{cl}(\emptyset)$ is modular.

Theorem 3.12. *The following are equivalent:*

(i) D is 1-based;

(ii) D is linear;

(iii) D is locally modular.

Proof.

(i) $\Rightarrow$ *(ii)* : Let a, b, C be as in the definition of linearity. By 1-basedness, $d := \mathrm{Cb}(ab/C) \subseteq \mathrm{acl}^{\mathrm{eq}}(ab)$. So $U(d) = U(abd) - U(ab/d) \leq 2 - 1 = 1$.

(ii) $\Rightarrow$ *(iii)* : Let $e \in D \setminus \mathrm{acl}_D(\emptyset)$. We prove modularity of the localisation to e via Lemma 3.2. So suppose $(a, b) \vDash \mathfrak{p}{\restriction}_e^{(2)}$ and $e \in C = \mathrm{acl}_D(C) \subseteq D$, and let $I := \mathrm{acl}_D(abe) \cap C$; we must show $ab \mathop{\smile\!\!\!\!\!\mid}_I C$. This is clear if $U(ab/C) = 0$ or $U(ab/C) = 2$, so suppose $U(ab/C) = 1$. So we must show

$$\mathrm{acl}_D(e) \lneq I.$$

Let $d := \mathrm{Cb}(ab/C)$. By linearity, $U(d) \leq 1$. Since $ab \not\mathop{\smile\!\!\!\!\!\mid} C$, we must have $U(d) = 1$. Note $U(d/ab) = U(dab) - U(ab) = U(ab/d) + U(d) - U(ab) = 1 + 1 - 2 = 0$, so $d \in \mathrm{acl}^{\mathrm{eq}}(ab)$. So $\mathrm{acl}^{\mathrm{eq}}(ed) \cap D \subseteq I$. So it suffices to show

$$\mathrm{acl}_D(e) \lneq \mathrm{acl}^{\mathrm{eq}}(ed) \cap D. \qquad (*)$$

If $a \not\mathop{\smile\!\!\!\!\!\mid} d$ then a witnesses (*). Else, $\mathrm{tp}(a/d) = \mathrm{tp}(e/d) = \mathfrak{p}{\restriction}_d$, since $e \mathop{\smile\!\!\!\!\!\mid} d$ as $d \in \mathrm{acl}^{\mathrm{eq}}(ab)$. So say $ab \equiv_d ef$. Then f witnesses (*).

(iii) $\Rightarrow$ *(i)* : Add parameters for an infinite Morley sequence in $\mathfrak{p}{\restriction}_\emptyset$. Then D has wEI by Lemma 3.10, and D is modular, so D is 1-based by Lemma 3.9. But then D was 1-based in the original language, by Lemma 3.7.

□

3.4 Trichotomy

Combining the two dividing lines of Fact 3.4 and Theorem 3.12, we obtain the following form of Zilber's Weak Trichotomy Theorem:

Theorem 3.13. *If D is a minimal set in a stable theory, then precisely one of the following holds:*

- *The pregeometry of D is trivial.*

- *The pregeometry of D is locally modular but non-trivial; equivalently, after localising at a non-algebraic point, the geometry of D is ("definably") projective geometry over a division ring.*
- *The pregeometry of D is not locally modular; equivalently, D defines a 2-dimensional family of plane curves (and in fact interprets a pseudoplane - see below).*

Historical Remark. In our Example 3.11 of a 2-dimensional family of plane curves, the same classical results of geometry which we referred to in Fact 3.4 apply to allow one to definably reconstruct the field from the incidence relation. Meanwhile, with a little extra work, one can extract from nonlinearity a particularly canonical kind of 2-dimensional family of "plane curves" known as a pseudoplane, which dimension-theoretically looks just like Example 3.11 (or rather its projective counterpart) - namely, each curve has infinitely many points, each point lies on infinitely many curves, a pair of curves has finite intersection, and a pair of points lie on finitely many common curves[1]. Given also Macintyre's theorem that any ω-stable division ring is an algebraically closed field, one might reasonably hope that Example 3.11 is characteristic of strongly minimal sets which are not locally modular, and hence that any such is bi-interpretable with an algebraically closed field in this way, and so in particular that the geometry of a minimal set which is not locally modular is that of an algebraically closed field. This is known as Zilber's Trichotomy Conjecture.

Reality is more complicated. Hrushovski found a counterexample to this conjecture in the early 90s, with a combinatorial construction of a strongly minimal set which is not locally modular but which doesn't even interpret an infinite group. He also produced a variety of "pathological" strongly minimal sets, such as two algebraically closed field structures of different characteristics on the same set. At the time of writing, there is no clear path to a classification of strongly minimal sets, or even of their geometries; certainly we know it must be much more involved than a trichotomy.

However, there are situations where the Trichotomy Conjecture does go through. Most notably, this is the case for Zariski Geometries [HZ96]. We will not discuss Zariski Geometries in this course, but instead another earlier incarnation of the same principle - unimodular minimal sets, which we will see are locally modular by proving that the Trichotomy Conjecture is valid for them and then seeing that the field case is impossible.

Fact 3.14. *If D is locally modular but not modular, its geometry is affine geometry over a division ring ([Pil96, Proposition 5.2.4]).*

[1] Generically, this means that (possibly after adding parameters) we have b_1, b_2 such that for $\{i, j\} = \{1, 2\}$, we have $U(b_i) = 2$, $U(b_i/b_j) = 1$, and $\mathrm{Cb}(b_i/b_j) = b_j$. In particular, $r := \mathrm{tp}(b_1 b_2)$ satisfies that $r(a, D)$ and $r(D, b)$ are infinite when non-empty, and $r(a, D) \cap r(a', D)$ and $r(D, b) \cap r(D, b')$ are finite for $a \neq a'$ and $b \neq b'$. If D is strongly minimal, by compactness and elimination of $\exists^\infty$ one can find a formula $I(x, y) \in r$ with the same properties; by definition, I is then the incidence relation of an intepretable pseudoplane.

Nonlinearity gives b_1, b_2 with all the above properties except that we only have $U(b_2) \geq 2$ and don't necessarily have $\mathrm{Cb}(b_2/b_1) = b_1$. We can reduce to $U(b_2) = 2$ by working over a Morley sequence in $\mathrm{tp}(b_1/b_2)$ of length $U(b_2) - 2$ independent from b_1 over b_2, and by finiteness of canonical bases (Lemma 3.15 below) we can then replace b_1 with $\mathrm{Cb}(b_2/b_1)$ (which is interalgebraic with b_1). Note that unlike the family of plane curves given by nonlinearity, the points of the pseudoplane we obtain this way are not necessarily in D^2.

3.5 Finiteness of canonical bases in D^{eq}

If D is strongly minimal, it is totally transcendental, and so types of elements of D^{eq} have finite canonical bases. In this technical subsection, we prove that this conclusion holds for arbitrary minimal D. We will make repeated use of this lemma in later sections, but a reader who is happy to restrict attention to strongly minimal D may omit it.

Lemma 3.15. *Let $C \subseteq \mathbb{M}^{\mathrm{eq}}$ and $a \in D^{\mathrm{eq}}$, and let $q := \mathrm{tp}(a/C)$.*

(a) If q is stationary, then $\mathrm{Cb}(q)$ is ($\mathrm{dcl}^{\mathrm{eq}}$ of) an element of D^{eq}.

(b) q has finitely many global non-forking extensions.

Proof. We first prove (a) and (b) under the assumption that $a \in D^n$.

(a) If $a = (a_1, \ldots, a_n)$, after an appropriate co-ordinate permutation say $\overline{a} := (a_1, \ldots, a_k) \models \mathfrak{p}\restriction_C^{(k)}$ and $\overline{b} := (a_{k+1}, \ldots, a_n) \in \mathrm{acl}^{\mathrm{eq}}(C, \overline{a})$.

Say $\phi_c(\overline{a}, y) \in \mathrm{tp}(\overline{b}/C\overline{a})$ with $c \in \mathrm{dcl}^{\mathrm{eq}}(C)$ and $|\phi_c(\overline{a}, \mathbb{M})|$ minimal, so $\phi_c(\overline{a}, y)$ isolates $\mathrm{tp}(\overline{b}/C\overline{a})$. Then clearly $q(x, y)$ is equivalent to $\mathfrak{p}\restriction_C^{(k)}(x) \cup \{\phi_c(x, y)\}$, and the global non-forking extension $\mathfrak{q}$ of q is equivalent to $\mathfrak{p}^{(k)}(x) \cup \{\phi_c(x, y)\}$. So $\mathfrak{q}$ is invariant over c, and so $\mathrm{Cb}(q) \in \mathrm{dcl}^{\mathrm{eq}}(c)$. Since also $\mathrm{Cb}(q)$ is in $\mathrm{dcl}^{\mathrm{eq}}$ of a Morley sequence in q (by [Pal17, Proposition 4.19]), we conclude $\mathrm{Cb}(q) \in D^{\mathrm{eq}}$, as required[2].

(b) If $\mathfrak{q}$ is a global non-forking extension, then $\mathrm{Cb}(\mathfrak{q}) \in \mathrm{acl}^{\mathrm{eq}}(C)$, and the global non-forking extensions of q are the conjugates of $\mathfrak{q}$ over C, which are in bijective correspondence with the finitely many conjugates of $\mathrm{Cb}(\mathfrak{q})$ over C.

Now for arbitrary $a \in D^{\mathrm{eq}}$, say $a = f(b)$ where $b \in D^n$ and f is $\emptyset$-definable, (b) holds since it holds for $\mathrm{tp}(b/C)$.

It remains to deduce (a). So suppose q is stationary.

Claim 3.16 ([Pil96, Lemma 1.3.19, first part])**.** *Let q be a stationary type in a stable theory with $U(q\restriction_\emptyset) < \infty$. Then there is $e \in \mathrm{Cb}(q)$ with $\mathrm{Cb}(q) \subseteq \mathrm{acl}^{\mathrm{eq}}(e)$.*

Proof. Suppose there is no $e \in \mathrm{Cb}(q)$ over which q does not fork. We build an infinite sequence $\emptyset = e_0, e_1, \ldots \in \mathrm{Cb}(q)$ such that $\mathrm{dcl}^{\mathrm{eq}}(e_i) \subseteq \mathrm{dcl}^{\mathrm{eq}}(e_{i+1})$ and $q\restriction_{e_{i+1}}$ forks over e_i, contradicting U-rankedness of $q\restriction_\emptyset$. Indeed, given $e_i \in \mathrm{Cb}(q)$, $q\restriction_{\mathrm{Cb}(q)}$ forks over e_i; then since forking is witnessed by a formula, already $q\restriction_{e_{i+1}}$ forks over e_i for some $e_{i+1} \in \mathrm{Cb}(q)$, and we may assume $e_i \in \mathrm{dcl}^{\mathrm{eq}}(e_{i+1})$. □

Now $U(q\restriction_\emptyset) = U(a) \leq U(b) \leq n$, so let e be as in the Claim for $q = \mathrm{tp}(a/C)$; then by (b) applied to $\mathrm{tp}(a/e)$, $\mathrm{Cb}(q)$ has only finitely many conjugates over e, and hence $\mathrm{Cb}(q) = \mathrm{dcl}^{\mathrm{eq}}(ee')$ for some finite part e' of $\mathrm{acl}^{\mathrm{eq}}(e)$, as required. □

[2]Thanks to the anonymous reviewer for a suggestion which led to an optimisation of this part of the proof.

4 Unimodularity

4.1 Preliminaries

Throughout this section, D is a minimal set over $\emptyset$.

Definition. If $a \in \operatorname{acl}^{\mathrm{eq}}(B)$, then $\operatorname{mult}(a/B) = |\{a' \in \mathbb{M}^{\mathrm{eq}} \mid a' \equiv_B a\}|$.

Remark. Multiplicity is multiplicative: $\operatorname{mult}(ab/C) = \operatorname{mult}(a/bC)\operatorname{mult}(b/C)$.

Lemma 4.1. *If* $\operatorname{tp}(a/c)$ *is stationary and* $b \in \operatorname{acl}^{\mathrm{eq}}(c)$*, then*

$$\operatorname{mult}(b/ac) = \operatorname{mult}(b/c).$$

Proof. Suppose $\sigma \in \operatorname{Aut}(\mathbb{M}^{\mathrm{eq}}/c)$. By stationarity, $\operatorname{tp}(a/c) \vDash \operatorname{tp}(a/cb)$. So $ab \equiv_c a^{\sigma^{-1}} b \equiv_c ab^{\sigma}$, so $b \equiv_{ac} b^{\sigma}$. □

4.2 Unimodularity

Definition. D is <u>unimodular</u> if whenever $a_i \vDash \mathfrak{p}\restriction_{\emptyset}^{(n)}$, $i = 1, 2$, and $\operatorname{acl}_D(a_1) = \operatorname{acl}_D(a_2)$, then $\operatorname{mult}(a_1/a_2) = \operatorname{mult}(a_2/a_1)$.

Example. An algebraically closed field is <u>not</u> unimodular: consider a and a^2 where a is generic.

In Section 7, we will show

Theorem 4.2 (Hrushovski)**.** *If D is unimodular, then D is 1-based.*

ω-categorical theories provide the motivating example of unimodular minimal sets:

Lemma 4.3. *If D is a minimal set in an ω-categorical theory, then D is unimodular.*

Proof. By Ryll-Nardzewski, acl_D is locally finite, i.e. $\operatorname{acl}_D(A)$ is finite for A finite. Let $a_i \vDash \mathfrak{p}\restriction_{\emptyset}^{(n)}$, $i = 1, 2$, be interalgebraic, and let $X := \operatorname{acl}_D(a_i)$.

For $c \in X^k$, let $\operatorname{mult}_X(c) := |\{c' \in X \mid c' \equiv c\}|$.

Then

$$\operatorname{mult}(a_1/a_2)\operatorname{mult}_X(a_2) = \operatorname{mult}_X(a_1a_2) = \operatorname{mult}(a_2/a_1)\operatorname{mult}_X(a_1)$$

and $\operatorname{mult}_X(a_1) = \operatorname{mult}_X(a_2)$ since $a_1 \equiv a_2$. So $\operatorname{mult}(a_1/a_2) = \operatorname{mult}(a_2/a_1)$. □

So by Theorem 4.2, minimal sets in ω-categorical theories are locally modular. This result is due originally to Zilber, and forms a key part of his study of totally categorical theories (uncountably categorical theories which are also ω-categorical); Hrushovski defined unimodularity as a way of understanding Zilber's proof (specifically, what Hrushovski terms "a series of computations of puzzling success").

It is also worth mentioning that this yields the purely combinatorial consequence that any homogeneous locally finite pregeometry is locally modular, by considering it as an ω-categorical strongly minimal structure in the language with a predicate for $x \in \operatorname{cl}(y_1, \ldots, y_n)$ for each n (see [Pil96, Proposition 2.4.22]).

Pseudofinite strongly minimal sets provide another example of unimodularity, via the pigeonhole principle. This was observed by Pillay in [Pil14]; we give a proof following the lines of his argument.

Lemma 4.4. *If D is a strongly minimal set definable in a pseudofinite theory T, then D is unimodular.*

Proof. T is pseudofinite so has a model which is an ultraproduct of finite structures, say $M = \Pi_{\mathcal{U}} M_i \vDash T$ with M_i finite. For X $\emptyset$-definable, let $|X| := \Pi_{\mathcal{U}} \mid X^{M_i} \mid \in \mathbb{N}^{\mathcal{U}}$.

Claim 4.5. *Let $X \subseteq D^n$ be $\emptyset$-definable. Then there exists a unique polynomial $p_X(q) \in \mathbb{Z}[q]$ of degree* $\mathrm{RM}(X)$ *such that*

$$|X| = p_X(|D|)$$

Proof. It is clear that if such a polynomial exists, it is unique. We show existence.

If $X = \emptyset$, then $p_X(q) := 0$ is as required.

Else, say $\mathrm{RM}(X) = d \geq 0$. We can find $k \in \mathbb{N}$ and $\emptyset$-definable $X' \subseteq X$ with $\mathrm{RM}(X') = d$, and a co-ordinate projection map $\pi : X' \to D^d$ with all fibres of size k and $\mathrm{RM}(\pi(X')) = d$.

By induction on Morley rank and degree, we have polynomials as in the statement for the cardinalities of $X \setminus X'$ and $D^d \setminus \pi(X')$. Then

$$p_X(q) := p_{X \setminus X'}(q) + k(q^d - p_{D^d \setminus \pi(X')}(q))$$

is as required. □

Now let $a_i \in D^n$, $i = 1, 2$, be interalgebraic generics. Let X have minimal Morley degree among $\emptyset$-definable sets with $(a_1, a_2) \in X$ and $\mathrm{RM}(X) = n$.

Let $k_i := \mathrm{mult}(a_1 a_2 / a_i)$. Then for $i = 1, 2$, there exists $\emptyset$-definable $X_i' \subseteq X$ with $\mathrm{RM}(X_i') = n$ and a projection $\pi_i : X_i' \to D^n$ with fibres of size k_i and $\mathrm{RM}(\pi_i(X')) = n$. Then $\mathrm{RM}(X \setminus X_i') < n$ by the minimality assumption on $\mathrm{dM}(X)$, and $\mathrm{RM}(D^n \setminus \pi(X_i')) < n$, so the corresponding polynomials have degree $< n$. So by the construction in the proof of the claim, the leading term of $p_X(q)$ is $k_i q^n$. But $p_X(q)$ is well-defined, so $k_1 = k_2$. □

So by Theorem 4.2, strongly minimal pseudofinite sets are locally modular.

In fact, the theorem of Zilber and Hrushovski that ω-categorical strongly minimal sets are quasifinitely, but not finitely, axiomatisable implies that they are pseudofinite, so in the end this second example is a generalisation of the first.

Using two further results which we won't have time to develop in this course - that 1-basedness of a finite U-rank theory is equivalent to 1-basedness of its minimal types, and Buechler's dichotomy which says that any minimal type in a superstable theory which is not locally modular is actually strongly minimal, Pillay [Pil14] deduces:

Corollary 4.6. *Any pseudofinite stable theory of finite U-rank is 1-based.*

5 Germs of definable functions, and Hrushovski-Weil

Notation. Due to alphabetic exhaustion, in this section and the next we will often use x, y, z as elements of $\mathbb{M}^{\mathrm{eq}}$ rather than variables.

5.1 Germs of definable functions

Definition. Let $\mathfrak{p}, \mathfrak{q} \in S(\mathbb{M})$. If f_1, f_2 are definable partial functions defined at $\mathfrak{p}$, meaning $\mathfrak{p}(x) \vDash x \in \operatorname{dom}(f_i)$, then say they have the same germ at $\mathfrak{p}$ if $\mathfrak{p}(x) \vDash f_1(x) = f_2(x)$.

The germ of f at $\mathfrak{p}$ is the equivalence class $\widetilde{f}$ under this equivalence relation.

Write $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$ if $\mathfrak{p}(x) \vDash \mathfrak{q}(f(x))$ for some (any) representative f (i.e. $f_*(\mathfrak{p}) = \mathfrak{q}$).

Given $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$ and $\sigma \in \operatorname{Aut}(\mathbb{M})$, we obtain a well-defined germ $\widetilde{f}^\sigma : \mathfrak{p}^\sigma \to \mathfrak{q}^\sigma$. Say a (possibly long) tuple $\bar{b}$ in $\mathbb{M}^{\mathrm{eq}}$ is a code for $\widetilde{f}$ if $\forall \sigma \in \operatorname{Aut}(\mathbb{M}).(\bar{b} = \bar{b}^\sigma \Leftrightarrow \widetilde{f} = \widetilde{f^\sigma})$, and then define $\ulcorner \widetilde{f} \urcorner := \operatorname{dcl}^{\mathrm{eq}}(\bar{b})$. Here, $\widetilde{f} = \widetilde{f^\sigma}$ should be understood as implying $\mathfrak{p} = \mathfrak{p}^\sigma$. We will see that in a stable theory, $\widetilde{f}$ has a code $b \in \mathbb{M}^{\mathrm{eq}}$ if $\operatorname{Cb}(\mathfrak{p}) \in \mathbb{M}^{\mathrm{eq}}$.

If p and q are stationary types and $\mathfrak{p}, \mathfrak{q}$ their global non-forking extensions, a germ at p is defined as a germ at $\mathfrak{p}$, and $\widetilde{f} : p \to q$ means $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$.

Remark. Composition of germs on global types is well-defined, hence composition of germs on stationary types is well-defined. So we have a category with objects the stationary types, and morphisms the germs. Write $\widetilde{f} : \mathfrak{p} \xrightarrow{\sim} \mathfrak{q}$ if $\widetilde{f}$ is invertible in this category. This is equivalent to f being injective on $\mathfrak{p}$: indeed, this is clearly required, and if it holds then by compactness f is already injective on some $\phi \in \mathfrak{p}$, and so f has a well-defined definable inverse.

Note that $\ulcorner \widetilde{f} \circ \widetilde{g} \urcorner \subseteq \operatorname{dcl}^{\mathrm{eq}}(\ulcorner \widetilde{f} \urcorner, \ulcorner \widetilde{g} \urcorner)$, and $\ulcorner \widetilde{f}^{-1} \urcorner = \ulcorner \widetilde{f} \urcorner$.

Remark. Since we work in a stable theory, $\mathfrak{p}$ and $\mathfrak{q}$ are definable. Suppose $\mathfrak{p}$ is $\emptyset$-definable. Then given a $\emptyset$-definable family f_z of partial functions, equality of germs at $\mathfrak{p}$ is definable by the equivalence relation $E(b, c)$ defined as

$$d_{\mathfrak{p}} x.(x \in \operatorname{dom}(f_b) \cap \operatorname{dom}(f_c) \wedge f_b(x) = f_c(x)),$$

i.e. $E(b, c) \Leftrightarrow \widetilde{f_b} = \widetilde{f_c}$, and then since $\widetilde{f_b}^\sigma = \widetilde{f_{b^\sigma}}$, we have

$$\ulcorner \widetilde{f_b} \urcorner = \operatorname{dcl}^{\mathrm{eq}}(b/E).$$

Definition. If $p, q \in S(\emptyset)$ are stationary, and a germ $\widetilde{f} : p \to q$ has a representative f defined over b, and $a \vDash p{\restriction}_b$, we define $\widetilde{f}(a) := f(a) \vDash q{\restriction}_b$. This is well-defined: if $g = g_c$ is another representative defined over some c such that also $a \vDash p{\restriction}_c$, then let $e := \ulcorner \widetilde{f} \urcorner \subseteq \operatorname{dcl}^{\mathrm{eq}}(b) \cap \operatorname{dcl}^{\mathrm{eq}}(c)$ and let $c' \equiv_e c$ with $c' \mathop{\smile\!\!\!\!\!\!\mid}_e abc$. Then $a \vDash p{\restriction}_{bc'}$ and $a \vDash p_{cc'}$, so $f(a) = g_{c'}(a) = g_c(a)$.

Definition. If $p, q, s \in S(\emptyset)$ are stationary, a family $\widetilde{f_s}$ of germs $p \to q$ is the family $\widetilde{f_s} := (\widetilde{f_b})_{b \vDash s}$ of germs at p of a $\emptyset$-definable family f_z of partial functions, which is such that $\widetilde{f_b} : p \to q$ whenever $b \vDash s$.

The family is canonical if b is a code for $\widetilde{f_b}$, for all $b \vDash s$.

The family is generically transitive if $\widetilde{f_b}(x) \mathop{\smile\!\!\!\!\!\!\mid} x$ for some (any) b and x such that $b \vDash s$ and $x \vDash p{\restriction}_b$.

Remark. $\widetilde{f_s}$ is generically transitive iff when $x \vDash p$ and $y \vDash q{\restriction}_x$, there exists $b \vDash s$ such that $\widetilde{f_b}(x) = y$.

Remark. Suppose $p, q, s \in S(\emptyset)$ are stationary, and $\widetilde{f_s}$ is a family of germs $p \to q$. Let $b \vDash s$ and $x \vDash p\restriction_b$, and let $y = \widetilde{f_b}(x)$. Then

$$x \perp b;\ y \perp b;\ y \in \mathrm{dcl}^{\mathrm{eq}}(bx). \tag{2}$$

Conversely, if (b, x, y) satisfy (2), and $s := \mathrm{tp}(b)$, $p := \mathrm{tp}(x)$, and $q := \mathrm{tp}(y)$ are stationary, let $f_b(x) = y$ be a formula witnessing $y \in \mathrm{dcl}^{\mathrm{eq}}(bx)$. Then $\widetilde{f_s}$ is a family of germs $p \to q$.

Lemma 5.1. *In the correspondence of the previous remark,*

(i) $\widetilde{f_b}$ can be taken to be invertible iff also $x \in \mathrm{dcl}^{\mathrm{eq}}(by)$;

(ii) the family $\widetilde{f_s}$ is generically transitive iff $x \perp y$;

(iii) $\ulcorner \widetilde{f_b} \urcorner = \mathrm{Cb}(xy/b)$ (so $\widetilde{f_s}$ is canonical iff $\mathrm{Cb}(xy/b) = b$).

Proof.

(i) Exercise.

(ii) Immediate from the definition.

(iii) Let $\sigma \in \mathrm{Aut}(\mathbb{M})$. Let $\mathfrak{p}$ be the global nonforking extension of p, so $\mathfrak{p}^\sigma = \mathfrak{p}$. Let $\mathfrak{r}$ be the global nonforking extension of $\mathrm{stp}(xy/b)$. Then $\mathfrak{r}$ is equivalent to $\mathfrak{p}(x) \cup \{y = f_b(x)\}$.

$$\begin{aligned}
\mathrm{Cb}(xy/b)^\sigma = \mathrm{Cb}(xy/b) &\Leftrightarrow \mathrm{Cb}(\mathfrak{r})^\sigma = \mathrm{Cb}(\mathfrak{r}) \\
&\Leftrightarrow \mathfrak{r}^\sigma = \mathfrak{r} \\
&\Leftrightarrow \mathfrak{p}(x) \cup \{y = f_{b^\sigma}(x)\} \equiv \mathfrak{p}(x) \cup \{y = f_b(x)\} \\
&\Leftrightarrow \mathfrak{p}(x) \vDash f_{b^\sigma}(x) = f_b(x) \\
&\Leftrightarrow \widetilde{f_b}^\sigma = \widetilde{f_b}.
\end{aligned}$$

□

Remark. Replacing b with $\mathrm{Cb}(xy/b)$ preserves the conditions of (2). By Lemma canonFamCB(iii), this shows that any germ is part of a canonical family. This is sometimes known as the existence of *strong codes* for germs in stable theories.

5.2 Hrushovski-Weil

Recall that a $\bigwedge$-definable homogeneous space (G, S) consists of a $\bigwedge$-definable group G and a $\bigwedge$-definable set S, and a relatively definable transitive action of G on S. We say G is connected if it has no relatively definable proper subgroup of finite index, and we say (G, S) is connected if G is connected. We say (G, S) is faithful if the action is faithful, i.e. only the identity element of G acts trivially on S. Recall also the notion of a generic type of a homogeneous space [Pal17, Definition 6.1].

Lemma 5.2 (Hrushovski-Weil)**.** *Let $p, s \in S(\emptyset)$ be stationary, and suppose $\widetilde{f_s}$ is a generically transitive canonical family of invertible germs $p \to p$. Suppose that $\widetilde{f_s}$ is closed under inverse and generic composition, meaning that for $b \vDash s$*

there exists $b' \models s$ *such that* $\widetilde{f}_b^{-1} = \widetilde{f}_{b'}$*, and for* $b_1 b_2 \models s^{(2)}$*, there exists* b_3 *such that*

$$\widetilde{f}_{b_1} \circ \widetilde{f}_{b_2} = \widetilde{f}_{b_3}$$

and $b_i b_3 \models s^{(2)}$ *for* $i = 1, 2$.

Then there exists a connected faithful $\bigwedge$*-definable/*$\emptyset$ *homogeneous space* (G, S)*, a definable embedding of* s *into* G *as its unique generic type, and a definable embedding of* p *into* S *as its unique generic type, such that the generic action of* s *on* p *agrees with that of* G *on* S*, i.e.* $\widetilde{f}_b(a) = b * a$ *for* $b \models s$ *and* $a \models p\restriction_b$.

Remark. This is essentially the Hrushovski-Weil "group chunk" theorem. There, one starts with a generically associative binary operation $*$, and applies the lemma to the germs of $x \mapsto a * x$ to obtain a group structure extending $*$.

Proof. Let G be the group of germs generated by $\widetilde{f}_s$.

Claim 5.3. *Any element of* G *is a composition of two generators.*

Proof. Since the family is closed under inverses, the identity is the composition of two generators. Since s is a complete type, it follows from generic composability that any generator is the composition of two generators. So it suffices to see that any composition of three generators

$$\widetilde{f}_{b_1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$$

is the composition of two.

Let $b' \models s\restriction_{b_1 b_2 b_3}$. Then

$$\widetilde{f}_{b_1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3} = \widetilde{f}_{b_1} \circ \widetilde{f}_{b'} \circ \widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$$

Now $b' \mathop{\smile\!\!\!\!\!\mid} b_2$, so say $\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} = \widetilde{f}_{b''}$ with $b'' \models s$ independent from b' and from b_2. Now $b' \mathop{\smile\!\!\!\!\!\mid}_{b_2} b_3$, so $b'' \mathop{\smile\!\!\!\!\!\mid}_{b_2} b_3$, since $b'' \in \mathrm{dcl}^{\mathrm{eq}}(b' b_2)$, so since $b'' \mathop{\smile\!\!\!\!\!\mid} b_2$, we have $b'' \mathop{\smile\!\!\!\!\!\mid} b_3$. Also $b' \mathop{\smile\!\!\!\!\!\mid} b_1$. So the germs $\widetilde{f}_{b_1} \circ \widetilde{f}_{b'}$ and $\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$ appear in the family $\widetilde{f}_s$. □

Now G is $\bigwedge$-definable as pairs of realisations of s, modulo equality of the corresponding composition of germs, and the group operation is defined by composition of germs. We identify s with its image in G under the embedding $b \mapsto \widetilde{f}_b$, which is a completion of G as a $\bigwedge$-definable group.

We show that G is connected with generic type s by showing that if $g \in G$, then for some (and hence any) $b \models s\restriction_g$, we have $g * b \models s\restriction_g$. This holds for $g \models s$, by assumption. Let $g \in G$, say $g = g_1 * g_2$ with $g_1, g_2 \models s$. Let $b \models s\restriction_{g_1, g_2}$. Then $g_2 * b \models s\restriction_{g_1, g_2}$, and so $g_1 * g_2 * b \models s\restriction_{g_1, g_2}$. Now $g = g_1 * g_2 \in \mathrm{dcl}^{\mathrm{eq}}(g_1, g_2)$, so $b \models s\restriction_g$ and $g * b \models s\restriction_g$, as required.

G acts generically on p by application of germs, i.e. $g * a := g(a)$ if $a \models p\restriction_g$.

Now define $S := (G \times p)/E$ where $(g, a)E(g', a')$ iff $(h * g) * a = (h * g') * a'$ for $h \models s\restriction_{aa'gg'}$, which is definable by definability of s. Define the action of G by $h * ((g, a)/E) := (h * g, a)/E$. This is well-defined, since if $(g, a)E(g', a')$ and $h \in G$, then if $h' \models s\restriction_{g, g', a, a', h}$, then also $h' * h \models s_{g, g', a, a', h}$ by genericity, and we have $h' * h * g * a = h' * h * g' * a'$, so $(h * g, a)E(h * g', a')$. p embeds via $a \mapsto (1, a)/E$.

We show transitivity. Let $a, a' \vDash p$, and we show $(1, a')/E \in G * (1, a)/E$; this suffices for transitivity, since clearly $(G, a')/E \subseteq G*(1, a')/E$. Let $c \vDash p\restriction_{aa'}$. Then by generic transitivity of $\widetilde{f_s}$, there exist $g, g' \vDash s$ such that $g * a = c$ and $g' * c = a'$. Then $(h * g) * a = h * c$ for $h \vDash s\restriction_{acg}$, so $(g, a)E(1, c)$. Similarly $(g', c)E(1, a')$. So $(g' * g) * (1, a)/E = g' * (g, a)/E = g' * (1, c)/E = (1, a')/E$.

For faithfulness of the action: suppose g acts trivially, and let $a \vDash p\restriction_g$. Then $(g, a)E(1, a)$, so let $h \vDash s\restriction_{ag}$; then $(h*g)*a = h*a$. But $h \perp\!\!\!\perp_g a$, so $h, h*g \perp\!\!\!\perp_g a$, so $h, h*g \perp\!\!\!\perp a$ since $g \perp\!\!\!\perp a$. So $h*g = h$ as germs, so $g = 1$.

Finally, we claim that p is the unique generic type. If $g \in G$ and $a \vDash p\restriction_g$, then $g * a \vDash p\restriction_g$ as this is the action of a germ. But G acts transitively on the generic types [Pal17, Proposition 6.13(2)]. So p is the unique generic type. $\square$

In the context of the group configuration, we work with definable families of bijections between two types, rather than from a type to itself. The following key lemma gives a condition for this to give rise to a $\bigwedge$-definable group.

Lemma 5.4 ("Hrushovski-Weil for bijections")**.** *Suppose* $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$, *let* $p, q, r \in S(\emptyset)$, *and suppose* $\widetilde{f_r}$ *is a generically transitive canonical family of invertible germs* $p \to q$. *Let* b_1 *and* b_2 *be independent realisations of* r *and say*

$$\widetilde{f_{b_1}}^{-1} \circ \widetilde{f_{b_2}} = \widetilde{g_c}$$

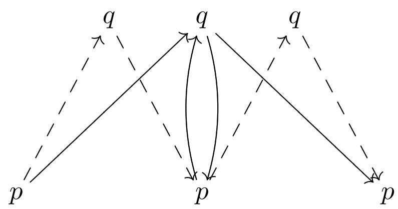
with $\widetilde{g_s}$ *a canonical family of invertible germs* $p \to p$, *where* $s = \mathrm{tp}(c)$, *and suppose*

$$c \perp\!\!\!\perp b_i \textit{ for } i = 1, 2. \tag{3}$$

Then $\widetilde{g_s}$ *satisfies the assumptions, and hence the conclusions, of Lemma 5.2.*

Proof. Let $c' \vDash s\restriction_c$. Let $b \vDash r\restriction_{c,c'}$. Then by (3), $bc \equiv b_2c$, so say $b_1'bc \equiv b_1b_2c$; similarly, $bc' \equiv b_1c$, so say $bb_2'c' \equiv b_1b_2c$. Then

$$\widetilde{g_c} \circ \widetilde{g_{c'}} = \widetilde{f_{b_1'}}^{-1} \circ \widetilde{f_b} \circ \widetilde{f_b}^{-1} \circ \widetilde{f_{b_2'}} = \widetilde{f_{b_1'}}^{-1} \circ \widetilde{f_{b_2'}}.$$



Now $b_i' \perp\!\!\!\perp b$ by choice of b_i', since $b_1 \perp\!\!\!\perp b_2$. Also $b_1' \perp\!\!\!\perp_b b_2'$, since $c \perp\!\!\!\perp_b c'$, since $c \perp\!\!\!\perp c'$ and $b \perp\!\!\!\perp cc'$. Hence $b_1' \perp\!\!\!\perp bb_2'$, so $b_1' \perp\!\!\!\perp b_2'$ and $b \perp\!\!\!\perp b_1'b_2'$.

So $(b_1', b_2') \vDash r^{(2)}$, so say $\widetilde{f_{b_1'}}^{-1} \circ \widetilde{f_{b_2'}} = \widetilde{g_{c''}}$ with $c'' \vDash s\restriction_{b_i'}$.

Then since $b \perp\!\!\!\perp b_1'b_2'$, we have $b \perp\!\!\!\perp c''b_1'$, hence $c'' \perp\!\!\!\perp b_1'b$, and so $c'' \perp\!\!\!\perp c$. Similarly $c'' \perp\!\!\!\perp c'$.

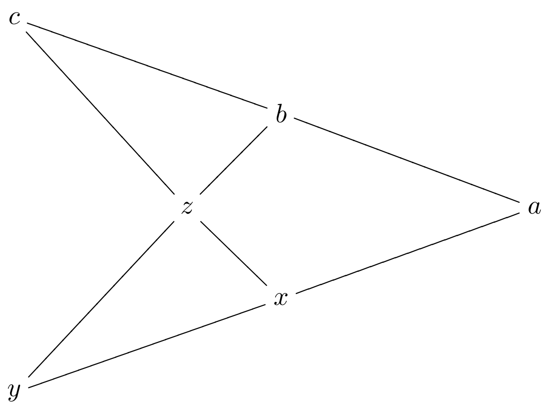
Finally, we must check that $\widetilde{g_s}$ is generically transitive. So let $x \vDash p$, let $b_2' \vDash r\restriction_x$, and let $b_1' \vDash r\restriction_{xb_2'}$. Let $y := \widetilde{f_{b_2'}}(x)$ and $z := \widetilde{f_{b_1'}}^{-1}(y)$. Then $y \perp\!\!\!\perp x$ by generic transitivity, and $b_1' \perp\!\!\!\perp yx$, so $y \perp\!\!\!\perp xb_1'$, i.e. $y \vDash q\restriction_{xb_1'}$, and so $z \vDash p\restriction_{xb_1'}$, and in particular $x \perp\!\!\!\perp z$. Since $b_1'b_2' \equiv b_1b_2$, this proves generic transitivity of $\widetilde{g_s}$. $\square$

Exercise. Assuming finite U-rank, (3) holds iff $U(s) = U(r)$.

6 The Group Configuration Theorem

We continue to work in a monster model $\mathbb{M}^{\mathrm{eq}}$ of an arbitrary stable theory T.

Definition.



(a, b, c, x, y, z) forms a group configuration if

- any non-collinear triple in the above diagram is independent,
- $\mathrm{acl}^{\mathrm{eq}}(ab) = \mathrm{acl}^{\mathrm{eq}}(bc) = \mathrm{acl}^{\mathrm{eq}}(ac)$,
- $\mathrm{acl}^{\mathrm{eq}}(ax) = \mathrm{acl}^{\mathrm{eq}}(ay)$ and $\mathrm{acl}^{\mathrm{eq}}(a) = \overline{\mathrm{Cb}}(xy/a)$; similarly for bzy and czx.

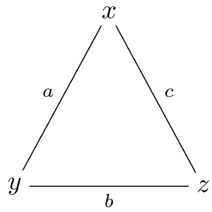
Remark. Suppose (G, S) is a connected faithful $\bigwedge$-definable/$\emptyset$ homogeneous space. Let (a, b, x) be an independent triple with a, b generics of G and x a generic of S, and define $b := c * a$, $x := a * y$, $z := b * y$ (so $z = c * a * y = c * x$). Then (a, b, c, x, y, z) forms a group configuration. (It follows from faithfulness that e.g. $\mathrm{Cb}(xy/a) = a$; see [Pil96, Remark 4.1] for a proof.)

Call such an (a, b, c, x, y, z) a group configuration of (G, S).

Theorem 6.1 (Group Configuration Theorem)**.** *Suppose (a, b, c, x, y, z) forms a group configuration. Then, after possibly expanding the language by parameters B with $B \perp\!\!\!\perp abcxyz$, there is a connected faithful $\bigwedge$-definable/$\emptyset$ homogeneous space (G, S), and a group configuration (a', b', c', x', y', z') of (G, S), such that each unprimed element is interalgebraic with the corresponding primed element.*

Example. In ACF, we can restate as follows: (b, z, y) extends to a group configuration (a, b, c, x, y, z) iff it is a generic point of a "pseudo-action", i.e. iff there is an algebraic group G acting birationally on a variety S, and there are generically finite-to-finite algebraic correspondences $f : G' \leftrightarrow G$, $g_1 : S'_1 \leftrightarrow S$ and $g_2 : S'_2 \leftrightarrow S$, such that (b, z, y) is a generic point of the image under (f, g_1, g_2) of the graph $\Gamma_* \subseteq G \times S \times S$ of the action. (c.f. [HZ96, 6.2].)

Remark. The above diagram is the traditional one, but the following alternative diagram is also suggestive (think of it as a commuting triangle)



Proof. We prove this by repeatedly applying the following two operations to transform (a, b, c, x, y, z) into the group configuration of a homogeneous space:

- add independent parameters to the language - we refer to this as "base-changing" to the parameters;
- replace any point of the configuration with an interalgebraic point of $\mathbb{M}^{\mathrm{eq}}$ - we refer to this as e.g. "interalgebraically replacing" b with b', implicitly claiming that $\mathrm{acl}^{\mathrm{eq}}(b) = \mathrm{acl}^{\mathrm{eq}}(b')$.

Note that these operations transform a group configuration into a group configuration.

By base-changing whenever necessary, we will assume throughout that $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$, so types over $\emptyset$ are stationary.

The proof will comprise three steps:

(I) "reduce $\mathrm{acl}^{\mathrm{eq}}$ to $\mathrm{dcl}^{\mathrm{eq}}$" to show we may assume (b, z, y) to define a generically transitive canonical family of invertible germs via Lemma 5.1;

(II) prove this family satisfies the independence assumption of Lemma 5.4;

(III) connect the resulting homogeneous space to the group configuration.

And so it begins.

(I) First, a claim.

Claim 6.2. *If (a, b, c, x, y, z) is a group configuration and if we let $\widetilde{z} \in \mathbb{M}^{\mathrm{eq}}$ be the set $\widetilde{z} = \{z_1, \ldots, z_d\}$ of conjugates z_i of z over $ybxc$, then $\widetilde{z}$ is interalgebraic with z.*

(Here, $\{z_1, \ldots, z_d\} := (z_1, \ldots, z_d)/S_d$ is the quotient of $(z_1, \ldots, z_d)$ under the action by permutations of the symmetric group S_d.)

Proof. It suffices that the conjugates be interalgebraic, $\mathrm{acl}^{\mathrm{eq}}(z_i) = \mathrm{acl}^{\mathrm{eq}}(z_j)$. Indeed, then $\mathrm{acl}^{\mathrm{eq}}(\widetilde{z}) \subseteq \mathrm{acl}^{\mathrm{eq}}(z_1, \ldots, z_n) = \mathrm{acl}^{\mathrm{eq}}(z)$; and $z \in \mathrm{acl}^{\mathrm{eq}}(\widetilde{z})$, since it satisfies the algebraic formula $z \in \widetilde{z}$.

But indeed: $c \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_z b$, so $cx \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\widetilde{z}} by$, so setting $B := \mathrm{acl}^{\mathrm{eq}}(cx) \cap \mathrm{acl}^{\mathrm{eq}}(by)$, we have $B \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{\widetilde{z}} B$ so $B \subseteq \mathrm{acl}^{\mathrm{eq}}(\widetilde{z})$. But $z \in B$. So $\mathrm{acl}^{\mathrm{eq}}(z) = B$, and so $\mathrm{acl}^{\mathrm{eq}}(z_i) = B$ for each z_i. So $\mathrm{acl}^{\mathrm{eq}}(z_i) = \mathrm{acl}^{\mathrm{eq}}(z)$. $\square$

Now let $a' \models \mathrm{tp}(a)\restriction_{abcxyz}$. Say $a'x'c' \equiv_{ybz} axc$. So (a', b, c', x', y, z) is also a group configuration. So by the Claim, the set $\widetilde{z}'$ of conjugates of z over $ybx'c'$ is interalgebraic with z. Note that $\widetilde{z}' \in \mathrm{dcl}^{\mathrm{eq}}(ybx'c')$.

So base-change to a', and interalgebraically replace y with yx', b with bc', and z with $\widetilde{z}'$. The group configuration now satisfies

$$z \in \mathrm{dcl}^{\mathrm{eq}}(by).$$

Repeating this procedure by base-changing to an independent copy of b and enlarging a and y and interalgebraically replacing x with an $\widetilde{x}$, we can also ensure that

$$x \in \mathrm{dcl}^{\mathrm{eq}}(ay).$$

Repeat once more: base-change to an independent copy c' of c, let $a'x'c' \equiv_{ybz} axc$, let $\widetilde{y}$ be the set of conjugates of y over $ba'zx'$.

Now since $x' \in \mathrm{dcl}^{\mathrm{eq}}(a'y)$ and $z \in \mathrm{dcl}^{\mathrm{eq}}(by)$, we have $zx' \in \mathrm{dcl}^{\mathrm{eq}}(ba'y)$ and so (e.g. by considering automorphisms) $zx' \in \mathrm{dcl}^{\mathrm{eq}}(ba'\widetilde{y})$. So after interalgebraically replacing b with ba', z with zx', and y with $\widetilde{y}$, as in the previous two cases $y \in \mathrm{dcl}^{\mathrm{eq}}(bz)$, and now also $z \in \mathrm{dcl}^{\mathrm{eq}}(by)$.

Finally, interalgebraically replace b with $\mathrm{Cb}(yz/b)$, noting that $\mathrm{Cb}(yz/b)$ is an element of $\mathbb{M}^{\mathrm{eq}}$ by Lemma 5.1(iii), and is interalgebraic with b by definition of a group configuration.

(II) Setting $p := \mathrm{tp}(y)$, $q := \mathrm{tp}(z)$, $r := \mathrm{tp}(b)$, (b, y, z) now corresponds via Lemma 5.1 to a generically transitive canonical family $\widetilde{f_r}$ of invertible germs $p \to q$.

We aim to apply Lemma 5.4 to obtain a group, so we must show that for $b' \vDash r\restriction_b$, if $\mathrm{dcl}^{\mathrm{eq}}(d) = \ulcorner \widetilde{f}_{b'}^{-1} \circ \widetilde{f_b} \urcorner$, we have $d \mathop{\smile\!\!\!\!\!\mid} b$ and $d \mathop{\smile\!\!\!\!\!\mid} b'$.

This holds for all $b' \vDash r\restriction_b$ or none, so we may assume $b' \mathop{\smile\!\!\!\!\!\mid} abcxyz$. Then $b' \equiv_{xcz} b$, so say $b'y'a' \equiv_{xcz} bya$.

Claim 6.3. *$y' \mathop{\smile\!\!\!\!\!\mid} bb'$.*

Proof. $b' \mathop{\smile\!\!\!\!\!\mid} bz$ and so $b' \mathop{\smile\!\!\!\!\!\mid}_z b$, and $b \mathop{\smile\!\!\!\!\!\mid} z$, so $b \mathop{\smile\!\!\!\!\!\mid} b'z$, hence $z \mathop{\smile\!\!\!\!\!\mid}_{b'} b$. Now $y' \in \mathrm{acl}^{\mathrm{eq}}(zb')$, so $y' \mathop{\smile\!\!\!\!\!\mid}_{b'} b$. But $y' \mathop{\smile\!\!\!\!\!\mid} b'$, so $y' \mathop{\smile\!\!\!\!\!\mid} bb'$. ☐

We also have $y \mathop{\smile\!\!\!\!\!\mid} bb'$ and $f_b'^{-1}(f_b(y)) = y'$, so by Lemma 5.1, $\mathrm{dcl}^{\mathrm{eq}}(d) = \ulcorner \widetilde{f}_b'^{-1} \circ \widetilde{f_b} \urcorner = \mathrm{Cb}(yy'/bb')$.

Now $y \mathop{\smile\!\!\!\!\!\mid} abc$, and $b' \mathop{\smile\!\!\!\!\!\mid} yabc$, so $y \mathop{\smile\!\!\!\!\!\mid} abcb'$, and since $a' \in \mathrm{acl}^{\mathrm{eq}}(cb')$, we have $y \mathop{\smile\!\!\!\!\!\mid} aa'bb'$. Since also $y' \in \mathrm{acl}^{\mathrm{eq}}(yaa')$, we have

$$yy' \mathop{\smile\!\!\!\!\!\mid}_{aa'} bb'.$$

Similarly,

$$yy' \mathop{\smile\!\!\!\!\!\mid}_{bb'} aa'.$$

So $\overline{\mathrm{Cb}}(yy'/bb') = \overline{\mathrm{Cb}}(yy'/aa'bb') = \overline{\mathrm{Cb}}(yy'/aa')$, so $d \in \mathrm{acl}^{\mathrm{eq}}(aa')$.

Claim 6.4. *$b \mathop{\smile\!\!\!\!\!\mid} aa'$ and $b' \mathop{\smile\!\!\!\!\!\mid} aa'$.*

Proof. $b \mathop{\perp}_c b'$, and then since $a \in \mathrm{acl}^{\mathrm{eq}}(cb)$ and $a' \in \mathrm{acl}^{\mathrm{eq}}(cb')$, we have $ab \mathop{\perp}_c a'b'$,

So $a' \mathop{\perp}_c ab$; but $a' \mathop{\perp} c$, so $a' \mathop{\perp} ab$, so $b \mathop{\perp}_a a'$, and then since $b \mathop{\perp} a$, we have $b \mathop{\perp} aa'$. Similarly, it follows from $a \mathop{\perp}_c a'b'$ that $b' \mathop{\perp} aa'$. $\square$

So $b \mathop{\perp} d$ and $b' \mathop{\perp} d$, as required.

(III) Let (G, S) be the connected faithful $\bigwedge$-definable connected homogeneous space obtained by Lemma 5.4 from (II). So p is the generic type of S. Let s_G be the generic type of G.

Finally, we must show that the original group configuration is interalgebraic with a group configuration of (G, S). This will involve further base-change.

First, let $b' \models \mathrm{tp}(b)\restriction_{abcxyz}$. Say $y'b' \equiv_z yb$. Say $\mathrm{dcl}^{\mathrm{eq}}(g) = \ulcorner \widetilde{f}_{b'}^{-1} \circ \widetilde{f}_b \urcorner$. By the construction of (G, S), we have $g \models s_G$, and $y, y' \models p$, and $y' = g * y$. Base-change to b', and interalgebraically replace b with $g \models s_G$ and z with $g * y \models p$, the latter interalgebraicity holding as $g * y = \widetilde{f}_{b'}^{-1}(\widetilde{f}_b(y)) = \widetilde{f}_{b'}^{-1}(z)$.

Now let $c'' \models \mathrm{tp}(c)\restriction_{abcxyz}$, and say $b''z''c'' \equiv_{axy} bzc$. Base-change to c'', and interalgebraically replace a by $b'' \models s_G$ and x by $z'' = b'' * y \models p$.

Let $h := b * a^{-1}$. Then $x = a * y$ and $z = b * y$, so $z = h * x$. By definition of G, (h, x, z) is as in Lemma 5.1, so $\mathrm{Cb}(xz/ab) = h$. But also x and z are interalgebraic over c, so $\overline{\mathrm{Cb}}(xz/ab) = \overline{\mathrm{Cb}}(xz/c) = \mathrm{acl}^{\mathrm{eq}}(c)$. So interalgebraically replace c with h.

Then (a, b, c, x, y, z) is a group configuration of (G, S), as required.

$\square$

Example. Suppose D is minimal and locally modular but non-trivial. Fact 3.4 claimed the existence of a definable group. This group can be found by applying the group configuration theorem. Expand by parameters to make D modular. By non-triviality, after possibly expanding by further parameters there are $a, b, c \in D$ such that $U(ab) = U(bc) = U(ca) = U(abc) = 2$. Let $b'c' \models \mathrm{stp}(bc/a)\restriction_{abc}$. Then $U(bcb'c') = 3$, so by modularity $U(\mathrm{acl}(bc') \cap \mathrm{acl}(b'c)) = 1$, say $d \in \mathrm{acl}(bc') \cap \mathrm{acl}(b'c) \setminus \mathrm{acl}(\emptyset)$. Then (a, b, c, b', c', d) is a group configuration.

Fact 6.5. *If S is strongly minimal, any connected faithful $\emptyset$-definable homogeneous space (G, S) is of one of the following forms:*

- *$U(G) = 1$, G is commutative, and the action is regular;*
- *$U(G) = 2$, $S = K$ is the universe of a $\emptyset$-definable algebraically closed field, and G is the affine group $K^+ \rtimes K^*$ acting as $(a, b) * x = a + bx$.*
- *$U(G) = 3$, $S = \mathbb{P}^1(K)$ is the projective line of a $\emptyset$-definable algebraically closed field, and G is the group $\mathrm{PSL}_2(K)$ of Möbius transformations.*

See [Poi01, Theorem 3.27] for a proof.

Corollary 6.6 (Field Configuration Theorem)**.** *Suppose (a, b, c, x, y, z) forms a field configuration, that is, a group configuration with $U(x) = U(y) = U(z) = 1$ and $U(a) = U(b) = U(c) = k$ where $k > 1$. Then we obtain a rank k group acting on a minimal type. So $k = 2$ or 3, and $\mathrm{tp}(x)$ is interalgebraic with the generic type of a $\bigwedge$-definable algebraically closed field.*

Proof. In the ω-stable case, or more generally whenever the $\bigwedge$-definable homogeneous space obtained from Theorem 6.1 is actually definable, this follows from Fact 6.5. The general result is stated in [Hru92, p395]; it should follow from the $\bigwedge$-definable version of Fact 6.5 in [Hru89], but proving this would go beyond the scope of this note, and I have not personally verified all the details. □

Definition. D is k-pseudolinear if whenever p is a complete minimal type with $p(x) \vDash x \in D^2$, we have $U(\mathrm{Cb}(p)) \leq k$.

Remark. 1-pseudolinear ⇔ linear.

Theorem 6.7. *Let $k > 1$. Suppose D is minimal and k-pseudolinear. Then D is locally modular.*

Proof. We may assume $\mathrm{dcl}^{\mathrm{eq}}(\emptyset) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

Suppose D is k-pseudolinear. We show D is $(k-1)$-pseudolinear.

Suppose not. So (using Lemma 3.15(a)) say $(a_2, a_3) \in D^2$, $\mathrm{tp}(a_2a_3/b_1)$ is minimal, $b_1 = \mathrm{Cb}(a_2a_3/b_1)$, $U(b_1) = k$. Now $a_2 \mathop{\smile\!\!\!\!\!\!|}\, b_1$, since else $a_2 \in \mathrm{acl}^{\mathrm{eq}}(b_1)$, and then $a_2a_3 \mathop{\smile\!\!\!\!\!\!|}_{a_2} b_1$ and so $b_1 \in \mathrm{acl}^{\mathrm{eq}}(a_2)$, contradicting $k > 1$. Similarly, $a_3 \mathop{\smile\!\!\!\!\!\!|}\, b_1$.

Let $b_2 \vDash \mathrm{stp}(b_1)\restriction_{b_1a_2a_3}$. Then $b_2a_3 \equiv b_1a_3$, so say a_1 is such that $b_2a_1a_3 \equiv b_1a_2a_3$. Let $b_3 := \mathrm{Cb}(a_1a_2/b_1b_2)$.

$U(a_1a_2/b_1b_2) = 1$ since $a_2 \in \mathrm{acl}^{\mathrm{eq}}(a_3b_1)$ and $a_3 \in \mathrm{acl}^{\mathrm{eq}}(a_1b_2)$. So by k-pseudolinearity, $U(b_3) \leq k$.

Similarly, $U(a_2a_3/b_2b_3) = 1$. Since $a_2 \mathop{\smile\!\!\!\!\!\!|}\, b_1$ and $b_2 \mathop{\smile\!\!\!\!\!\!|}\, b_1a_2$ and $b_3 \in \mathrm{acl}^{\mathrm{eq}}(b_1b_2)$, we have $a_2 \mathop{\smile\!\!\!\!\!\!|}\, b_1b_2b_3$, so $U(a_2a_3/b_1b_2b_3) = 1$. So $U(a_2a_3/b_2b_3) = U(a_2a_3/b_1b_2b_3) = U(a_2a_3/b_1) = 1$, so $a_2a_3 \mathop{\smile\!\!\!\!\!\!|}_{b_2b_3} b_1$ and $a_2a_3 \mathop{\smile\!\!\!\!\!\!|}_{b_1} b_2b_3$, so since $b_1 = \mathrm{Cb}(a_2a_3/b_1) = \mathrm{Cb}(a_2a_3/b_1b_2b_3)$, $b_1 \in \mathrm{acl}^{\mathrm{eq}}(b_2b_3)$. Similarly, $b_2 \in \mathrm{acl}^{\mathrm{eq}}(b_1b_3)$.

So $U(b_2b_3) = U(b_1b_2b_3) = U(b_1b_2) = 2k$, so $U(b_3) = k$, and similarly $U(b_1b_3) = 2k$.

Then $(b_1, b_2, b_3, a_2, a_3, a_1)$ is a field configuration, so by the Field Configuration Theorem (Corollary 6.6), there is a $\bigwedge$-definable algebraically closed field with generic type interalgebraic with $\mathfrak{p}$. But e.g. $y = 1 + b_1x + b_2x^2 + \ldots + b_{k+1}x^{k+1}$ is a $(k+1)$-dimensional family of plane curves in the field, which contradicts k-pseudolinearity (as in Example 3.11). □

7 Unimodular minimal sets are locally modular

In this section, we prove Theorem 4.2. So assume D is a unimodular minimal type; we will show that D is 1-based. Let $\mathfrak{p}$ be the unrealised global completion of D.

Lemma 7.1. *D remains unimodular on adding to the language a Morley sequence in $\mathfrak{p}\restriction_{\emptyset}$.*

Proof. Say $\bar{b}$ is such a Morley sequence. Work in the unexpanded language.

Suppose a, a' are as in the definition of unimodularity in the expanded language, so $a, a' \vDash \mathfrak{p}{\restriction}_{\bar{b}}^{(n)}$, and $a \in \operatorname{acl}^{\mathrm{eq}}(a'\bar{b})$ and $a' \in \operatorname{acl}^{\mathrm{eq}}(a\bar{b})$.

Let $\bar{b}'$ be a finite subtuple such that $\operatorname{mult}(a/a'\bar{b}) = \operatorname{mult}(a/a'\bar{b}')$ and $\operatorname{mult}(a'/a\bar{b}) = \operatorname{mult}(a'/a\bar{b}')$. Now $a\bar{b}'$ and $a'\bar{b}'$ are interalgebraic finite Morley sequences in $\mathfrak{p}{\restriction}_{\emptyset}$, so by unimodularity in the original language, $\operatorname{mult}(a\bar{b}'/a'\bar{b}') = \operatorname{mult}(a'\bar{b}'/a\bar{b}')$. So $\operatorname{mult}(a/a'\bar{b}) = \operatorname{mult}(a'/a\bar{b})$ as required. □

So since 1-basedness is invariant under adding parameters, for the purposes of proving Theorem 4.2, we may add such a Morley sequence.

So by Lemma 3.10, we may and will assume that D has wEI, and hence gEI.

Definition. For $b \in D^{\mathrm{eq}}$, define the Zilber degree by

$$Z(b) := \frac{\operatorname{mult}(b/c)}{\operatorname{mult}(c/b)},$$

where $c \vDash \mathfrak{p}{\restriction}_{\emptyset}^{(<\omega)}$ is interalgebraic with b; equivalently, using gEI, c is an acl_D-basis for $\operatorname{acl}^{\mathrm{eq}}(b) \cap D$.

Then define $Z(a/b) := Z(ab)/Z(b)$.

Lemma 7.2. *$Z(b)$ is well-defined.*

Proof. This follows from unimodularity. Indeed, if $c' \vDash \mathfrak{p}{\restriction}_{\emptyset}^{(<\omega)}$ is another acl_D-basis for $\operatorname{acl}^{\mathrm{eq}}(b) \cap D$, then

$$\begin{aligned}
\frac{\operatorname{mult}(b/c)}{\operatorname{mult}(c/b)} &= \frac{\operatorname{mult}(b/c)\operatorname{mult}(c'/bc)}{\operatorname{mult}(c/b)\operatorname{mult}(c'/bc)} \\
&= \frac{\operatorname{mult}(bc'/c)}{\operatorname{mult}(cc'/b)} \\
&= \frac{\operatorname{mult}(c'/c)\operatorname{mult}(b/cc')}{\operatorname{mult}(cc'/b)} \\
&= \frac{\operatorname{mult}(c/c')\operatorname{mult}(b/c'c)}{\operatorname{mult}(c'c/b)} \\
&= [\text{reversing above steps}] \\
&= \frac{\operatorname{mult}(b/c')}{\operatorname{mult}(c'/b)}.
\end{aligned}$$

□

Remark. For $b \in D^n$, $Z(b) \in \mathbb{N}$, since we may take c to be a subtuple of b. In the pseudofinite strongly minimal case, $Z(b)$ is the leading coefficient (not the degree!) of the polynomial p_X of Lemma 4.4, for X a definable set of minimal rank and degree in $\operatorname{tp}(b)$.

Remark. Analogous polynomials, called Zilber polynomials, can also be defined in the locally finite case, and again $Z(b)$ is then the leading coefficient. The Zilber polynomial p_X of a definable set X has the defining property that for all sufficiently large n, if $a \vDash \mathfrak{p}{\restriction}_{\emptyset}^{(n)}$ then $|X \cap \operatorname{acl}_D(a)| = p(|D \cap \operatorname{acl}_D(a)|)$.

Lemma 7.3.

(i) Z *is* $\mathrm{Aut}(\mathbb{M}^{\mathrm{eq}})$*-invariant.*

(ii) $Z(ab/c) = Z(a/bc)Z(b/c)$.

(iii) $a \in \mathrm{acl}^{\mathrm{eq}}(b) \Rightarrow Z(a/b) = \mathrm{mult}(a/b)$.

(iv) $a \in D \setminus \mathrm{acl}^{\mathrm{eq}}(b) \Rightarrow Z(a/b) = 1$.

Proof.

(i) Clear.

(ii)

$$Z(a/bc)Z(b/c) = \frac{Z(abc)}{Z(bc)}\frac{Z(bc)}{Z(c)} = \frac{Z(abc)}{Z(c)} = Z(ab/c)$$

(iii) Let c be a basis for $\mathrm{acl}^{\mathrm{eq}}(b) \cap D = \mathrm{acl}^{\mathrm{eq}}(ab) \cap D$. Then

$$\begin{aligned} Z(a/b) &= Z(ab)/Z(b) \\ &= \frac{\mathrm{mult}(ab/c)\,\mathrm{mult}(c/b)}{\mathrm{mult}(c/ab)\,\mathrm{mult}(b/c)} \\ &= \frac{\mathrm{mult}(a/bc)\,\mathrm{mult}(c/b)}{\mathrm{mult}(c/ab)} \\ &= \frac{\mathrm{mult}(ac/b)}{\mathrm{mult}(c/ab)} \\ &= \mathrm{mult}(a/b). \end{aligned}$$

(iv) Let c be a basis for $\mathrm{acl}^{\mathrm{eq}}(b) \cap D$. Then ac is a basis for $\mathrm{acl}^{\mathrm{eq}}(ab) \cap D$, so by stationarity of $\mathrm{tp}(a/b) = \mathfrak{p}\restriction_b$ and of $\mathrm{tp}(a/c) = \mathfrak{p}\restriction_c$ and by Lemma 4.1,

$$\begin{aligned} Z(ab) &= \frac{\mathrm{mult}(ab/ac)}{\mathrm{mult}(ac/ab)} \\ &= \frac{\mathrm{mult}(b/ac)}{\mathrm{mult}(c/ab)} \\ &= \frac{\mathrm{mult}(b/c)}{\mathrm{mult}(c/b)} \\ &= Z(b). \end{aligned}$$

□

Remark. In fact, the function $Z(x/y)$ is uniquely determined by (i)-(iv) (see [Hru92] for a proof).

Definition. For $p = \mathrm{tp}(a/B)$ a stationary type, let $Z(p) := Z(a/\operatorname{Cb}(p))$.

Lemma 7.4. *Suppose* $a, b, c \in D^{\mathrm{eq}}$.

(I) If $\mathrm{tp}(a/c)$ *is stationary and* $a \mathop{\smile\!\!\!\!\!\!\!\;\mid}_c b$*, then* $Z(a/bc) = Z(a/c)$.
In particular, if $\mathrm{tp}(a/c)$ *is stationary then* $Z(\mathrm{tp}(a/c)) = Z(a/c)$.

(II) $Z(a/c) = \sum_{i \in I} Z(\mathfrak{q}_i)$ *where* $(\mathfrak{q}_i)_{i \in I}$ *enumerates the global non-forking extensions of* $\mathrm{tp}(a/c)$.

(III) $Z(a/c) = \sum_i Z(a_i/bc)$ *where* $(\mathrm{tp}(a_i/bc))_i$ *enumerates the nonforking extensions of* $\mathrm{tp}(a/c)$ *to* bc*.*

Proof.

(I) We first consider two special cases.

Claim 7.5. *If* $\mathrm{tp}(a/c)$ *is stationary, then* $Z(a/bc) = Z(a/c)$

(a) when $b \in D \setminus \mathrm{acl}^{\mathrm{eq}}(ac)$*, and*

(b) when $b \in \mathrm{acl}^{\mathrm{eq}}(c)$*.*

Proof. Since $Z(a/bc) = \frac{Z(ab/c)}{Z(b/c)} = \frac{Z(b/ac)}{Z(b/c)} Z(a/c)$, it suffices to see that $Z(b/ac) = Z(b/c)$.

In case (a), by Lemma 7.3(iv) we have $Z(b/ac) = 1 = Z(b/c)$.

In case (b), by Lemma 7.3(iii) and Lemma 4.1 we have $Z(b/ac) = \mathrm{mult}(b/ac) = \mathrm{mult}(b/c) = Z(b/c)$. □

Now let b be such that $a \mathop{\smile\!\!\!\!\!|}_c b$ and let b' be a basis for $\mathrm{acl}^{\mathrm{eq}}(b) \cap D$ over $\mathrm{acl}^{\mathrm{eq}}(c) \cap D$. Using that $a \mathop{\smile\!\!\!\!\!|}_c b'$ and applying case (a) iteratively, $Z(a/b'c) = Z(a/c)$. Then since bc and $b'c$ are interalgebraic, and $\mathrm{tp}(a/bc)$ and $\mathrm{tp}(a/b'c)$ are both stationary, by case (b) twice we have $Z(a/bc) = Z(a/bb'c) = Z(a/b'c) = Z(a/c)$.

(II) Using Lemma 3.15(a), let $b := \mathrm{Cb}(a/c) \in D^{\mathrm{eq}}$, so $(\mathrm{Cb}(\mathfrak{q}_i))_{i \in I}$ enumerates the conjugates of b over c (and so $|I| = \mathrm{mult}(b/c)$ is finite). Then $a \mathop{\smile\!\!\!\!\!|}_b bc$, so by (I) we have $\sum_i Z(\mathfrak{q}_i) = \mathrm{mult}(b/c) Z(a/bc) = Z(b/c)Z(a/bc) = Z(ab/c) = Z(a/c)Z(b/ac) = Z(a/c)$, using that $b = \mathrm{Cb}(a/c) \in \mathrm{dcl}^{\mathrm{eq}}(ac)$ (as one sees by considering automorphisms).

(III) This follows from (II), since the set of global non-forking extensions of $\mathrm{tp}(a/c)$ is the union of the sets of global non-forking extensions of its non-forking extensions to bc.

□

Lemma 7.6 ("Relaxation")**.** *Suppose* a, b_1, b_2 *are such that* $U(a) = 2$*,* $U(a/b_i) = 1$*,* $b_i = \mathrm{Cb}(a/b_i)$*,* $U(b_1) \geq 1$*, and* $U(b_2) \geq 2$*.*

Then $b_1 \mathop{\smile\!\!\!\!\!|} b_2$ *iff* $b_1 \mathop{\smile\!\!\!\!\!|}_a b_2$*.*

(In words: the pairs of curves which are independent and happen to both pass through a are precisely the pairs of curves which are independent given that they both pass through a.)

Proof. First we see that this equivalence holds if $a \not\mathop{\smile\!\!\!\!\!|}_{b_1} b_2$. Indeed, then $a \in \mathrm{acl}^{\mathrm{eq}}(b_1 b_2)$, and so

$$\begin{aligned} U(b_1/b_2a) - U(b_1/a) &= U(b_1b_2a) - U(b_2a) - U(b_1a) + U(a) \\ &= U(b_1b_2) - U(a/b_2) - U(b_2) - U(a/b_1) - U(b_1) + U(a) \\ &= U(b_1b_2) - U(b_2) - U(b_1) - 1 - 1 + 2 \\ &= U(b_1/b_2) - U(b_1); \end{aligned}$$

in particular one side is zero iff the other is, as required.

Similarly, we are done if $a \not\perp_{b_2} b_1$.

If neither dependence holds, then $\mathrm{dcl}^{\mathrm{eq}}(b_1) = \mathrm{Cb}(a/b_1) = \mathrm{Cb}(a/b_1b_2) = \mathrm{Cb}(a/b_2) = \mathrm{dcl}^{\mathrm{eq}}(b_2)$. Then $b_1 \perp_a b_2$ implies $b_2 \in \mathrm{acl}^{\mathrm{eq}}(a)$, so $1 = U(a/b_2) = U(a) - U(b_2) \leq 2 - 2$, which is a contradiction, so $b_1 \not\perp_a b_2$, and similarly we find $b_1 \not\perp b_2$. $\square$

Lemma 7.7 ("Bézout"). *Suppose $p_1 \in S(b_1c)$ and $p_2 \in S(b_2c)$ are minimal types, $p_i(x) \vDash x \in D^2$, $b_i = \mathrm{Cb}(p_i)$, $c \in D^{\mathrm{eq}}$. Suppose $U(b_1/c) \geq 1$ and $U(b_2/c) \geq 2$, and $b_1 \perp_c b_2$. Then $|p_1 \cup p_2| := |\{a \mid a \vDash p_1 \cup p_2\}| = Z(p_1)Z(p_2)$.*

Remark. Thinking of p_i as plane curves, $p_1 \cup p_2$ is the intersection, so the lemma can be read as saying that for curves in "generic position" within sufficiently large families, the size of the intersection is the product of the degrees. Bézout's theorem in algebraic geometry makes an analogous claim. However, that theorem concerns complete curves rather than complete types, so all points are counted rather than only those generic on both curves. We will see that this difference is crucial.

Remark. It will turn out that this lemma can never actually be applied: using the lemma, we will show that D is linear, so no such p_2 exists.

Proof. First note that we may assume that each $\mathrm{tp}(b_i/c)$ is stationary, by replacing c with $c' := c\,\mathrm{Cb}(b_1/c)\,\mathrm{Cb}(b_2/c) \in \mathrm{acl}^{\mathrm{eq}}(c) \cap D^{\mathrm{eq}}$. Indeed, each p_i is stationary so has a unique extension p_i' to $c'b_i$, so $|p_1' \cup p_2'| = |p_1 \cup p_2|$ and $Z(p_i') = Z(p_i)$.

In the interests of notational sanity, we will assume $c = \emptyset$. It is straightforward to check that the arguments below go through without this assumption, by working everywhere over c. (Note that $p_1{\restriction_c} = \mathfrak{p}{\restriction_c^{(2)}} = p_2{\restriction_c}$.)

In this proof, to emphasise the distinction between them, we use capital letters for variables and lower case for realisations.

Let $p_i'(X,Y) := \mathrm{tp}(a_i, b_i)$ where $a_i \vDash p_i$, and let $q(Y_1,Y_2) := \mathrm{tp}(b_1,b_2)$. Let $Q(X,Y_1,Y_2)$ be the incomplete type $p_1'(X,Y_1) \cup p_2'(X,Y_2) \cup q(Y_1,Y_2)$.

Claim 7.8. *If $a \vDash p_2$, then the completions of $Q(a,Y_1,b_2)$ in $S(ab_2)$ are precisely the non-forking extensions of $p_1'(a,Y_1)$. In particular, Q is consistent.*

Proof. It suffices to show that if $\vDash p_1'(a,b_1')$ then $b_1' \perp_a b_2$ iff $b_1' \perp b_2$. This is immediate from Lemma 7.6. $\square$

So let $a \vDash Q(X,b_1,b_2)$. Let $(\mathrm{tp}(a^i/b_1b_2))_{i\in I}$ enumerate the completions of $Q(X,b_1,b_2) = p_1(X) \cup p_2(X)$. By completeness of q, $(\mathrm{tp}(a^ib_1/b_2))_{i\in I}$ enumerates the completions of $Q(X,Y_1,b_2)$. Say $a^ib_1 \equiv_{b_2} ab_1^i$, so $(\mathrm{tp}(ab_1^i/b_2))_{i\in I}$ also enumerates the completions of $Q(X,Y_1,b_2)$, so $(\mathrm{tp}(b_1^i/ab_2))_{i\in I}$ enumerates the completions of $Q(a,Y_1,b_2)$, and hence by Claim 7.8 enumerates the non-forking extensions of $p_1'(a,Y_1) = \mathrm{tp}(b_1/a)$; in particular, by Lemma 3.15(b), I is finite.

Claim 7.9.

(a) $Z(a) = 1$

(b) $\sum_i Z(b_1^i/ab_2) = Z(b_1/a) = Z(ab_1)$

(c) $Z(b_1^ib_2) = Z(b_1)Z(b_2)$

Proof.

(a) By Lemma 7.3(iv) twice.

(b) By Claim 7.8 and Lemma 7.4(III), and (a).

(c) By $b_1^i \equiv_{b_2} b_1$ and Lemma 7.4(I), we have $Z(b_1^i/b_2) = Z(b_1/b_2) = Z(b_1)$.

□

Now we calculate

$$
\begin{aligned}
|\, p_1 \cup p_2 \,| &= \sum_i \mathrm{mult}(a^i/b_1 b_2) \\
&= \sum_i \mathrm{mult}(a/b_1^i b_2) \\
&= \sum_i Z(a/b_1^i b_2) \\
&= \sum_i \frac{Z(ab_1^i b_2)}{Z(b_1^i b_2)} \\
&= \sum_i \frac{Z(b_1^i/ab_2) Z(ab_2)}{Z(b_1^i b_2)} \\
&= \frac{Z(ab_1) Z(ab_2)}{Z(b_1) Z(b_2)} \\
&= Z(a/b_1) Z(a/b_2).
\end{aligned}
$$

□

Lemma 7.10. *D is 2-pseudolinear.*

Proof. Suppose p is minimal complete, $p(x) \vDash x \in D^2$, $b := Cb(p)$, $U(b) \geq 3$. Let $b' \vDash \mathrm{stp}(b){\restriction}_{\mathrm{acl}^{\mathrm{eq}}(b)}$. Say $b^\sigma = b'$, $\sigma \in \mathrm{Aut}(\mathbb{M})$, and let $p' := p^\sigma$. So p' is minimal and $\mathrm{Cb}(p') = b'$. Let $a \vDash p \cup p'$ (which is consistent by Lemma 7.7). We have $U(a) = 2$ since $U(Cb(p)) = U(b) > 0$ and $a \in D^2$. By Lemma 7.6, $b \mathop{\smile\!\!\!\!\!\!\mid}_a b'$. Note also that $U(b/a) \geq 2 \leq U(b'/a)$. Let $q := p{\restriction}_{ba}$ and $q' := p'{\restriction}_{b'a}$. Then by Lemma 7.7,

$$
|\, q \cup q' \,| = Z(q)Z(q') = Z(p)Z(p') = |\, p \cup p' \,| ,
$$

which contradicts the fact that $a \vDash p \cup p'$ but $a \nvDash q \cup q'$. □

Applying Theorem 6.7, this concludes the proof of Theorem 4.2.

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