Logik II: Model theory

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Contents

†Dieses Skript ist auch auf Deutsch verfügbar.

1 Overview

- A key goal of model theory is to understand the *definable sets* of a structure $\mathcal{M}$, the subsets of powers $\mathcal{M}^n$ defined by formulas.

- Gödel vs Tarski: Some structures have uncontrollably wild definable sets, becoming ever more complicated as we allow more quantifiers; for example, in $(\mathbb{N}; +, \cdot)$, already with one unbounded existential quantifier we can define arbitrary recursively enumerable sets.

  However, it is a remarkable fact that many structures important to mathematics avoid these Gödelian phenomena and are "tame": their definable sets are well-controlled (in particular, all definable by formulas with only a few quantifiers[1]), they often have decidable theories, and sometimes the models of the theory being classifiable up to isomorphism.

Examples include:

- $(\mathbb{C}; +, \cdot)$, $(\mathbb{F}_p^{\text{alg}}; +, \cdot)$, $(\mathbb{R}; +, \cdot)$, $(\mathbb{Q}_p; +, \cdot)$;
- $(\mathbb{Q}; <)$;
- Vector spaces;
- $(\mathbb{N}; +)$, $(\mathbb{N}; +, <)$, $(\mathbb{N}; \cdot)$ (but not $(\mathbb{N}; \cdot, <)$!);
- $(\mathbb{R}; +, \cdot, x \mapsto e^x)$;
- $(\mathbb{C}; +, \cdot, x \mapsto x^\zeta)$ (for most $\zeta \in \mathbb{C}$);
- Compact Lie groups e.g. $(\mathrm{SO}_3(\mathbb{R}); *)$;
- $(\mathbb{F}_{p^*}; +, \cdot)$ for an "infinite" (pseudofinite) prime p^*;
- Compact complex manifolds (complex tori, Calabi-Yau manifolds, etc);
- Differential and difference equations (in a certain sense);

We will examine only a fraction of this richness in this course, but we will develop tools with wide applicability.

- We often study a structure by considering other models of its theory. In particular, "tameness" of the class of models often corresponds to "tameness" of the definable sets. We will examine in detail the following strong form of this correspondence.

  A theory T is **κ-categorical** if it has a unique model of cardality κ. Let $\mathcal{M}$ be an infinite structure in a countable language.

  - Ryll-Nardzewski: $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical iff for each $n \in \omega$ there are only finitely many $\mathcal{L}$-definable subsets of $\mathcal{M}^n$.
  - Baldwin-Lachlan: for an uncountable cardinal κ, $\mathrm{Th}(\mathcal{M})$ is κ-categorical iff $\mathcal{M}$ is prime and minimal over a strongly minimal set defined over the prime model.

    (We'll define these terms later; a strongly minimal set is a particularly straightforward structure, and being prime and minimal over it implies that the definable sets of $\mathcal{M}$ are "constructed" from its definable sets in a certain sense.)

2 Preliminaries

We work in ZFC throughout the course. A set A is *countable* if $|A| \leq \aleph_0$.

[1] We consider e.g. $\exists x, y.$ as one quantifier rather than two.

2.1 Syntax and structure

- A (first-order) (1-sorted) **language** is a set $\mathcal{L}$ of relation symbols, function symbols, and constants.
- An **$\mathcal{L}$-structure** $\mathcal{M} = (M; (R^{\mathcal{M}})_R, (f^{\mathcal{M}})_f, (c^{\mathcal{M}})_c)$ is a non-empty set M equipped with interpretations of the symbols of $\mathcal{L}$:
  - $R^{\mathcal{M}} \subseteq \mathcal{M}^n$ for $R \in \mathcal{L}$ an n-ary relation symbol;
  - $f^{\mathcal{M}} : \mathcal{M}^n \to \mathcal{M}$ for $f \in \mathcal{L}$ an n-ary function symbol;
  - $c^{\mathcal{M}} \in \mathcal{M}$ for $c \in \mathcal{L}$ a constant.

  Often, we write $\mathcal{M}$ to refer to the underlying set M.

  We sometimes consider constants as 0-ary functions.
- An **$\mathcal{L}$-term** is a variable, a constant, or $f(t_1, \dots, t_n)$ where t_i are $\mathcal{L}$-terms and f is an n-ary function symbol.
- An **atomic $\mathcal{L}$-formula** is $t_1 \doteq t_2$ or $R(t_1, \dots, t_n)$ or $\top$, where t_i are $\mathcal{L}$-terms and R is an n-ary relation symbol. Here $\top$ is the *always true* sentence; $\mathcal{M} \vDash \top$ for any structure $\mathcal{M}$.
- An **$\mathcal{L}$-formula** is an atomic $\mathcal{L}$-formula or $\neg\phi$ or $(\phi \wedge \phi')$ or $\exists x.\, \phi$ where ϕ, ϕ' are $\mathcal{L}$-formulas and x is a variable.
- An **$\mathcal{L}$-sentence** is an $\mathcal{L}$-formula with no free variables. For $\mathcal{M}$ an $\mathcal{L}$-structure and σ an $\mathcal{L}$-sentence, $\mathcal{M} \vDash \sigma$ is defined recursively.
- Abbreviations:

$$\begin{aligned}
(\phi \vee \psi) &\mapsto \neg(\neg\phi \wedge \neg\psi) \\
(\phi \to \psi) &\mapsto (\neg\phi \vee \psi) \\
(\phi \leftrightarrow \psi) &\mapsto ((\phi \to \psi) \wedge (\psi \to \phi)) \\
\forall x.\, \phi &\mapsto \neg\exists x.\, \neg\phi \\
x \not\doteq y &\mapsto \neg x \doteq y \\
\bot &\mapsto \neg\top
\end{aligned}$$

- We also define abbreviations for conjunctions and disjunctions of finite sets of formulas :

$$\begin{aligned}
\bigwedge \emptyset &:= \top \\
\bigwedge (\Phi \cup \{\phi\}) &:= \left(\bigwedge \Phi \wedge \phi\right) \\
\bigvee \emptyset &:= \bot \\
\bigvee (\Phi \cup \{\phi\}) &:= \left(\bigvee \Phi \vee \phi\right).
\end{aligned}$$

- If $\mathcal{L}' \supseteq \mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}'$-structure, we write $\mathcal{M}{\restriction_{\mathcal{L}}}$ for the corresponding $\mathcal{L}$-structure, and call $\mathcal{M}{\restriction_{\mathcal{L}}}$ the **reduct** of $\mathcal{M}$ to $\mathcal{L}$, and $\mathcal{M}$ an **expansion** of $\mathcal{M}{\restriction_{\mathcal{L}}}$ to $\mathcal{L}'$.

- If $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq \mathcal{M}$, the **expansion by constants for** A is the expansion $\mathcal{M}_A$ of $\mathcal{M}$ to $\mathcal{L}(A) := \mathcal{L}\dot{\cup}A$ defined by $a^{\mathcal{M}_A} := a$.

- We write a tuple $(a_1, a_2, \ldots, a_n)$ $(n \geq 0)$ as $\overline{a}$, and we write $|\overline{a}|$ for its length, $|\overline{a}| = n$. For A a set, we define $A^{<\omega} := \bigcup_{n\in\omega} A^n$, the set of tuples from A

- We write an $\mathcal{L}$-formula ϕ as $\phi(\overline{x})$ if $\overline{x}$ is a tuple of distinct variables and the free variables of ϕ are among $x_1, \ldots, x_{|\overline{x}|}$. Then if $\mathcal{M}$ is an $\mathcal{L}$-structure and $\overline{a} \in \mathcal{M}^{|\overline{x}|}$, we write $\phi(\overline{a})$ for the $\mathcal{L}(\mathcal{M})$-sentence obtained by substituting a_i for x_i (for $i = 1, \ldots, |\overline{x}|$). Then $\mathcal{M} \vDash \phi(\overline{a})$ means $\mathcal{M}_{\mathcal{M}} \vDash \phi(\overline{a})$.

  Then the set **defined by** $\phi(\overline{x})$ in $\mathcal{M}$ is

  $$\phi(\mathcal{M}) := \{\overline{a} \in \mathcal{M}^{|\overline{x}|} : \mathcal{M} \vDash \phi(\overline{a})\} \subseteq \mathcal{M}^{|\overline{x}|}.$$

  (Technically, this depends on the choice of tuple $\overline{x}$ and not just on ϕ).

  Similarly, if $\phi(\overline{x}, \overline{y})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M}^{|\overline{x}|}$, we write $\phi(\overline{a}, \overline{y})$ for the $\mathcal{L}(\mathcal{M})$-formula obtained by substituting a_i for x_i.

- A **partial isomorphism** of $\mathcal{L}$-structures is a partial function $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ such that for any atomic $\mathcal{L}$-formula $\phi(\overline{x})$ and any $\overline{a} \in \operatorname{dom}\theta^{|\overline{x}|}$, $\mathcal{M} \vDash \phi(\overline{a})$ iff $\mathcal{N} \vDash \phi(\theta(\overline{a}))$. If this holds for *any* $\mathcal{L}$-formula ϕ, we call θ a **partial elementary map**.

- An **embedding** is a *total* partial isomorphism $\theta : \mathcal{M} \hookrightarrow \mathcal{N}$.

- An **elementary embedding** is a *total* partial elementary map $\theta : \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{N}$.

- An **isomorphism** is a *surjective* embedding $\theta : \mathcal{M} \overset{\cong}{\to} \mathcal{N}$.

- A **substructure** (resp. **elementary substructure**), of an $\mathcal{L}$-structure $\mathcal{N}$ is an $\mathcal{L}$-structure $\mathcal{M}$ on a subset of $\mathcal{N}$ such that the inclusion $\iota : \mathcal{M} \to \mathcal{N}$ is an embedding (resp. an elementary embedding).

  <u>Convention</u>: If $\mathcal{L}$ has no constants, we also allow the "empty structure" $\emptyset$ as an $\mathcal{L}$-substructure of any $\mathcal{L}$-structure (even though $\emptyset$ is not an $\mathcal{L}$-structure![2]).

- We write $\mathcal{M} \leq \mathcal{N}$ for a substructure[3], and $\mathcal{M} \preceq \mathcal{N}$ for an elementary substructure.

- If $A \subseteq \mathcal{M}$ is a subset of an $\mathcal{L}$-structure $\mathcal{M}$, let

  $$\langle A \rangle^{\mathcal{M}}_{\mathcal{L}} := \bigcap \{\mathcal{B} : A \subseteq \mathcal{B} \leq \mathcal{M}\} \leq \mathcal{M}$$

  be the $\mathcal{L}$-substructure generated by A.

  Note: when $\mathcal{L}$ has no constant symbols, $\langle \emptyset \rangle^{\mathcal{M}}_{\mathcal{L}} = \emptyset$.

  Lemma 2.1. $|\langle A \rangle^{\mathcal{M}}_{\mathcal{L}}| \leq \max(|A|, |\mathcal{L}|, \aleph_0)$.

[2] In many ways it would be preferable to allow the empty structure to be a structure, and some authors do this.

[3] Many authors write $\mathcal{M} \subseteq \mathcal{N}$ for the substructure relation.

Proof. Let $A_0 := A$ and, considering constants as 0-ary functions,

$$A_{i+1} := A_i \cup \{f^{\mathcal{M}}(\overline{a}) : f \in \mathcal{L} \text{ an } n\text{-ary function symbol}, \ \overline{a} \in A_i^n, \ n \geq 0\}.$$

Then $\langle A \rangle_{\mathcal{L}}^{\mathcal{M}} = \bigcup_{i \in \omega} A_i$, and $|A_{i+1}| \leq |A_i| + |\mathcal{L}| \cdot \max(|A|, \aleph_0) \leq \max(|A_i|, |\mathcal{L}|, \aleph_0)$. Hence $|A_i| \leq \max(|A|, |\mathcal{L}|, \aleph_0)$ for all i. So $|\bigcup_i A_i| \leq \max(|A|, |\mathcal{L}|, \aleph_0)$. $\square$

- We can always make an embedding into an inclusion by applying an isomorphism:

  Lemma 2.2. *Suppose $\theta : \mathcal{A} \hookrightarrow \mathcal{B}$ is an embedding of $\mathcal{L}$-structures. Then there is an isomorphism $\sigma : \mathcal{B} \to \mathcal{B}'$ such that $\mathcal{A} \leq \mathcal{B}'$ and $\sigma \circ \theta = \mathrm{id}_{\mathcal{A}}$.*

  Proof. First, let $\sigma : \mathcal{B} \to B'$ be a bijection with a set $B' \supseteq \mathcal{A}$ such that $\sigma \circ \theta = \mathrm{id}_{\mathcal{A}}$. Let $\mathcal{B}'$ be the $\mathcal{L}$-structure on B' such that σ is an isomorphism. Then $\mathrm{id}_{\mathcal{A}} = \sigma \circ \theta$ an embedding, so $\mathcal{A} \leq \mathcal{B}'$. $\square$

2.2 Theories

- An **$\mathcal{L}$-theory** is a set of $\mathcal{L}$-sentences.
- The **theory** of an $\mathcal{L}$-structure $\mathcal{M}$ is

$$\mathrm{Th}(\mathcal{M}) := \{\sigma : \mathcal{M} \models \sigma, \ \sigma \text{ is an } \mathcal{L}\text{-sentence}\}.$$

- An $\mathcal{L}$-structure $\mathcal{M}$ is a **model** of an $\mathcal{L}$-theory T,

$$\mathcal{M} \models T,$$

  if $\mathcal{M} \models \sigma$ for all $\sigma \in T$.

- $T \models \sigma$ means: $\mathcal{M} \models \sigma$ for any $\mathcal{M} \models T$. We also write $T \vdash \sigma$.

  $T \models T'$ or $T \vdash T'$ means: $T \models \sigma$ for all $\sigma \in T'$.

  $T \models_{T''} T'$ or $T \vdash_{T''} T'$ means: $T \cup T'' \models T'$.

- T is **consistent** if it has a model.

  Remark 2.3. T is consistent iff $T \not\models \bot$.

- A consistent $\mathcal{L}$-theory T is **complete** if for any $\mathcal{L}$-sentence σ

$$T \models \sigma \quad \text{or} \quad T \models \neg\sigma.$$

- $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ are **elementarily equivalent**, $\mathcal{M} \equiv \mathcal{N}$, if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

  Remark 2.4. A consistent theory T is complete iff $\mathcal{M} \equiv \mathcal{N}$ whenever $\mathcal{M}, \mathcal{N} \models T$.

  Remark 2.5. If A is a common subset of $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, then $\mathrm{id}_A : \mathcal{M} \dashrightarrow \mathcal{N}$ is partial elementary iff $\mathcal{M}_A \equiv \mathcal{N}_A$.

  In particular, if $\mathcal{M} \subseteq \mathcal{N}$, we have $\mathcal{M} \preceq \mathcal{N}$ iff $\mathcal{M}_{\mathcal{M}} \equiv \mathcal{N}_{\mathcal{M}}$.

  The $\mathcal{L}_{\mathcal{M}}$-theory $\mathrm{Th}(\mathcal{M}_{\mathcal{M}})$ is called the **elementary diagram** of $\mathcal{M}$.

3 Ultraproducts

Let I be a set. A non-empty set $\mathcal{B} \subseteq \mathcal{P}(I)$ is a **filter base** if

- $X, Y \in \mathcal{B} \Rightarrow \exists Z \in \mathcal{B}.\, X \cap Y \supseteq Z$;
- $\emptyset \notin \mathcal{B}$.

A maximal filter base $\mathcal{U}$ is an **ultrafilter**; equivalently: $\mathcal{U}$ is a filter base, and for any $X \subseteq I$ we have

$$X \in \mathcal{U} \text{ or } I \setminus X \in \mathcal{U}.$$

As an immediate consequence of Zorn's Lemma, we have

Fact 3.1. *Any filter base $\mathcal{B} \subseteq \mathcal{P}(I)$ extends to an ultrafilter $\mathcal{B} \subseteq \mathcal{U} \subseteq \mathcal{P}(I)$.*

Remark 3.2. Fact 3.1 is not a theorem of ZF, and it is strictly weaker than the axiom of choice modulo ZF.

Remark 3.3. Ultrafilters are *upwards-closed*: if $X \subseteq Y$ then $X \in \mathcal{U} \Rightarrow Y \in \mathcal{U}$. An upwards-closed filter base is a *filter*.

If $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter and a_i are elements of sets A_i ($i \in I$), the **ultralimit** is $\lim_{i\to\mathcal{U}} a_i$ the equivalence class $(a_i)_i/\sim^{\mathcal{U}}$ of the sequence $(a_i)_i$ under the equivalence relation

$$(a_i)_i \sim^{\mathcal{U}} (a'_i)_i \text{ iff } \{i : a_i = a'_i\} \in \mathcal{U}.$$

If A_i are sets, the **ultraproduct** is the set of all ultralimits,

$$\prod_{i\to\mathcal{U}} A_i := {}^{\prod_{i\in I} A_i}/_{\sim^{\mathcal{U}}} = \left\{\lim_{i\to\mathcal{U}} a_i : a_i \in A_i\right\}.$$

We have $\lim_{i\to\mathcal{U}}(a_i, b_i) = (\lim_{i\to\mathcal{U}} a_i, \lim_{i\to\mathcal{U}} b_i)$.

For functions $f_i : A_i \to B_i$ we define

$$\lim_{i\to\mathcal{U}} f_i : \prod_{i\to\mathcal{U}} A_i \to \prod_{i\to\mathcal{U}} B_i$$

by

$$(\lim_{i\to\mathcal{U}} f_i)(\lim_{i\to\mathcal{U}} a_i) := \lim_{i\to\mathcal{U}} f_i(a_i).$$

If $(\mathcal{M}_i : i \in I)$ are $\mathcal{L}$-structures, the **ultraproduct** $\mathcal{M} = \prod_{i\to\mathcal{U}} \mathcal{M}_i$ is the $\mathcal{L}$-structure such that

- as sets, $\mathcal{M} := \prod_{i\to\mathcal{U}} \mathcal{M}_i$;
- $\mathcal{M} \vDash R(\lim_{i\to\mathcal{U}} \overline{a}_i) \quad\Leftrightarrow\quad \{i : \mathcal{M}_i \vDash R(\overline{a}_i)\} \in \mathcal{U}$ (for $R \in \mathcal{L}$ a relation symbol);
- $f^{\mathcal{M}} := \lim_{i\to\mathcal{U}} f^{M_i}$ (for $f \in \mathcal{L}$ a function symbol);
- $c^{\mathcal{M}} := \lim_{i\to\mathcal{U}} c^{M_i}$ (for $c \in \mathcal{L}$ a constant symbol).

Theorem 3.4 (Łoś)**.** *For $\mathcal{L}$-structures $\mathcal{M}_i$, an $\mathcal{L}$-formula $\phi(\overline{x})$, and $\overline{a}_i \in \mathcal{M}_i$,*

$$\prod_{i\to\mathcal{U}} M_i \vDash \phi(\lim_{i\to\mathcal{U}} \overline{a}_i) \textit{ iff } \{i : M_i \vDash \phi(\overline{a}_i)\} \in \mathcal{U}.$$

Proof. Exercise. □

An ultrafilter $\mathcal{U}$ is **principal** if there is $i_0 \in I$ such that $\mathcal{U} = \{X \subseteq I : i_0 \in X\}$. Then $\prod_{i \to \mathcal{U}} \mathcal{M}_i \cong \mathcal{M}_{i_0}$.

Example 3.5. Let $\mathcal{U} \subseteq \mathcal{P}(P)$ be a non-principal ultrafilter on the set $P \subseteq \mathbb{N}$ of primes Then the ultraproduct of finite fields $\prod_{p \to \mathcal{U}} (\mathbb{F}_p; +, \cdot)$ is a field of characteristic 0 (a "pseudofinite field").

If $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter and $\mathcal{M}$ is an $\mathcal{L}$-structure, the **ultrapower** is defined as $\mathcal{M}^{\mathcal{U}} := \prod_{i \to \mathcal{U}} \mathcal{M}$.

Lemma 3.6. *With respect to the* **diagonal embedding** $a \mapsto \lim_{i \to \mathcal{U}} a$, *$\mathcal{M}$ is an elementary substructure, $\mathcal{M} \preceq \mathcal{M}^{\mathcal{U}}$.*

Proof. Exercise. □

Example 3.7. Let $\mathcal{U} \subseteq \mathcal{P}(\omega)$ be a non-principal ultrafilter. Let $(\mathbb{R}^*; +, \cdot) := (\mathbb{R}; +, \cdot)^{\mathcal{U}}$ (a "non-standard real field"). Let

$$\epsilon := \lim_{n \to \mathcal{U}} \frac{1}{n+1} \in \mathbb{R}^*.$$

Then $0 < \epsilon < r$ for all $r \in \mathbb{R} \subseteq \mathbb{R}^*$.

4 Compactness

Theorem 4.1 (Compactness)**.** *Suppose every finite subset of an $\mathcal{L}$-theory T is consistent. Then T is consistent.*

Proof. Let $\mathcal{P}^{\text{fin}}(T) := \{T' \in \mathcal{P}(T) : |T'| < \aleph_0\}$ be the set of finite subsets of T. Say $\mathcal{M}_{T'} \vDash T'$ for $T' \in \mathcal{P}^{\text{fin}}(T)$.

For $T' \in \mathcal{P}^{\text{fin}}(T)$, let $[T'] = \{T'' \in \mathcal{P}^{\text{fin}}(T) : T' \subseteq T''\} \subseteq \mathcal{P}^{\text{fin}}(T)$. Let $\mathcal{B} = \{[T'] : T' \in \mathcal{P}^{\text{fin}}(T)\} \subseteq \mathcal{P}(\mathcal{P}^{\text{fin}}(T))$. Then $\mathcal{B}$ is a filter base, since $[T'] \cap [T''] = [T' \cup T'']$, and $[T'] \neq \emptyset$ since $T' \in [T']$, and $\mathcal{B} \neq \emptyset$ since $[\emptyset] \in \mathcal{B}$. By Fact 3.1, let $\mathcal{U} \supseteq \mathcal{B}$ be an ultrafilter on $\mathcal{P}^{\text{fin}}(T)$.

Let $\mathcal{M} = \prod_{T' \to \mathcal{U}} \mathcal{M}_{T'}$.

Then for $\sigma \in T$,

$$\{T' : \mathcal{M}_{T'} \vDash \sigma\} \supseteq \{T' : \sigma \in T'\} = [\{\sigma\}] \in \mathcal{U},$$

so by Łoś, $\mathcal{M} \vDash \sigma$.

So $\mathcal{M} \vDash T$, so T is consistent. □

Lemma 4.2 (Separation)**.** *Let T_1 and T_2 be consistent $\mathcal{L}$-theories, and Σ a set of $\mathcal{L}$-sentences closed under $\wedge$ and $\vee$. Then TFAE:*

(i) Given $\mathcal{M}_1 \vDash T_1$ and $\mathcal{M}_2 \vDash T_2$, there exists $\sigma \in \Sigma$ with $\mathcal{M}_1 \vDash \sigma$ and $\mathcal{M}_2 \vDash \neg\sigma$.

(ii) There exists $\sigma \in \Sigma$ with $T_1 \vDash \sigma$ and $T_2 \vDash \neg\sigma$.

We then say Σ **separates** *T_1 from T_2.*

Proof.

(i) $\Leftarrow$ (ii): Clear.

(i) $\Rightarrow$ (ii): Let $\mathcal{M}_1 \models T_1$. For $\mathcal{M}_2 \models T_2$ there is by (i) $\sigma_{\mathcal{M}_2} \in \Sigma$ with $\mathcal{M}_1 \models \sigma_{\mathcal{M}_2}$ and $\mathcal{M}_2 \models \neg\sigma_{\mathcal{M}_2}$. Then $T_2 \cup \{\sigma_{\mathcal{M}_2} : \mathcal{M}_2 \models T_2\}$ is inconsistent, so by compactness there is a finite conjunction $\sigma_{\mathcal{M}_1}$ of the $\sigma_{\mathcal{M}_2}$, such that $T_2 \models \neg\sigma_{\mathcal{M}_1}$. Then $\sigma_{\mathcal{M}_1} \in \Sigma$ by closedness of Σ under $\wedge$, and $\mathcal{M}_1 \models \sigma_{\mathcal{M}_1}$.

Now $T_1 \cup \{\neg\sigma_{\mathcal{M}_1} : \mathcal{M}_1 \models T_1\}$ is inconsistent, so by compactness and closedness of Σ under $\vee$, there is a finite disjunction $\sigma \in \Sigma$ of the $\sigma_{\mathcal{M}_1}$, such that $T_1 \models \sigma$. Then $T_2 \models \neg\sigma$.

□

5 Quantifier elimination

5.1 Definitions

Definition 5.1. A formula is **quantifier free** (**qf**) if it contains no quantifiers.

Definition 5.2. $\mathcal{L}$-formulas $\phi(\overline{x})$ and $\psi(\overline{x})$ are **equivalent modulo** an $\mathcal{L}$-theory T, written $\phi(\overline{x}) \leftrightarrow_T \psi(\overline{x})$, if

$$T \models \forall\overline{x}.\ (\phi(\overline{x}) \leftrightarrow \psi(\overline{x})).$$

Similarly, $\phi(\overline{x}) \rightarrow_T \psi(\overline{x})$ if $T \models \forall\overline{x}.\ (\phi(\overline{x}) \rightarrow \psi(\overline{x}))$.

(As always, we allow here the case $|\overline{x}| = 0$, i.e. the case that ϕ and ψ are sentences.)

Definition 5.3. An $\mathcal{L}$-theory T has **quantifier elimination** (**QE**) if any $\mathcal{L}$-formula $\phi(\overline{x})$ is equivalent modulo T to a quantifier free formula $\psi(\overline{x})$.

An $\mathcal{L}$-structure $\mathcal{M}$ has QE if $\mathrm{Th}(\mathcal{M})$ has QE.

5.2 Discussion

Remark 5.4. An $\mathcal{L}$-structure $\mathcal{M}$ has QE iff every $\mathcal{L}$-definable set is defined by a qf $\mathcal{L}$-formula.

Example 5.5. $(\mathbb{R}; +, -, \cdot, 0, 1)$ does not have QE: The order $x < y$ is definable by $\exists z.\ (z \neq 0 \wedge x + z \cdot z = y)$ but not by any qf formula.

Remark 5.6. If $\mathcal{L}$ has no constants, then the only qf sentences up to equivalence are $\top$ and $\bot$. So any consistent $\mathcal{L}$-theory with QE is complete.

Example 5.7. Let $\mathcal{L}_\emptyset$ be the empty language $\mathcal{L}_\emptyset := \emptyset$. We will see below that if X is an infinite set, the $\mathcal{L}_\emptyset$-structure $(X;)$ has QE. Moreover, we will see that $T_\infty := \{\exists x_1, \ldots, x_n.\ \bigwedge_{i \neq j} x_i \neq x_j : n \in \omega\}$ has QE, and so axiomatises $\mathrm{Th}((X;))$.

If $\mathcal{M}$ is an $\mathcal{L}$-structure and Φ is a set of $\mathcal{L}$-formulas, the **expansion by relations for** Φ is the expansion of $\mathcal{M}$ to $\mathcal{L} \cup \{R_\phi : \phi \in \Phi\}$, where R_ϕ is interpreted as $\phi(\mathcal{M})$. It has the same definable sets as $\mathcal{M}$.

If $\mathcal{M}$ does not have QE, we can try to expand by relations for non-qf formulas until we obtain QE, e.g.:

Fact 5.8 (Tarski-Seidenberg)**.** $(\mathbb{R}; +, -, \cdot, 0, 1, <)$ *has QE.*

We can always do this in a trivial way:

Remark 5.9. For any $\mathcal{L}$-structure $\mathcal{M}$, the expansion by relations for *all* $\mathcal{L}$-formulas (the "Morleyisation" of $\mathcal{M}$) has QE.

For some particularly intractable structures, we can't really do any better than Morleyising:

Fact 5.10 ("The arithmetic hierarchy is strict.")**.** *For any $n \in \omega$, the expansion of $(\mathcal{N}; +, \cdot)$ by relations for all formulas with at most n unbounded quantifiers (equivalently, all formulas of the form $\exists \overline{x}_1. \neg\exists\overline{x}_2. \ldots \neg\exists\overline{x}_n. \phi$, where ϕ has no unbounded quantifier) does not have QE.*

5.3 Criterion for QE

Definition 5.11.

- A **basic** formula[4] is an atomic formula or the negation of an atomic formula.
- A formula $\phi(\overline{x})$ is **primitive existential** if it is of the form $\exists y. \bigwedge_i \psi_i(y, \overline{x})$, where each ψ_i is basic.

Lemma 5.12. *Let T be an $\mathcal{L}$-theory. If any primitive existential $\mathcal{L}$-formula $\phi(\overline{x})$ is equivalent modulo T to a quantifier free formula $\psi(\overline{x})$, then T has QE.*

Proof. We show by induction on complexity that any $\phi(\overline{x})$ is equivalent modulo T to a quantifier free formula $\psi(\overline{x})$. For atomic ϕ this is clear. For $\phi = \phi' \wedge \phi''$ or $\phi = \neg\phi'$ it is immediate by induction.

For $\phi = \exists y. \phi'$: by the inductive hypothesis $\phi' \leftrightarrow_T \psi$ with ψ quantifier-free. We may assume ψ is in disjunctive normal form : $\psi = \bigvee_i \bigwedge_j \psi_{ij}$, where each ψ_{ij} is basic. So $\phi \leftrightarrow_T \bigvee_i \exists y. \bigwedge_j \psi_{ij}$. Each formula $\exists y. \bigwedge_j \psi_{ij}$ is primitive existential, and we conclude by the assumption. □

Definition 5.13. The **diagram** of an $\mathcal{L}$-substructure $\mathcal{A}$ of an $\mathcal{L}$-structure is the $\mathcal{L}(\mathcal{A})$-theory

$$\mathrm{Diag}(\mathcal{A}) := \mathrm{qfTh}(\mathcal{A}_\mathcal{A}) := \{\sigma : \sigma \text{ qf } \mathcal{L}(\mathcal{A})\text{-sentence}, \ \mathcal{A}_\mathcal{A} \vDash \sigma\}.$$

Lemma 5.14 (Method of diagrams)**.** *Up to isomorphism, the models of* $\mathrm{Diag}(\mathcal{A})$ *are precisely $\mathcal{M}_\mathcal{A}$, where $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{A} \leq \mathcal{M}$.*

Theorem 5.15. *For an $\mathcal{L}$-theory T, TFAE:*

(i) T has QE.

(ii) If $\mathcal{M}, \mathcal{N} \vDash T$ have a common $\mathcal{L}$-substructure $\mathcal{A}$, then $\mathcal{M}_\mathcal{A} \equiv \mathcal{N}_\mathcal{A}$.

(ii') "T is **substructure complete***": If $\mathcal{M} \vDash T$ and $\mathcal{A} \leq \mathcal{M}$ is an $\mathcal{L}$-substructure, then $T \cup \mathrm{Diag}(\mathcal{A})$ is complete.*

(iii) If $\mathcal{M}, \mathcal{N} \vDash T$ have a common $\mathcal{L}$-substructure $\mathcal{A}$, and $\overline{a} \in \mathcal{A}^{<\omega}$ and $\phi(\overline{x})$ is a primitive existential $\mathcal{L}$-formula, then

$$\mathcal{M} \vDash \phi(\overline{a}) \Leftrightarrow \mathcal{N} \vDash \phi(\overline{a}).$$

[4] Also known as a *literal*.

(iii') If $\mathcal{M}, \mathcal{N} \vDash T$ have a common finitely generated $\mathcal{L}$-substructure $\mathcal{A}$, and σ is a primitive existential $\mathcal{L}(\mathcal{A})$-sentence, then

$$\mathcal{M}_\mathcal{A} \vDash \sigma \Leftrightarrow \mathcal{N}_\mathcal{A} \vDash \sigma.$$

Proof.

(i) $\Rightarrow$ (ii): Let $\phi(\overline{a})$ be an $\mathcal{L}(\mathcal{A})$-sentence. By (i), $\phi(\overline{x})$ is equivalent to some qf $\phi'(\overline{x})$ modulo T. Then $\mathcal{M}_\mathcal{A} \vDash \phi'(\overline{a}) \Leftrightarrow \mathcal{A} \vDash \phi'(\overline{a}) \Leftrightarrow \mathcal{M}_\mathcal{B} \vDash \phi'(\overline{a})$. So $\mathcal{M}_\mathcal{A} \vDash \phi(\overline{a}) \Leftrightarrow \mathcal{M}_\mathcal{B} \vDash \phi(\overline{a})$.

(ii) $\Leftrightarrow$ (ii'): By Lemma 5.14, the models of $T \cup \mathrm{Diag}(\mathcal{A})$ are exactly $\mathcal{M}_\mathcal{A}$, where $\mathcal{M} \vDash T$ and $\mathcal{A} \leq \mathcal{M}$. So (ii) exactly says that $T \cup \mathrm{Diag}(\mathcal{A})$ is complete.

(ii) $\Rightarrow$ (iii): Clear.

(iii) $\Leftrightarrow$ (iii'): Clear.

(iii) $\Rightarrow$ (i): Let $\phi(\overline{x})$ be primitive existential. By Lemma 5.12, it suffices to show that ϕ is equivalent modulo T to a qf formula .

Let $\overline{c}$ be a tuple of new constants with $|\overline{c}| = |\overline{x}|$.
Let $T_1 := T \cup \{\phi(\overline{c})\}$ and $T_2 := T \cup \{\neg\phi(\overline{c})\}$.
If T_1 is inconsistent, then $T \vDash \forall \overline{x}.\, \neg\phi(\overline{x})$, and so $\phi(\overline{x}) \leftrightarrow_T \bot$.
If T_2 is inconsistent, then $T \vDash \forall \overline{x}.\, \phi(\overline{x})$, and so $\phi(\overline{x}) \leftrightarrow_T \top$.
So assume T_1 and T_2 are consistent.

Suppose $\Sigma := \{\psi(\overline{c}) : \psi(\overline{x}) \text{ qf } \mathcal{L}\text{-formula}\}$ does not separate T_1 from T_2. Then by Lemma 4.2, there are $\mathcal{M}_1, \mathcal{M}_2 \vDash T$ and $\overline{a}_i \in \mathcal{M}_i$ with $\mathcal{M}_1 \vDash \phi(\overline{a}_1)$ and $\mathcal{M}_2 \vDash \neg\phi(\overline{a}_2)$, but for $\psi(\overline{x})$ qf, $\mathcal{M}_1 \vDash \psi(\overline{a}_1) \Leftrightarrow \mathcal{M}_2 \vDash \psi(\overline{a}_2)$.

But then the map $\overline{a}_1 \mapsto \overline{a}_2$ extends to an isomorphism $\langle \overline{a}_1 \rangle_\mathcal{L}^{\mathcal{M}_1} \xrightarrow{\cong} \langle \overline{a}_2 \rangle_\mathcal{L}^{\mathcal{M}_2}$ (namely $t^{\mathcal{M}_1}(\overline{a}_1) \mapsto t^{\mathcal{M}_2}(\overline{a}_2)$ for t an $\mathcal{L}$-term), which itself extends to an isomorphism $\mathcal{M}_1 \xrightarrow{\cong} \mathcal{M}_1' \geq \langle \overline{a}_2 \rangle_\mathcal{L}^{\mathcal{M}_2}$ (by Lemma 2.2). Then $\mathcal{M}_1' \vDash \phi(\overline{a}_2)$ and $\mathcal{M}_2 \vDash \neg\phi(\overline{a}_2)$. But this contradicts (iii).

So there is $\psi(\overline{c}) \in \Sigma$ such that $T_1 \vDash \psi(\overline{c})$ and $T_2 \vDash \neg\psi(\overline{c})$, i.e. $\phi(\overline{x}) \rightarrow_T \psi(\overline{x})$ and $\neg\phi(\overline{x}) \rightarrow_T \neg\psi(\overline{x})$. So $\phi(\overline{x}) \leftrightarrow_T \psi(\overline{x})$.

□

We can now generalise Remark 5.6 to arbitrary languages:

Corollary 5.16. *Let T be a consistent $\mathcal{L}$-theory with QE. Then TFAE:*

(i) T is complete.

(ii) For any $\mathcal{M}, \mathcal{N} \vDash T$, we have $\langle \emptyset \rangle_\mathcal{L}^\mathcal{M} \cong \langle \emptyset \rangle_\mathcal{L}^\mathcal{N}$.

Proof.

(i) $\Rightarrow$ (ii): The map $t^\mathcal{M} \mapsto t^\mathcal{N}$ for t an $\mathcal{L}$-term with no variables is an $\mathcal{L}$-isomorphism.

(ii) $\Rightarrow$ (i): Let $\mathcal{M} \vDash T$ and $\mathcal{A} := \langle \emptyset \rangle_\mathcal{L}^\mathcal{M} \leq \mathcal{M}$. Let $\mathcal{N} \vDash T$. By (ii) there is (by Lemma 2.2) $\mathcal{N}' \cong \mathcal{N}$ with $\mathcal{A} \leq \mathcal{N}'$. Then by Theorem 5.15(ii) $\mathcal{M}_\mathcal{A} \equiv \mathcal{N}'_\mathcal{A}$. Hence $\mathcal{M} \equiv \mathcal{N}' \cong \mathcal{N}$. So T is complete.

□

5.4 Examples

5.4.1 T_∞

Let T_∞ be the $\mathcal{L}_\emptyset$-theory

$$T_\infty := \{\exists x_1, \dots, x_n. \bigwedge_{i \neq j} x_i \dot{\neq} x_j : n \in \omega\}.$$

Proposition 5.17. *T_∞ is complete and admits quantifier elimination.*

Proof. Completeness follows from QE, since the language has no constants.

For QE, we show Theorem 5.15(iii').

Let $\mathcal{M}, \mathcal{N} \vDash T_\infty$ and let $\mathcal{A} \leq \mathcal{M}, \mathcal{N}$ be a finite common subset

Let $\exists y. \psi(y)$ be a primitive existential $\mathcal{L}_\infty(\mathcal{A})$-sentence. Suppose there exists $b \in \mathcal{M}$ such that $\mathcal{M}_\mathcal{A} \vDash \psi(b)$. We must find $b' \in \mathcal{N}$ such that $\mathcal{N}_\mathcal{A} \vDash \psi(b')$.

If $b \in \mathcal{A}$: set $b' := b$. If $b \notin \mathcal{A}$: since $\mathcal{N} \vDash T_\infty$, $\mathcal{N}$ is infinite; set $b' \in \mathcal{N} \setminus \mathcal{A}$.

Then $\mathrm{id}_\mathcal{A} \cup \{b \mapsto b'\}$ is a bijection, and hence an $\mathcal{L}_\infty(\mathcal{A})$-isomorphism. So $\mathcal{N}_\mathcal{A} \vDash \psi(b')$. □

5.4.2 DLO

Let $\mathcal{L}_< := \{<\}$ and let DLO be the $\mathcal{L}_<$-theory of dense linear orderings without endpoints:

$$\begin{aligned} \mathrm{DLO} := \{\forall x, y, z. \, (&\neg x < x \\ &\wedge (x < y \vee x = y \vee y < x) \\ &\wedge ((x < y \wedge y < z) \rightarrow x < z) \\ &\wedge (x < y \rightarrow \exists w. \, (x < w \wedge w < y)) \\ &\wedge \exists w. \, w < x \\ &\wedge \exists w. \, x < w)\}. \end{aligned}$$

Proposition 5.18. DLO *is complete and admits quantifier elimination.*

In particular, DLO *axiomatises* $(\mathbb{Q}; <)$. *Hence* $\mathrm{Th}((\mathbb{Q}; <))$ *is decidable.*

Proof. Completeness follows from quantifier elimination, since the language has no constants.

Decidability follows from completeness, since DLO is a recursive set, and hence $\mathrm{Th}((\mathbb{Q}; <)) = \{\sigma : \mathrm{DLO} \vDash \sigma\}$ and its complement $\{\sigma : \mathrm{DLO} \vDash \neg\sigma\}$ are recursively enumerable, from which it follows that $\mathrm{Th}((\mathbb{Q}; <))$ is recursive.

Let $\mathcal{M}, \mathcal{N} \vDash \mathrm{DLO}$ and let $\mathcal{A} = \{a_1, \dots, a_n\} \leq \mathcal{M}, \mathcal{N}$ be a common finite substructure. Without loss of generality, we may assume $a_1 < a_2 < \dots < a_n$.

¡—Let¿— $\exists y. \psi(y)$ be a primitive existential $\mathcal{L}_<(\mathcal{A})$-sentence. Suppose there exists $b \in \mathcal{M}$ with $\mathcal{M}_\mathcal{A} \vDash \psi(b)$. We find $b' \in \mathcal{N}$ with $\mathcal{N}_\mathcal{A} \vDash \psi(b')$.

There are four cases:

(i) $b \in A$: set $b' = b$.

(ii) $b < a_1$: let $b' \in \mathcal{N}$ be such that $b' < a_1$ (b' exists, since $\mathcal{N}$ has no endpoint).

(iii) $b > a_n$: let $b' \in \mathcal{N}$ be such that $b' > a_n$ (b' exists, since $\mathcal{N}$ has no endpoint).

(iv) $a_i < b < a_{i+1}$: let $b' \in \mathcal{N}$ be such that $a_i < b' < a_{i+1}$ (b' exists, since $\mathcal{N}$ is dense).

In all cases, $A \cup \{b\}$ is isomorphic to $A \cup \{b'\}$ over A as ordered sets. Hence $\mathcal{N}_{\mathcal{A}} \vDash \psi(b')$. □

5.4.3 $(\mathbb{Z}; S)$

Let $\mathcal{L}_S := \{S\}$, where S is a unary function.

Let T_S be the $\mathcal{L}_S$-theory of cycle-free bijections

$$T_S := \{\forall x, y. ((S(x) = S(y) \to x = y) \wedge \exists z. S(z) = x)\} \cup \{\forall x. S^n(x) \neq x : n \geq 1\}$$

(where $S^{n+1}(x) := S(S^n(x))$; $S^1(x) := S(x)$).

$(\mathbb{Z}; S) \vDash T_S$ (where $S^{\mathbb{Z}}(n) := n + 1$).

Proposition 5.19. *T_S is complete and admits quantifier elimination.*

In particular, $(\mathbb{Z}; S)$ is decidable and axiomatised by T_S.

Proof. Completeness follows from quantifier elimination, since the language has no constants.

Let $\mathcal{M}, \mathcal{N} \vDash T_S$ and let $\mathcal{A}$ be a common finitely generated substructure.

We may assume that $S(\mathcal{A}) = \mathcal{A}$: Indeed, $\bigcup_n (S^{\mathcal{M}})^{-n}(\mathcal{A})$ is isomorphic to $\bigcup_n (S^{\mathcal{N}})^{-n}(\mathcal{A})$ over $\mathcal{A}$, since $\mathcal{M}$ and $\mathcal{N}$ are cycle-free.

Then every atomic $\mathcal{L}_S(\mathcal{A})$-formula $\phi(x)$ is equivalent modulo $T_S \cup \mathrm{Diag}(\mathcal{A})$ to $x \doteq a$ for some $a \in \mathcal{A}$, or to $\top$, or to $\bot$. Indeed:

$$S^n(x) \doteq S^m(x) \leftrightarrow_{T_S} \begin{cases} \top & (n = m) \\ \bot & (n \neq m) \end{cases};$$

$$S^n(x) \doteq S^m(a) \leftrightarrow_{T_S \cup \mathrm{Diag}(\mathcal{A})} x \doteq S^{m-n}(a) (\in \mathcal{A}).$$

Hence any primitive existential $\mathcal{L}_S(\mathcal{A})$-formula σ is equivalent modulo $T_S \cup \mathrm{Diag}(\mathcal{A})$ to $\top$, or $\bot$, or

$$\exists y. \bigwedge_{i<k} y \doteq a_i \wedge \bigwedge_{i<l} y \neq b_i$$

. Since $\mathcal{M}$ and $\mathcal{N}$ are infinite, we have $\mathcal{M}_{\mathcal{A}} \vDash \sigma \Leftrightarrow \mathcal{N}_{\mathcal{A}} \vDash \sigma$.

The result now follows by Theorem 5.15(iii'). □

5.4.4 ACF

Let $\mathcal{L}_{\mathrm{ring}} := \{+, -, \cdot, 0, 1\}$. Let ACF be the $\mathcal{L}_{\mathrm{ring}}$-theory of algebraically closed fields:

$$\mathrm{ACF} := \text{[Körperaxiome]} \cup \{\forall z_0, \ldots, z_n. \exists x. \sum_{i=0}^{n} z_i x^i \doteq 0 : n \geq 1\}.$$

Proposition 5.20. ACF *admits quantifier elimination.*

Proof. Let $K_i \vDash \mathrm{ACF}$ and let $R = \langle a_1, \ldots, a_n \rangle \leq K_1, K_2$ be a finitely generated common subring.

Let $\exists y.\, \psi(y)$ be a primitive existential $\mathcal{L}_{\mathrm{ring}}(R)$-sentence. Suppose there is $b \in K_1$ such that $K_1 \vDash \psi(b)$. We show that $K_2 \vDash \exists y.\, \psi(y)$.

Let F_i be the quotient field of R in K_i. Then id_R extends to an isomorphism $f : F_1 \xrightarrow{\cong} F_2$ ($F_1 \ni \frac{r}{s} \mapsto \frac{r}{s} \in F_2$).

Now let G_i be the algebraic closure of F_i in K_i, namely the set of all solutions to polynomial equations with coefficients in F_i.

Since the algebraic closure of F_i is uniquely determined up to F_i-isomorphism, f extends to an isomorphism $g : G_1 \xrightarrow{\cong} G_2$.

If $b \in G_1$, we have $K_2 \vDash \psi(g(b))$.

Otherwise: b is transcendental over G_1. Then $G_1(b)$ is isomorphic over G_1 to the rational function field $G_1(X)$. Let K_2' be a proper elementary extension of K_2. Then say $b' \in K_2' \setminus G_2$. Then $G_2(b')$ is again isomorphic over G_2 to the rational function field $G_2(X)$. Hence g extends to an isomorphism $h : G_1(b) \to G_2(b')$ with $h(b) = b'$. Hence $K_2' \vDash \psi(b')$ and so $K_2' \vDash \exists y.\, \psi(y)$. Finally, we conclude $K_2 \vDash \exists y.\, \psi(y)$. □

For $p \in \mathbb{N}$ prime, let $\mathrm{ACF}_p := \mathrm{ACF} \cup \{\bar{p} \doteq 0\}$, where $\bar{n}$ is the term $1+1+\ldots+1$ (n times).

Let $\mathrm{ACF}_0 := \mathrm{ACF} \cup \{\bar{n} \not\doteq 0 : n \geq 1\}$.

Theorem 5.21. *The completions of* ACF *are precisely* ACF_p *for p prime or 0.*

Proof. The characteristic of a field is either prime or 0. For K a field of characteristic p,

$$\langle \emptyset \rangle^K_{\mathcal{L}_{\mathrm{ring}}} = \begin{cases} \mathbb{F}_p & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}.$$

So quantifier elimination implies by Corollary 5.16 completeness of each ACF_p. □

Theorem 5.22 (Ax)**.** *Any injective polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ (i.e. $F(\overline{a}) = (F_1(\overline{a}), \ldots, F_n(\overline{a})$, where $F_i \in \mathbb{C}[\overline{X}]$) is surjective.*

Proof.

Claim 5.23. *Let p be prime. Any injective polynomial map $F : (\mathbb{F}_p^{\mathrm{alg}})^n \to (\mathbb{F}_p^{\mathrm{alg}})^n$ is surjective.*

Proof. Recall: $\mathbb{F}_p^{\mathrm{alg}} = \bigcup_k \mathbb{F}_{p^k}$.

Let k_0 be such that the coefficients of F are in $\mathbb{F}_{p^{k_0}}$.

Let $k \geq k_0$. Dann $F(\mathbb{F}^n_{p^k}) \subseteq \mathbb{F}^n_{p^k}$, and so by injectivity and the pigeonhole principle, $F(\mathbb{F}^n_{p^k}) = \mathbb{F}^n_{p^k}$.

Hence $F((\mathbb{F}_p^{\mathrm{alg}})^n) = (\mathbb{F}_p^{\mathrm{alg}})^n$. □

Let $n, d \in \omega$. Let $\sigma_{n,d}$ be an $\mathcal{L}_{\mathrm{ring}}$-sentence expressing that any injective polynomial map $F : K^n \to K^n$ consisting of polynomials of degree $\leq d$ is surjective:

$$\sigma_{n,d} := \forall z_{1,0}, \ldots, z_{n,d}.(\forall \overline{x}, \overline{y}.((\bigwedge_i \sum_j z_{i,j} x_i^j \doteq \sum_j z_{i,j} y_i^j) \to \bigwedge_i x_i \doteq y_i) \to \forall \overline{y}.\exists \overline{x}. \bigwedge_i \sum_j z_{i,j} x_i^j \doteq y_i).$$

Suppose $\mathbb{C} \not\models \sigma_{n,d}$. Then by completeness of ACF_0, $\mathrm{ACF}_0 \models \neg\sigma_{n,d}$. Then by compactness, for some $m \in \omega$,

$$\mathrm{ACF} \models \bigwedge_{0<i<m} \bar{i} \neq 0 \to \neg\sigma_{n,d}.$$

So if $p > m$ is prime, $\mathrm{ACF}_p \models \neg\sigma_{n,d}$. But this contradicts the Claim. □

5.4.5 Presburger Arithmetik

Beispiel 5.24. (ohne Beweis) In $(\mathbb{Z}; +, <)$ sind $n\mathbb{Z} \subseteq \mathbb{Z}$ $(n \geq 2)$ existentiell definierbar aber nicht qf definierbar. Jedoch hat $(\mathbb{Z}; 0, 1, +, -, <, 2\mathbb{Z}, 3\mathbb{Z}, 4\mathbb{Z}, \dots)$ QE.

6 Elementary extensions

Theorem 6.1 (Tarski-Test)**.** *Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and A is a subset. TFAE:*

(i) A is the domain of an elementary substructure;

(ii) for every $\mathcal{L}(A)$-formula in one free variable $\phi(x)$ gilt: if $\mathcal{M} \models \exists x.\, \phi(x)$, then $\mathcal{M} \models \phi(a)$ for some $a \in A$.

Proof.

(i) ⇒ (ii): Let $\mathcal{A}$ be the elementary substructure with domain A. Then if $\mathcal{M} \models \exists x.\, \phi(x)$, then $\mathcal{A} \models \exists x.\, \phi(x)$ by elementarity; so say $a \in A$ and $\mathcal{A} \models \phi(a)$; but then $\mathcal{M} \models \phi(a)$ by elementarity.

(ii) ⇒ (i): By (ii) with $\phi(x) := x \doteq f(\overline{a})$ we have that A is closed under ($\geq$ 0-ary) functions. So A is the domain of a substructure $\mathcal{A}$.

We show by induction on complexity that for any $\mathcal{L}(A)$-sentence σ,

$$\mathcal{A} \models \sigma \Leftrightarrow \mathcal{M} \models \sigma. \tag{1}$$

(1) holds for σ atomic since $\mathcal{A}$ is a substructure, and if it holds for σ and σ' then clearly it holds for $\neg\sigma$ and $(\sigma \wedge \sigma')$.

So suppose $\sigma = \exists x.\, \phi(x)$, and (1) holds for $\phi(a)$ for any $a \in A$.

If $\mathcal{A} \models \sigma$, then $\mathcal{A} \models \phi(a)$ for some $a \in A$, so $\mathcal{M} \models \phi(a)$ by the induction hypothesis, so $\mathcal{M} \models \sigma$. Conversely, if $\mathcal{M} \models \sigma$, then $\mathcal{M} \models \phi(b)$ for some $b \in \mathcal{M}$, so, by (i), $\mathcal{M} \models \phi(a)$ for some $a \in A$, so $\mathcal{A} \models \phi(a)$ by the induction hypothesis, so $\mathcal{A} \models \sigma$.

□

Definition 6.2. An $\mathcal{L}$-theory T has **built-in Skolem functions**, if for every $\mathcal{L}$-formula $\phi(x, \overline{y})$ there is $f_{\phi(x,\overline{y})} \in \mathcal{L}$ such that

$$T \models \forall \overline{y}.\, (\exists x.\, \phi(x, \overline{y}) \to \phi(f_{\phi(x,\overline{y})}(\overline{y}), \overline{y})).$$

Lemma 6.3. *If T is an $\mathcal{L}$-theory with built-in Skolem functions, then substructures of models are elementary: if $\mathcal{N} \leq \mathcal{M} \models T$ then $\mathcal{N} \preceq \mathcal{M}$.*

Proof. Let $\phi(x, \overline{a})$ be an $\mathcal{L}(\mathcal{N})$-formula, and suppose $\mathcal{M} \vDash \exists x.\ \phi(x, \overline{a})$. Then $f_{\phi(x,\overline{y})}(\overline{a}) \in \mathcal{N}$, since $\mathcal{N} \leq \mathcal{M}$, and $\mathcal{M} \vDash \phi(f_{\phi(x,\overline{y})}(\overline{a}), \overline{a})$. So by Theorem 6.1 $\mathcal{N} \preceq \mathcal{M}$. □

Lemma 6.4. *Let T be an $\mathcal{L}$-theory. Then T has a* **skolemisation** $T^* \supseteq T$*, a theory in a language $\mathcal{L}^* \supseteq \mathcal{L}$ with $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$, such that T^* has built-in Skolem functions, and any model of T expands to a model of T^*.*

Proof. Let $\mathcal{L}_0 := \mathcal{L}$ and $\mathcal{L}_{i+1} := \mathcal{L}_i \cup \{f_{\phi(x,\overline{y})} : \phi(x, \overline{y}) \text{ an } \mathcal{L}_i\text{-formula}\}$ and $\mathcal{L}^* := \bigcup_i L_i$.

Let $T^* := T \cup \{\forall \overline{y}.\ (\exists x.\ \phi(x, \overline{y}) \to \phi(f_{\phi(x,\overline{y})}(\overline{y}), \overline{y})) \mid \phi(x, \overline{y}) \text{ an } \mathcal{L}^*\text{-formula}\}$.

If $\mathcal{M} \vDash T$, recursively define the $f_{\phi(x,\overline{y})}$ to witness the existentials (using the axiom of choice) to obtain an expansion to a model of T^*. □

Theorem 6.5 (Löwenheim-Skolem)**.** *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure.*

(i) "Downwards": If $A \subseteq \mathcal{M}$ is a subset with $|A| \geq |\mathcal{L}| + \aleph_0$, then there exists an elementary substructure $\mathcal{N} \preceq \mathcal{M}$ containing A, with $|\mathcal{N}| = |A|$.

(ii) "Upwards": For any cardinal $\kappa \geq |\mathcal{L}| + |\mathcal{M}|$ there exists an elementary extension $\mathcal{N} \succeq \mathcal{M}$ with $|\mathcal{N}| = \kappa$.

In particular, for any $\kappa \geq |\mathcal{L}| + \aleph_0$, there is $\mathcal{N} \equiv \mathcal{M}$ with $|\mathcal{N}| = \kappa$.

Proof. (i) First, assume $T := \mathrm{Th}(\mathcal{M})$ has built-in skolem functions.

Let $\mathcal{N} := \langle A \rangle^{\mathcal{M}}_{\mathcal{L}^*}$. By Lemma 2.1, $|\mathcal{N}| = |A|$. By Lemma 6.3, $\mathcal{N} \preceq \mathcal{M}$.

Now for the general case, let $\mathcal{L}^*$ and T^* be as in Lemma 6.4, and let $\mathcal{M}^* \vDash T^*$ be an expansion of $\mathcal{M}$ to $\mathcal{L}^*$. Since $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$, we have $|A| \geq |\mathcal{L}^*| + \aleph_0$, so we obtain $A \subseteq \mathcal{N}^* \preceq \mathcal{M}^*$ with $|\mathcal{N}^*| = |A|$. Then $\mathcal{N} := \mathcal{N}^* \restriction_{\mathcal{L}}$ is as required.

(ii) Let $\mathcal{L}' := \mathcal{L}(\mathcal{M}) \dot\cup \{c_i : i \in \kappa\}$, where c_i are new constants, and $T' := \mathrm{Th}(\mathcal{M}_{\mathcal{M}}) \cup \{c_i \neq c_j : i \neq j \in \kappa\}$. Then T' is consistent since $\mathcal{M}$ is infinite Let $\mathcal{M}' \vDash T'$ with $\mathcal{M} \leq \mathcal{M}'$ and let $A := \{c_i^{\mathcal{M}'} : i \in \kappa\} \subseteq \mathcal{M}'$. Then by (i), there exists $\mathcal{N} \preceq \mathcal{M}'$ with $|\mathcal{N}| = |A| = \kappa$. Now $\mathcal{N} \succeq \mathcal{M}$, since $\mathcal{N} \vDash \mathrm{Th}(\mathcal{M}_{\mathcal{M}})$.

□

Corollary 6.6 ("Skolem's Paradox")**.** *If* ZFC *is consistent, it has a countable model. This is not a paradox!*

Corollary 6.7. *No infinite structure is determined uniquely up to isomorphism by its theory.*

Remark 6.8. In contrast, any *finite* structure *is* determined uniquely up to isomorphism by its theory.

Remark 6.9. $(\mathbb{R}; +, \cdot, <)$ *is* the unique complete ordered field; but "complete" (every bounded subset has a supremum) is not first-order expressible.

Definition 6.10. Let κ be an infinite cardinal. A theory T is κ**-categorical** if T has a unique model of cardinality κ up to isomorphism.

Theorem 6.11 (Cantor)**.** DLO *is $\aleph_0$-categorical.*

Proof. ("Back-and-forth argument")

Let $\mathcal{M}, \mathcal{N} \vDash$ DLO with $|\mathcal{M}| = \aleph_0 = |\mathcal{N}|$. Let $\mathcal{M} = (m_i)_{i\in\omega}$ and $\mathcal{N} = (n_i)_{i\in\omega}$. We recursively construct a chain of partial isomorphisms $\theta_i : \mathcal{M} \dashrightarrow \mathcal{N}$ such that

$$|\operatorname{dom}\theta_i| < \aleph_0 \text{ und for all } j < i, \text{ we have } m_j \in \operatorname{dom}\theta_i \text{ and } n_j \in \operatorname{im}\theta_i. \qquad (*)$$

Let $\theta_0 := \emptyset$.
Given θ_i satisfying (*),
exactly as in the proof of QE for DLO, θ_i extends to $\theta_i' : \mathcal{M} \dashrightarrow \mathcal{N}$ with $m_i \in \operatorname{dom}{\theta_i}'$;
similarly, $(\theta_i')^{-1} : \mathcal{N} \dashrightarrow \mathcal{M}$ extends to $\theta_i'' : \mathcal{N} \dashrightarrow \mathcal{M}$ with $n_i \in \operatorname{dom}{\theta_i}''$;
then $\theta_{i+1} := (\theta_i'')^{-1} : \mathcal{M} \dashrightarrow \mathcal{N}$ satisfies (*).

Then $\theta := \bigcup_i \theta_i : \mathcal{M} \xrightarrow{\cong} \mathcal{N}$ is an isomorphism. □

Theorem 6.12 (Vaughts Criterion)**.** *If an $\mathcal{L}$-theory T has no finite models and is κ-categorical for some $\kappa \geq |\mathcal{L}| + \aleph_0$, then T is complete.*

Proof. Let $\mathcal{M}, \mathcal{N} \vDash T$. Both $\mathcal{M}$ and $\mathcal{N}$ are infinite. By Theorem 6.5, there are $\mathcal{M}' \equiv \mathcal{M}$ and $\mathcal{N}' \equiv \mathcal{N}$ with $|\mathcal{M}'| = \kappa = |\mathcal{N}'|$. By κ-categoricity $\mathcal{M}' \cong \mathcal{N}'$. Hence $\mathcal{M} \equiv \mathcal{N}$. □

Notation 6.13. For T a complete $\mathcal{L}$-theory, we set

$$|T| := |\mathcal{L}| + \aleph_0,$$

being the cardinality of the set of all $\mathcal{L}$-sentences.

7 Types

Definition 7.1. Let $n \in \omega$, and let $x_1, \ldots, x_n$ be a tuple of distinct variables.

- Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The **type** (in variables $\overline{x}$) of a tuple $\overline{b} \in \mathcal{M}^n$ in $\mathcal{M}$ is

$$\operatorname{tp}^{\mathcal{M}}(\overline{b}) := \{\phi(\overline{x}) : \mathcal{M} \vDash \phi(\overline{b});\ \phi(\overline{x}) \text{ an } \mathcal{L}\text{-formula}\}.$$

- A **type** is the type of some tuple in some structure
- A **partial type** is a subset of type.
- Let T be a consistent $\mathcal{L}$-theory. The set of **n-types in T** is

$$S_n(T) := \{\operatorname{tp}^{\mathcal{M}}(\overline{b}) : \mathcal{M} \vDash T;\ \overline{b} \in \mathcal{M}^n\}.$$

  (Technically, this should be written as $S_{\overline{x}}(T)$ as it depends on the choice of variables $\overline{x}$ and not just on n. But "S_n" is traditional.)

- If A is a subset of a structure $\mathcal{M}$ and $\overline{b} \in \mathcal{M}^{<\omega}$, the type of $\overline{b}$ **over** A is

$$\operatorname{tp}^{\mathcal{M}}(\overline{b}/A) := \operatorname{tp}^{\mathcal{M}_A}(\overline{b}) \in S_n(\operatorname{Th}(\mathcal{M}_A)).$$

- Let $S_n^{\mathcal{M}}(A) := S_n(\operatorname{Th}(\mathcal{M}_A))$.

- We write $S(T)$ for the set of all types (in arbitrary free variables) in T, and $S^{\mathcal{M}}(A)$ for the set of all types over $A \subseteq \mathcal{M}$.

Lemma 7.2. *A set of $\mathcal{L}$-formulas $\Phi(\overline{x})$ is a partial type iff it is finitely satisfiable, i.e. for every finite subset $\Phi_0 \subseteq \Phi$ there exist an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \exists \overline{x}. \bigwedge_{\psi \in \Phi_0} \psi(\overline{x})$.*

A partial type $\Phi(\overline{x})$ is a type iff for every $\mathcal{L}$-formula $\psi(\overline{x})$, either $\psi(\overline{x}) \in \Phi(\overline{x})$ or $\neg\psi(\overline{x}) \in \Phi(\overline{x})$.

So types in T are precisely maximal consistent *(= finitely satisfiable in models of T) sets of formulas in a given tuple of variables.*

Proof. Compactness. □

Definition 7.3. $S_n(T)$ is a topological space with basis of open sets $\{[\phi] : \phi(\overline{x}) \text{ an } \mathcal{L}\text{-formula}\}$, where $[\phi] := \{p \in S_n(T) : \phi \in p\} \subseteq S_n(T)$.

Fact 7.4. *$S_n(T)$ is a* Stone space*, i.e. it is compact Hausdorff and totally disconnected.*

Proof. Exercise 4. □

Example 7.5 (Types in DLO). DLO has QE, so if $\mathcal{M} \vDash \text{DLO}$ and $\overline{b} \in \mathcal{M}^{<\omega}$, $\text{tp}^{\mathcal{M}}(\overline{b}/A)$ is determined by the basic $\mathcal{L}_<(A)$-formulas satisfied by $\overline{b}$.

- $S_1(\text{DLO})$: the only consistent basic formula in one variable x is $x = x$, so $|S_1(\text{DLO})| = 1$.
- $S_2(\text{DLO})$ consists of the three types implied (modulo DLO) respectively by $x < y$, $x = y$, and $y < x$.
- $|S_n(\text{DLO})| < \aleph_0$.
- $S_1^{\mathbb{Q}}(\mathbb{Z})$ consists of the types implied by
  - $x = n$ (some $n \in \mathbb{Z}$);
  - $n < x < n + 1$ (some $n \in \mathbb{Z}$);
  - $\{x < n : n \in \mathbb{Z}\}$;
  - $\{x > n : n \in \mathbb{Z}\}$.
- More generally, consider $p(x) = S_1(A)$, where $A \subseteq \mathcal{M} \vDash \text{DLO}$.

  If $x = a \in p(x)$ for some $a \in A$, then $x = a$ implies p.

  Else, let $L := \{a \in A : a < x \in p\}$ and $R := \{a \in A : a > x \in p\}$. Then (L, R) is a **cut** in A, i.e. $L \dot{\cup} R = A$ and for all $l \in L$ and $r \in R$, we have $l < r$. Then $p(x)$ is implied by $\{l < x : l \in L\} \cup \{x < r : r \in R\}$.

  Conversely, if (L, R) is a cut, $\{l < x : l \in L\} \cup \{x < r : r \in R\}$ is finitely satisfiable, so implies a type in $S_1(A)$.
- e.g. $S_1^{\mathbb{Q}}(\mathbb{Q})$ is in bijection with $\mathbb{R} \cup \{-\infty, +\infty\}$ (but the topology is certainly not the Euclidean topology!).

Definition 7.6. For $\Phi(\overline{x})$ a set of $\mathcal{L}$-formulas and $\overline{c}$ an $|\overline{x}|$-tuple of constants,

$$\Phi(\overline{c}) := \{\phi(\overline{c}) : \phi(\overline{x}) \in \Phi(\overline{x})\}.$$

If $\overline{y}$ is a $|\overline{x}|$-tuple of variables, we define $\Phi(\overline{y})$ similarly.

Remark 7.7. Let T be a consistent $\mathcal{L}$-theory.

- $S_0(T)$ consists of the completions of T.
- The map $S_n(T) \to S_0(T')$; $p(\overline{x}) \mapsto p(\overline{c})$ is a homeomorphism, where $\overline{c}$ is an n-tuple of new constants and and T' is the $\mathcal{L}(\overline{c})$-theory consisting of the same sentences as T.

Remark 7.8. Let $A \subseteq \mathcal{M}$ be a subset of an $\mathcal{L}$-structure $\mathcal{M}$.

If $\mathcal{N}$ is another $\mathcal{L}$-structure containing A and $\mathcal{N}_A \equiv \mathcal{M}_A$ (e.g. if $\mathcal{N} \succeq \mathcal{M}$) then $S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$.

We often just write $S_n(A)$ for $S_n^{\mathcal{M}}(A)$.

Definition 7.9. Let $\Phi(\overline{x})$ a set of formulas.

- If $\overline{a} \in \mathcal{M}$, we write
$$\overline{a} \vDash \Phi$$
to mean
$$\mathcal{M}_{\overline{a}} \vDash \Phi(\overline{a}).$$
Then $\overline{a}$ is called a **realisation** of the partial type Φ.
- We write
$$\Phi(\overline{x}) \vdash_T \Phi'(\overline{x})$$
to mean that
$$\Phi(\overline{c}) \vdash_T \Phi'(\overline{c}),$$
where $\overline{c}$ is a $|\overline{x}|$-tuple of new variables. Equivalently for any $\mathcal{M} \vDash T$ and $\overline{a} \in \mathcal{M}^{|\overline{x}|}$
$$\overline{a} \vDash \Phi(\overline{x}) \;\Rightarrow\; \overline{a} \vDash \Phi'(\overline{x}).$$
We write e.g. $\phi \vdash_T \Phi$ to mean $\{\phi\} \vdash_T \Phi$.

Definition 7.10. $\mathcal{M} \vDash T$ **realises** a set of formulas Φ in T, if some $\overline{b} \in \mathcal{M}^n$ is a realisation of Φ.

If $\mathcal{M}$ does not realise Φ, $\mathcal{M}$ **omits** Φ.

Definition 7.11. A type $p(\overline{x}) \in S_n(T)$ is **isolated** if there is $\phi(\overline{x}) \in p(\overline{x})$ such that $\phi(\overline{x}) \vdash_T p(\overline{x})$. We say then that ϕ **isolates** p. .

Lemma 7.12. *Let T be a complete theory. Let $p \in S_n(T)$ be isolated and $\mathcal{M} \vDash T$. Then $\mathcal{M}$ realises.*

Proof. Let $\phi(\overline{x}) \in p(\overline{x})$ with $\phi(\overline{x}) \vdash_T p(\overline{x})$. Since p is finitely satisfiable (by Lemma 7.2) and T is complete, we have $T \vDash \exists\overline{x}.\ \phi(\overline{x})$. So say $\overline{b} \in \mathcal{M}^{|\overline{x}|}$ with $\mathcal{M} \vDash \phi(\overline{b})$. Then $\overline{b} \vDash p$. □

Example 7.13. Let $K \vDash$ ACF and $F \subseteq K$ a subfield. Let $p(x) \in S_1(F)$. Since ACF has QE, p is determined by the polynomial equations over A it implies, i.e. by
$$I_p := \{f(X) \in F[X] : f(x) \doteq 0 \in p\}.$$
This is an ideal: $I_p \trianglelefteq F[X]$. Moreover it is a prime ideal, i.e. $f \cdot g \in I_p \Rightarrow (f \in I_p$ or $g \in I_p)$. Indeed, take a realisation $a \in K' \succeq K$; then if $(f \cdot g)(a) = 0$ then $f(a) \cdot g(a) = 0$, so $f(a) = 0$ or $g(a) = 0$ (since K' is an integral domain).

Now $F[X]$ is a principal ideal domain, so $I_p = m_p \cdot F[X]$ for some prime $m_p \in F[X]$. If $m_p = 0$, p is the type of a transcendental element over F, and p might not be realised in K (K could be F^{alg}). Else, p is an *algebraic* type, and it is isolated by $m_p(x) \doteq 0$, and is realised in every algebraically closed field extension of F.

Now consider $p \in S_n(F)$, where $n \geq 1$. As above,

$$I_p := \{f(\overline{X}) \in F[\overline{X}] : f(\overline{x}) \doteq 0 \in p\}$$

is a prime ideal in $F[\overline{X}]$.

Conversely, if $I \trianglelefteq F[\overline{X}]$ is a prime ideal, then $R := F[\overline{X}]/I$ is an integral domain. Let $K' \supseteq F$ be an algebraic closure of the fraction field of R, and let $a_i := X_i/I \in K'$. Let $p := \mathrm{tp}^{K'}(\overline{a}/F)$. Then $I_p = I$.

So $p \mapsto I_p$ is a bijection $S_n(F) \to \mathrm{Spec}(F[\overline{X}])$, where $\mathrm{Spec}(F[\overline{X}])$ is the set of prime ideals of $F[\overline{X}]$. (This map is continuous if $\mathrm{Spec}(F[\overline{X}])$ is equipped with its usual Zariski topology, but is not a homeomorphism.)

We can also think about this in terms of naive algebraic geometry. If F is a subfield of an algebraically closed field K, a *closed algebraic subset* of K^n over F is the common zero set $V = V(I) \subseteq K^n$ of an ideal $I \trianglelefteq F[\overline{X}]$. V is *irreducible* if it is not the union of two proper closed algebraic subsets over F; equivalently, if I is a prime ideal. A point $\overline{a} \in V$ is *generic* (over F) if it is contained in no proper closed algebraic subset over F. Then $\mathrm{tp}(\overline{a}/F) = p_{I_p}$, and conversely any $p \in S_n(F)$ is of this form for some $\overline{a} \in K^n$ for some K (e.g. by considering $F[\overline{X}]/I_p$ as above). In other words: the types in ACF are precisely the types of generics of irreducible closed algebraic sets.

Finally, consider an arbitrary subset $A \subseteq K$. Let $F \leq K$ be the subfield generated by A. Then a type over A determines a type over F. In other words, the restriction map $S_n(F) \to S_n(A)$ is a bijection.

7.1 Saturation

Lemma 7.14 ("Joint Consistency for Constants")**.** *Let T be a complete $\mathcal{L}$-theory. For $i \in I$ let C_i be a set of constants, and suppose $C_i \cap C_j = \emptyset = C_i \cap \mathcal{L}$ for $i \neq j$. Let $T_i \supseteq T$ be a consistent $\mathcal{L} \cup C_i$-theory. Then $\bigcup_{i \in I} T_i$ is consistent.*

Remark 7.15. In fact this holds if we add new relations and functions too. This is known as "Robinson's Joint Consistency Theorem" (a proof can be found in Chang&Keisler).

Proof. If $\bigcup_{i \in I} T_i$ is inconsistent then by compactness $T \cup \{\phi_i(\overline{c}_i) : i \in I_0\}$ is inconsistent where $I_0 \subseteq I$ is finite and $T_i \vDash \phi_i(\overline{c}_i)$ and $\overline{c}_i \in C_i^{<\omega}$. WLOG $I_0 = \{1, 2, \ldots, n\}$.

But then $T \vDash \forall \overline{x}_1, \ldots, \overline{x}_n. \neg \bigwedge_{1 \leq i \leq n} \phi_i(\overline{x}_i)$ with the $\overline{x}_i$ disjoint tuples.

Then $T \vDash \forall \overline{x}_1, \ldots, \overline{x}_n. \bigvee_{1 \leq i \leq n} \neg \phi_i(\overline{x}_i)$.

Then $T \vDash \bigvee_{1 \leq i \leq n} \forall \overline{x}_i. \neg \phi_i(\overline{x}_i)$. So $T \cup \{\phi_i(\overline{c}_i)\}$ is inconsistent for some i. This contradicts consistency of T_i. □

Lemma 7.16. *Given $\mathcal{M}$ an infinite $\mathcal{L}$-structure and $A \subseteq \mathcal{M}$, there exists $\mathcal{N} \succeq \mathcal{M}$ which realises every $p \in S(A)$,.*

Proof. For each $n \geq 1$ and each n-type $p \in S(A)$, let $\bar{c}_p$ be a new tuple of constants with $|\bar{c}_p| = n$.

We must show $T' := \mathrm{Th}(\mathcal{M}_{\mathcal{M}}) \cup \bigcup_{p \in S(A)} p(\bar{c}_p)$ is consistent.

By definition of $S(A)$, each $\mathrm{Th}(\mathcal{M}_A) \cup p(\bar{c}_p)$ is consistent. Also $\mathrm{Th}(\mathcal{M}_{\mathcal{M}}) \supseteq \mathrm{Th}(\mathcal{M}_{\mathcal{A}})$ is consistent. So T' is consistent by Lemma 7.14. □

Definition 7.17. Let κ be an infinite cardinal. An $\mathcal{L}$-structure $\mathcal{M}$ is **κ-saturated** if for any $A \subseteq \mathcal{M}$ with $|A| < \kappa$ every $p \in S_1(A)$ is realised in $\mathcal{M}$.

$\mathcal{M}$ is **saturated**, if it is $|\mathcal{M}|$-saturated.

Example 7.18. $(\mathbb{Q}; <)$ is $\aleph_0$-saturated.

$\mathbb{Q}^{\mathrm{alg}} \vDash \mathrm{ACF}$ is not $\aleph_0$-saturated (since it omits the transcendental type in $S_1(\emptyset)$).

Lemma 7.19. *If $\mathcal{M}$ is κ-saturated, for any $A \subseteq \mathcal{M}$ with $|A| < \kappa$ and every $n \geq 1$, every $p \in S_n(A)$ is realised in $\mathcal{M}$.*

Proof. By induction on n.

Sei $p(x_1, \ldots, x_n, y) \in S_{n+1}(A)$. Setze $q(x_1, \ldots, x_n) := \{\phi(x_1, \ldots, x_n) : \phi \in p\} \in S_n(A)$. Inductively, q is realised in $\mathcal{M}$. Say $\overline{a} \in \mathcal{M}^n$ with $\overline{a} \vDash q$.

Now $p(\overline{a}, y) = \{\phi(\overline{a}, y) : \phi \in p\} \in S_1^{\mathcal{M}}(A \cup \{a_1, \ldots, a_n\})$; Indeed, If $\Phi_0(\overline{x}, y) \subseteq_{\mathrm{fin}} p(\overline{x}, y)$ is a finite subset, Since $\mathrm{tp}^{\mathcal{M}}(a/A) = q(\overline{x}) \ni \exists y.\ \bigwedge_{\phi \in \Phi_0} \phi(\overline{x}, y)$, we have $\mathcal{M} \vDash \exists y.\ \bigwedge_{\phi \in \Phi_0} \phi(\overline{a}, y)$.

Hence (since $|A| + n < \kappa$) some $b \in \mathcal{M}$ with $b \vDash p(\overline{a}, y)$ and then $(\overline{a}, b) \vDash p(\overline{x}, y)$. □

Lemma 7.20. *Suppose $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ is partial elementary and $A \subseteq \operatorname{dom} \theta$ and $p(\overline{x}) \in S^{\mathcal{M}}(A)$.*

(i) The **conjugate** *of p by θ*

$$p^{\theta}(\overline{x}) := \{\phi(\overline{x}, \theta(\overline{a})) : \phi(\overline{x}, \overline{a}) \in p(\overline{x});\ \phi \text{ a } \mathcal{L}\text{-formula}\}$$

is a type $p^{\theta} \in S^{\mathcal{N}}(\theta(A))$.

(ii) For $b \in \mathcal{M}$, an extension θ' of θ with $\operatorname{dom} \theta' = \operatorname{dom} \theta \cup \{b\}$ is partial elementary iff $\theta'(b) \vDash \mathrm{tp}(b/\operatorname{dom} \theta)^{\theta}$.

Proof. (i) For a finite subset $\Phi_0(\overline{x}) \subseteq_{\mathrm{fin}} p(\overline{x})$, write $\bigwedge \Phi_0$ as $\psi(\overline{x}, \overline{a})$, where ψ is an $\mathcal{L}$-formula and $\overline{a} \in A^{<\omega}$. Then $\mathcal{M} \vDash \exists \overline{x}.\ \psi(\overline{x}, \overline{a})$. So by elementarity, $\mathcal{N} \vDash \exists \overline{x}.\ \psi(\overline{x}, \theta(\overline{a}))$.

(ii) Immediate.

□

Lemma 7.21. *If $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}| \geq \aleph_0$ and $\mathcal{M}$ and $\mathcal{N}$ are both saturated, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. Back-and-forth.

Let $\mathcal{M} = (m_\alpha)_{\alpha \in \lambda}$ and $\mathcal{N} = (n_\alpha)_{\alpha \in \lambda}$. We recursively construct a chain of partial isomorphisms $\theta_\alpha : \mathcal{M} \dashrightarrow \mathcal{N}$ for $\alpha \in \lambda$ such that

$|\operatorname{dom} \theta_\alpha| \leq 2 \cdot |\alpha|$ und for all $\beta < \alpha$, we have $m_\beta \in \operatorname{dom} \theta_\alpha$ and $n_\beta \in \operatorname{im} \theta_\alpha$. (*)

Let $\theta_0 := \emptyset$. For η a limit ordinal, let $\theta_\eta := \bigcup_{\alpha<\eta} \theta_\alpha$. We have $|\mathrm{dom}\,\theta_\eta| \leq |\eta| = 2 \cdot |\eta|$.

Given θ_α satisfying (*),

By saturation, $\mathrm{tp}(m_\alpha/\mathrm{dom}\,\theta_\alpha)^{\theta_\alpha} \in S_1(\mathrm{im}\,\theta_\alpha)$ is realised in $\mathcal{N}$.

Hence θ_α extends to $\theta'_\alpha : \mathcal{M} \dashrightarrow \mathcal{N}$ with $m_\alpha \in \mathrm{dom}\,\theta_\alpha{}'$; symmetrically, $(\theta'_\alpha)^{-1} : \mathcal{N} \dashrightarrow \mathcal{M}$ extends to $\theta''_\alpha : \mathcal{N} \dashrightarrow \mathcal{M}$ with $n_\alpha \in \mathrm{dom}\,\theta_\alpha{}''$; then $\theta_{\alpha+1} := (\theta''_\alpha)^{-1} : \mathcal{M} \dashrightarrow \mathcal{N}$ satisfies (*).

Then $\theta := \bigcup_\alpha \theta_\alpha : \mathcal{M} \xrightarrow{\cong} \mathcal{N}$ is an isomorphism. □

Definition 7.22. An $\mathcal{L}$-structure $\mathcal{M}$ is κ-universal if any $\mathcal{N} \equiv \mathcal{M}$ with $|\mathcal{N}| < \kappa$ elementarily embeds in $\mathcal{M}$.

Lemma 7.23. *Let $\mathcal{M}$ be an κ-saturated $\mathcal{L}$-structure. Then $\mathcal{M}$ κ^+-universal.*

Proof. Suppose $\mathcal{N} \equiv \mathcal{M}$ with $\lambda := |\mathcal{N}| < \kappa^+$. Say $\mathcal{N} = \{a_\alpha : \alpha \in \lambda\}$. We build a chain of partial elementary maps $\theta_\alpha : \mathcal{N} \dashrightarrow \mathcal{M}$ with $\mathrm{dom}\,\theta_\alpha = A_\alpha := \{a_\beta : \beta < \alpha\}$.

Set $\theta_0 := \emptyset$. For η a limit ordinal, set $\theta_\eta := \bigcup_{\alpha<\eta} \theta_\alpha$.

Since $|A_\alpha| = |\alpha| < \lambda \leq \kappa$, by κ-saturation, $\mathrm{tp}(a_\alpha/A_\alpha)^{\theta_\alpha}$ is realised in $\mathcal{M}$. Set $\theta_{\alpha+1}(a_\alpha)$ to be a realisation. □

8 Countable models of countable theories

8.1 Countable saturated models

Lemma 8.1 (Tarski's Chain Lemma). *If $(I; <)$ is a linear order and $(\mathcal{M}_i)_{i \in I}$ is an elementary chain, meaning $\mathcal{M}_i \preceq \mathcal{M}_j$ for $i < j$, then $\mathcal{M}_i \preceq \bigcup_{i \in I} \mathcal{M}_i$ for all i.*

Proof. Exercise 1.1(b). □

Definition 8.2. A theory T is **small** if $|S_n(T)| \leq \aleph_0$ for all $n \in \omega$.

Example 8.3. $(\mathbb{Q}; <)$ ist schmal. $(\mathbb{Q}; <)_{\mathbb{Q}}$ ist nicht schmal.

Theorem 8.4. *Let T be a countable (i.e. $|\mathcal{L}| \leq \aleph_0$) complete $\mathcal{L}$-theory with infinite models.*

Then T has a countable saturated model iff T is small.

Proof. $\Rightarrow$: Let $n \in \omega$. Every type in $S_n(T)$ is realised in the countable saturated model, so $|S_n(T)| \leq \aleph_0$.

$\Leftarrow$: If $A \subseteq_{\mathrm{fin}} \mathcal{M} \vDash T$, then $|S_1(A)| \leq |S_{|A|+1}(T)| \leq \aleph_0$; indeed, if $A = \{a_1, \ldots, a_n\}$ then $p(x, \overline{a}) \mapsto p(x, \overline{y})$ is an injection $S_1(A) \hookrightarrow S_{n+1}(T)$.

We build an elementary chain $(\mathcal{M}_i)_{i \in \omega}$ of countable models. Let $\mathcal{M}_0 \vDash T$ with $|\mathcal{M}_0| = \aleph_0$, which exists by Löwenheim-Skolem. Given, let $X := \bigcup_{A \subseteq_{\mathrm{fin}} \mathcal{M}_i} S_1(A)$. Then $|X| \leq \aleph_0$. So by Lemma 7.16 and Löwenheim-Skolem, there is a countable model $\mathcal{M}_{i+1} \succeq \mathcal{M}_i$ which realises all $p \in X$.

Now let $\mathcal{M} := \bigcup_{i \in \omega} \mathcal{M}_i \vDash T$. Then $|\mathcal{M}| \leq \aleph_0$ and if $A \subseteq_{\mathrm{fin}} \mathcal{M}$, then $A \subseteq_{\mathrm{fin}} \mathcal{M}_i$ for some $i \in \omega$. Hence every $p \in S_1(A)$ is realised in $\mathcal{M}_{i+1}$ and hence in $\mathcal{M} \succeq \mathcal{M}_{i+1}$.

□

8.2 Omitting types

Definition 8.5. Let T be a consistent $\mathcal{L}$-theory.

- A $\mathcal{L}$-formula $\phi(\overline{x})$ is **consistent** (with T), if $T \not\models \neg\exists x.\ \psi(\overline{x})$.
- A set of formulas $\Phi(\overline{x})$ in T is **isolated** if there exists a consistent $\phi(\overline{x})$ s.t. $\phi(\overline{x}) \vdash_T \Phi(\overline{x})$.

Theorem 8.6 (Omitting Types Theorem)**.** *Let T be a countable consistent theory.*

Let $(\Phi_k(\overline{x}_k))_{k\in\omega}$ be non-isolated sets of formulas. Then there exists a countable model $\mathcal{M} \models T$ which omits every Φ_k.

Proof. Let $C = \{c_i : i < \omega\}$ be a set of new constants.

Enumerate the $\mathcal{L}(C)$-formulas in x as $(\psi_i(x) : i < \omega)$.

Let $\xi : \{(k, \overline{c}) : k \in \omega;\ \overline{c} \in C^{|\overline{x}_k|}\} \to \omega$ be a bijection.

We construct an increasing chain $(\Sigma_i)_{i\in\omega}$ of sets of $\mathcal{L}(C)$-sentences such that

(i) $|\Sigma_i| < \aleph_0$;

(ii) $T_i := T \cup \Sigma_i$ is consistent;

(iii) if $j < i$, there is $c \in C$ such that $T_i \models \exists x.\ \psi_j(x) \to \psi_j(c)$;

(iv) if $\xi(k, \overline{c}) < i$, there is $\phi(\overline{x}_k) \in \Phi_k(\overline{x}_k)$ such that $T_i \models \neg\phi(\overline{c})$.

Let $\Sigma_0 := \emptyset$. Suppose Σ_i satisfies (i)-(iv).

Say $c \in C$ does not appear in Σ_i nor in ψ_i, and let $\Sigma'_{i+1} := \Sigma_i \cup \{\exists x. \psi_i(x) \to \psi_i(c)\}$. Note that $T \cup \Sigma'_{i+1}$ is consistent.

Say $\xi(k, \overline{c}) = i$, and let $\overline{x} := \overline{x}_k$. Let $\delta(\overline{x}, \overline{y})$ be such that $\bigwedge \Sigma'_{i+1} = \delta(\overline{c}, \overline{c}')$, where $\overline{c}' \in (C \setminus \{c_1, \dots, c_{|\overline{x}|}\})^{<\omega}$. Now $T \not\models \neg\exists\overline{x}.\ \exists\overline{y}.\ \delta(\overline{x}, \overline{y})$, so by non-isolation of Φ_k there is $\phi_i(\overline{x}) \in \Phi_k(\overline{x})$ such that

$$\exists\overline{y}.\ \delta(\overline{x}, \overline{y}) \not\vdash_T \phi_i(\overline{x})). \qquad (*)$$

Let $\Sigma_{i+1} := \Sigma'_{i+1} \cup \{\neg\phi_i(\overline{c})\}$. Then $T \cup \Sigma_{i+1}$ is consistent by (*).

Now let $T_\omega := T \cup \bigcup_{i\in\omega} \Sigma_i$. This is consistent by (ii) (since $(\Sigma_i)_i$ is an increasing chain), so let $\mathcal{M} \models T_\omega$. By (iii) and Tarski's test, $\{c^{\mathcal{M}} : c \in C\}$ is the domain of a countable elementary substructure $\mathcal{N} \preceq \mathcal{M}$. By (iv), $\mathcal{N}$ omits each Φ_k. □

8.3 Prime models

Definition 8.7. $\mathcal{M} \models T$ is a **prime** model of a theory T if it elementarily embeds in any model of T.

Example 8.8. $(\mathbb{Z}; S)$ is a prime model of its theory.

$(\mathbb{Q}; <)$ is a prime model of DLO.

Definition 8.9. An $\mathcal{L}$-structure $\mathcal{M}$ is **atomic** if $\mathrm{tp}(\overline{a})$ is isolated for every $\overline{a} \in \mathcal{M}^{<\omega}$.

Definition 8.10. A formula $\phi(\overline{x})$ is an **atom** modulo T if it is consistent with T and isolates a type in T; equivalently, for no $\psi(\overline{x})$ are both $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ consistent with T.

Notation 8.11. $\overline{a}\overline{b} := (a_1, \ldots, a_{|\overline{a}|}, b_1, \ldots, b_{|\overline{b}|})$.

Lemma 8.12 ("Monotonicity and transitivity of isolation")**.** *Let* $\overline{a}, \overline{b} \in \mathcal{M}^{<\omega}$*. Then* $\mathrm{tp}(\overline{a}\overline{b})$ *is isolated iff* $\mathrm{tp}(\overline{a}/\overline{b})$ *and* $\mathrm{tp}(\overline{b})$ *are isolated.*

Proof. $\Rightarrow$: Say, $\phi(\overline{x}, \overline{y})$ isolates $\mathrm{tp}(\overline{a}\overline{b})$. Then

- $\phi(\overline{x}, \overline{b})$ isolates $\mathrm{tp}(\overline{a}/\overline{b})$; indeed, if $\overline{a} \vDash \psi(\overline{x}, \overline{b})$ then $\overline{a}\overline{b} \vDash \psi(\overline{x}, \overline{y})$, and so $\phi(\overline{x}, \overline{y}) \vdash_T \psi(\overline{x}, \overline{y})$, and so $\phi(\overline{x}, \overline{b}) \vdash_{T_{\overline{b}}} \psi(\overline{x}, \overline{b})$.
- $\exists \overline{x}.\ \phi(\overline{x}, \overline{y})$ isolates $\mathrm{tp}(\overline{b})$; indeed, if $\overline{b} \vDash \psi(\overline{y})$ then $\phi(\overline{x}, \overline{y}) \vdash_T \psi(\overline{y})$, and so $\exists \overline{x}.\ \phi(\overline{x}, \overline{y}) \vdash_T \psi(\overline{y})$.

$\Leftarrow$: Say, $\phi(\overline{y})$ isolates $\mathrm{tp}(\overline{b})$ and $\xi(\overline{x}, \overline{b})$ isolates $\mathrm{tp}(\overline{a}/\overline{b})$ (where $\xi(\overline{x}, \overline{y})$ is an $\mathcal{L}$-formula).

Then $\xi(\overline{x}, \overline{y}) \wedge \phi(\overline{y})$ isolates $\mathrm{tp}(\overline{a}\overline{b})$. Indeed, if $\overline{a}\overline{b} \vDash \psi(\overline{x}, \overline{y})$ then $\overline{a} \vDash \psi(\overline{x}, \overline{b})$, hence $\xi(\overline{x}, \overline{b}) \vdash_{T_{\overline{b}}} \psi(\overline{x}, \overline{b})$, hence $\overline{b} \vDash \forall \overline{x}.(\xi(\overline{x}, y) \to \psi(\overline{x}, \overline{y}))$, hence $T \vDash \forall \overline{y}.(\phi(\overline{y}) \to \forall \overline{x}.\ (\xi(\overline{x}, y) \to \psi(\overline{x}, \overline{y})))$, hence $T \vDash \forall \overline{x}, \overline{y}.\ ((\xi(\overline{x}, \overline{y}) \wedge \phi(\overline{y})) \to \psi(\overline{x}, \overline{y})))$. ☐

Lemma 8.13. *Let* $\mathcal{M}$ *be an infinite* $\mathcal{L}$*-structure, where* $|\mathcal{L}| \leq \aleph_0$*.*

Then $\mathcal{M}$ *is a prime model of* $\mathrm{Th}(\mathcal{M})$ *iff* $\mathcal{M}$ *is countable and atomic.*

Proof. Suppose $\mathcal{M} \vDash T$ is prime. Then $\mathcal{M}$ is countable since it embeds in a countable model, by Löwenheim-Skolem. Let $\overline{a} \in \mathcal{M}^{<\omega}$. Then $\mathrm{tp}(\overline{a})$ is realised in any $\mathcal{M}' \vDash T$ (namely by $\theta(\overline{a})$, where $\theta : \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{M}'$). So by the omitting types theorem, $\mathrm{tp}(\overline{a})$ is isolated. Hence $\mathcal{M}$ is atomic.

Conversely, suppose $\mathcal{M} = (a_i)_{i \in \omega}$ is countable atomic, and let $\mathcal{M}' \vDash T$. We build a chain of partial elementary maps $\theta_i : \mathcal{M} \to \mathcal{M}'$ with $\operatorname{dom} \theta_i = \{a_j : j < i\}$.

Set $\theta_0 := \emptyset$.

Given θ_i, $p_i := \mathrm{tp}(a_i / a_0, \ldots, a_{i-1})$ is isolated by atomicity (and Lemma 8.12), and hence $p_i^{\theta_i}$ is isolated and hence realised by some $b_i \in \mathcal{M}'$. Let $\theta_{i+1}(a_i) := b_i$. Then θ_{i+1} is partial elementary (by Lemma 7.20(ii)).

Then $\bigcup_{i<\omega} \theta_i : \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{M}'$ is an elementary embedding. So $\mathcal{M}$ is prime. ☐

Lemma 8.14. *Let* $\mathcal{M} \equiv \mathcal{N}$ *be countable atomic elementarily equivalent structures. Then* $\mathcal{M}$ *is isomorphic to* $\mathcal{N}$*.*

Proof. Exercise. ☐

Proposition 8.15. *Let* T *be a countable theory. Then* T *has at most one prime model up to isomorphism.*

Proof. Exercise (it follows from Lemma 8.14 and Lemma 8.13). ☐

Definition 8.16. We say the **isolated types are dense in** $S(T)$ if for each formula $\phi(\overline{x})$ consistent with T, there is some isolated $p(\overline{x}) \in S(T)$ with $\phi \in p$.

Theorem 8.17. *A countable complete theory* T *has a prime model iff isolated types are dense in* $S(T)$*.*

Proof. $\Rightarrow$: Say (by Lemma 8.13) $\mathcal{M} \vDash T$ is countable atomic. Then any consistent $\phi(\overline{x})$ has a realisation in $\mathcal{M}$ (since T is complete), which by atomicity has isolated type.

$\Leftarrow$: Let $n \in \omega$ and $\Psi_n(x_1, \ldots, x_n) := \{\neg\psi(\overline{x}) : \psi(\overline{x}) \text{ an atom}\}$. Suppose a formula $\phi(\overline{x})$ isolates $\Psi_n(\overline{x})$. By density of isolated types, $\psi(\overline{x}) \vdash_T \phi(\overline{x})$ for some atom $\psi(\overline{x})$, but then $\psi(\overline{x}) \vdash_T \neg\psi(\overline{x})$, contradicting consistency of $\psi(\overline{x})$. So each $\Psi_n(\overline{x})$ is not isolated, so by the omitting types theorem, there is a countable $\mathcal{M} \vDash T$ which omits each Ψ_n. Then $\mathcal{M}$ is atomic: if $\overline{a} \in \mathcal{M}^{<\omega}$, there is an atom $\psi(\overline{x})$, such that $\overline{a} \nvDash \neg\psi(\overline{x})$, hence $\overline{a} \vDash \psi(\overline{x})$, hence $\psi(\overline{x})$ isolates $\mathrm{tp}(\overline{a})$. □

Notation 8.18. We write elements of $2^{<\omega}$ or 2^ω as binary strings, with $\emptyset$ for the empty string.

$s \lessdot t$ means that s is a prefix of t, i.e. $t = st'$ for some t'.

Definition 8.19. A **binary tree of formulas** for a theory T is a family of formulas $(\phi_s(\overline{x}))_{s\in 2^{<\omega}}$ such that for each $s \in 2^{<\omega}$:

- $\phi_s(\overline{x})$ is consistent with T;
- $\phi_{s0}(\overline{x}) \vdash_T \phi_s(\overline{x})$ and $\phi_{s1}(\overline{x}) \vdash_T \phi_s(\overline{x})$;
- $\phi_{s0}(\overline{x}) \vdash_T \neg\phi_{s1}(\overline{x})$.

Lemma 8.20. *If a countable theory T has a binary tree of formulas, then $S(T) = 2^{\aleph_0}$.*

Proof. Say $(\phi_s(\overline{x}))_{s\in 2^{<\omega}}$ is a binary tree and $|\overline{x}| = n$. For $t \in 2^\omega$, $\{P_s(\overline{x}) : s \lessdot t\}$ is consistent, so extends to a type $p_t(\overline{x}) \in S_n(T)$. Then if $t \neq t'$, there is $s \in 2^{<\omega}$ such that $s0 \lessdot t$ and $s1 \lessdot t'$ or the other way round, so $p_t(\overline{x}) \neq p_{t'}(\overline{x})$.

So $|S(T)| \geq |S_n(T)| \geq |2^\omega| = 2^{\aleph_0}$.

Since the language $\mathcal{L}$ of T is countable, $|S(T)| \leq |\mathcal{P}(\{\mathcal{L}\text{-formulas}\})| = 2^{\aleph_0}$. □

Example 8.21. Consider 2^ω as a structure in the language $\{P_s : s \in 2^{<\omega}\}$, where $P_s(2^\omega) := \{t : s \lessdot t\}$, and let T_{BT} be its theory.

Then $\{P_s(x) : s \in 2^{<\omega}\}$ is a binary tree of formulas in T_{BT}.

Exercise: T_{BT} has QE. It follows that $2^\omega \ni t \mapsto \mathrm{tp}(t) \in S_1(T_{\mathrm{BT}})$ is a bijection, and none of these types are isolated.

So T_{BT} is not small, and isolated types are not dense, and there is no prime model.

Bonus exercise: explicitly describe a countable model.

Lemma 8.22. *Let T be a consistent theory.*

(i) If the isolated types are not dense in $S(T)$, then T has a binary tree of formulas.

(ii) If T is small, the isolated types are dense in $S(T)$.

Proof. (i) Say $\phi_\emptyset(\overline{x})$ is consistent and in no isolated type.

Given $\phi_s(\overline{x})$ which is consistent and in no isolated type ($s \in 2^{<\omega}$), $\phi_s(\overline{x})$ is not an atom, so there is $\psi(\overline{x})$ such that $\phi_{s0} := \phi(\overline{x}) \wedge \neg\psi(\overline{x})$ and $\phi_{s1} := \phi(\overline{x}) \wedge \psi(\overline{x})$ are consistent, and each is in no isolated type since ϕ has this property.

So we may recursively construct a binary tree of formulas $(\phi_s(\overline{x}))_{s\in 2^{<\omega}}$.

(ii) Follows from (i) and Lemma 8.20.

□

Proposition 8.23. *Any countable complete small theory has a prime model.*

Proof. Immediate from Lemma 8.22(ii) and Theorem 8.17. □

Remark 8.24. The converse fails; consider $(\mathbb{Q}; <)_{\mathbb{Q}}$ (Exercise).

Example 8.25. Let k be a countable field. The theory of infinite k-vector spaces is complete, countable, and small. The countable models are the vector spaces V_d of dimensions $d \in (\omega \setminus 0) \cup \{\aleph_0\}$. V_1 is the prime model, and $V_{\aleph_0}$ is the countable saturated model.

Proposition 8.26. *For a countable theory T, TFAE:*

(i) T is not small, i.e. $|S(T)| > \aleph_0$;

(ii) T has a binary tree of formulas;

(iii) $|S(T)| = 2^{\aleph_0}$.

Furthermore, if T is not small, then

(iv) T has $2^{\aleph_0}$ countable models up to isomorphism.

Proof. Exercise. □

Remark 8.27. There do exist small theories with $2^{\aleph_0}$ countable models, e.g. $\mathrm{Th}((\omega \times \omega; (P_i)_i)_{\omega\times\omega})$, where $P_i(\omega \times \omega) = \{i\} \times \omega$.

Corollary 8.28. *If a countable theory T has countably many countable models, then T is small, and hence T has a prime countable model and a saturated countable model.*

Conjecture 8.29 (Vaught)**.** *If T is a countable theory with uncountably many countable models, it has $2^{\aleph_0}$ countable models.*

(Note: it's easy to see that a countable theory has at most $2^{\aleph_0}$ countable models, so this is immediate if we assume CH.)

8.4 Ryll-Nardzewski

Theorem 8.30 (Ryll-Nardzewski)**.** *Let T be a complete countable $\mathcal{L}$-theory with infinite models. TFAE:*

(A) T is $\aleph_0$-categorical.

- *(B1) For all $n \in \omega$ and $\mathcal{M} \models T$, there are only finitely many definable subsets of $\mathcal{M}^n$.*
  (B1') For all $n \in \omega$, we have $|\Phi_{n,T}| < \aleph_0$, where $\Phi_{n,T} := {}^{\{\phi(x_1,\ldots,x_n)\}}/_{\leftrightarrow_T}$ is the set of $(\leftrightarrow_T)$-equivalence classes of $\mathcal{L}$-formulas $\phi(x_1, \ldots, x_n)$.
  (B2) Every type in T is isolated.
  (B3) For all $n \in \omega$, we have $|S_n(T)| < \aleph_0$.

(C) Every countable model of T is saturated.

(D) Every countable model of T is prime.

(E) T has a countable model which is saturated and prime.

(The equivalence (A) $\Leftrightarrow$ (B1) is the key result here, and is what is most often referred to as the Ryll-Nardzewski Theorem.)

Proof.

(B1) $\Leftrightarrow$ (B1') If $\mathcal{M} \vDash T$, then $\phi(\overline{x}) \leftrightarrow_T \psi(\overline{x})$ iff $\phi(\mathcal{M}) = \psi(\mathcal{M})$.

(B1') $\Rightarrow$ (B2) Given $p(\overline{x}) \in S(T)$, $p(\overline{x})$ contains by (B1') only finitely many formulas $\phi_1(\overline{x}), \ldots, \phi_k(\overline{x})$ up to equivalence. Then $\bigwedge_i \phi_i \in p$ isolates p.

(B2) $\Rightarrow$ (B3) Suppose each $p(\overline{x}) \in S(T)$ is isolated, say by $\phi_p(\overline{x})$. Let $n \in \omega$.

Suppose $|S_n(T)|$ is infinite. Then $\{\neg\phi_p(\overline{x}) : p \in S_n(T)\}$ is finitely satisfiable and so can be completed to some $p \in S_n(T)$, but then $\phi_p \vdash_T p \ni \neg\phi_p$, contradicting consistency of ϕ_p.

(B3) $\Rightarrow$ (B1') The map $\phi(x_1, \ldots, x_n) \mapsto \{p \in S_n(T) : \phi(\overline{x}) \in p(\overline{x})\}$ induces an injection $\Phi_{n,T} \hookrightarrow \mathcal{P}(S_n(T))$; indeed, if $\phi(\overline{x}) \not\leftrightarrow_T \psi(\overline{x})$, then $T \vDash \exists \overline{x}. \neg(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$, so for some $p \in S_n(T)$, we have $\phi \in p \not\Leftrightarrow \psi \in p$. So $\Phi_{n,T}$ is finite if $S_n(T)$ is.

For the remaining equivalences, note first that since T is complete and has infinite models, it has no finite models, so countable models have cardinality $\aleph_0$.

(A) $\Rightarrow$ (D) By Löwenheim-Skolem, every model of T has a countable elementary submodel. So if $\mathcal{M}$ is the unique countable model, it elementarily embeds in every model, so is prime.

(D) $\Rightarrow$ (B2) Every type is (by Löwenheim-Skolem) realised in a countable model, which by (D) and Lemma 8.13 is atomic.

(B2) $\Rightarrow$ (C) Let $\mathcal{M} \vDash T$ be countable, and let $A \subseteq_{\text{fin}} \mathcal{M}$. Then every $p \in S(A)$ is isolated by (B2) (and Lemma 8.12), and hence realised in $\mathcal{M}$. So $\mathcal{M}$ is saturated.

(C) $\Rightarrow$ (A) Lemma 7.21.

((C) $\wedge$ (D)) $\Rightarrow$ (E) A countable model exists by Löwenheim-Skolem, so this is immediate.

(E) $\Rightarrow$ (D) If $\mathcal{M} \vDash T$ is a countable saturated prime model and $\mathcal{N} \vDash T$ is countable, then $\mathcal{N} \overset{\preceq}{\hookrightarrow} \mathcal{M}$ by Lemma 7.23. Then $\mathcal{N}$ is prime since $\mathcal{M}$ is. .

□

Remark 8.31. We can also give direct proofs of some of the other implications:

(A) $\Rightarrow$ (B) If some type is not isolated, by the omitting types theorem we have a countable model which omits it, but we also have a countable model which realises it.

(B2) $\Rightarrow$ (D) Lemma 8.13.

(D) $\Rightarrow$ (A) Proposition 8.15.

((A) $\wedge$ (B)) $\Rightarrow$ (E) By (B), T is small, so saturated and prime countable models exist; by (A) they are isomorphic.

(E) $\Rightarrow$ (B2) By (E), an atomic model realises every type.

8.5 Fraïssé constructions

Let $\mathcal{L}$ be finite and *relational*, i.e. containing only relation symbols. In this section, we consider the empty $\mathcal{L}$-structure to be an $\mathcal{L}$-structure.

Our aim is to find $\aleph_0$-categorical theories.

Remark 8.32. Let $n \in \omega$. There are up to isomorphism only finitely many $\mathcal{L}$-structures of cardinality n.

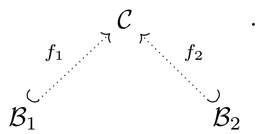
Definition 8.33. The **age** of an $\mathcal{L}$-structure $\mathcal{M}$ is the class of finite $\mathcal{L}$-structures which embed in $\mathcal{M}$,

$$\operatorname{age}(\mathcal{M}) := \{\mathcal{A} : |\mathcal{A}| < \aleph_0;\ \exists f : \mathcal{A} \hookrightarrow \mathcal{M}\} = \{\mathcal{A} : \mathcal{A} \cong \mathcal{A}' \leq_{\text{fin}} \mathcal{M}\}.$$

Lemma 8.34. *Any age $\mathcal{K}$ satisfies:*

(HP) If $\mathcal{A} \in \mathcal{K}$, then $\operatorname{age}(\mathcal{A}) \subseteq \mathcal{K}$*;*

(JEP) If $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{K}$, then there exist $\mathcal{C} \in \mathcal{K}$ and embeddings $f_i : \mathcal{B}_i \hookrightarrow \mathcal{C}$.



Proof. (HP) Clear.

(JEP) Say $f_i : \mathcal{B}_i \hookrightarrow \mathcal{M}$. Let $\mathcal{C} := \langle f_1(\mathcal{B}_1) \cup f_2(\mathcal{B}_2)\rangle^{\mathcal{M}} \leq \mathcal{M}$. Then $f_i : \mathcal{B}_i \hookrightarrow \mathcal{C} \in \mathcal{K}$.

□

Conversely:

Lemma 8.35. *Any non-empty class $\mathcal{K}$ of finite $\mathcal{L}$-structures satisfying (HP) and (JEP) is the age of a countable $\mathcal{L}$-structure.*

Proof. By Remark 8.32 we can find $\mathcal{A}_i \in \mathcal{K}$ for $i \in \omega$ such that any $\mathcal{A} \in \mathcal{K}$ is isomorphic to some $\mathcal{A}_i$.

We construct a countable chain $\mathcal{D}_0 \leq \mathcal{D}_1 \leq \dots$ with $\mathcal{D}_i \in \mathcal{K}$, such that each $\mathcal{A}_j$ for $j < i$ embeds in $\mathcal{D}_i$.

Let $\mathcal{D}_0 := \emptyset$ (which is in $\mathcal{K}$ by (HP) and $\mathcal{K} \neq \emptyset$). Suppose $\mathcal{D}_i$ has been constructed. By (JEP) there is $\mathcal{D}'_{i+1} \in \mathcal{K}$, such that $\mathcal{D}_i$ and $\mathcal{A}_i$ embed in $\mathcal{D}'_{i+1}$. Let (by Lemma 2.2) $\mathcal{D}_{i+1} \cong \mathcal{D}'_{i+1}$ with $\mathcal{D}_i \leq \mathcal{D}_{i+1}$. Then also $\mathcal{A}_i$ embeds in $\mathcal{D}_{i+1}$.

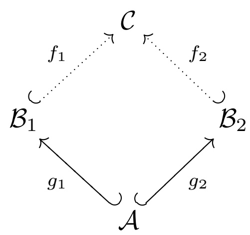
Let $\mathcal{M} := \bigcup_{i \in \omega} \mathcal{D}_i$, which is countable since each $\mathcal{D}_i$ is finite. Then each $\mathcal{A}_i$ embeds in $\mathcal{M}$, so $\mathcal{K} \subseteq \operatorname{age}(\mathcal{M})$. Conversely, if $\mathcal{A} \leq \mathcal{M}$, then $\mathcal{A} \leq \mathcal{D}_i$ for some i, so $\mathcal{A} \in \operatorname{age}(\mathcal{K})$ by (HP). □

Lemma 8.36. *Any consistent $\mathcal{L}$-theory with QE and with infinite models is $\aleph_0$-categorical.*

Proof. For any $n \in \omega$ there are only finitely many atomic $\mathcal{L}$-formulas in free variables $x_1, \ldots, x_n$, and hence only finitely many qf $\mathcal{L}$-formulas in $\overline{x}$ up to equivalence. So we conclude by Ryll-Nardzewski. □

Definition 8.37. A **Fraïssé class** is a class $\mathcal{K}$ of finite $\mathcal{L}$-structures which contains unboundedly large structures and satisfies (HP) and

(AP) If $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{K}$ and $g_i : \mathcal{A} \hookrightarrow \mathcal{B}_i$ are embeddings, then there exist $\mathcal{C} \in \mathcal{K}$ and embeddings $f_i : \mathcal{B}_i \hookrightarrow \mathcal{C}$, such that $f_1 \circ g_1 = f_2 \circ g_2$.



Remark 8.38. (AP) implies (JEP): take $\mathcal{A} := \emptyset$.

Theorem 8.39 (Fraïssé)**.**

(i) Let $\mathcal{K}$ be a Fraïssé class. Then there is a unique $\mathcal{L}$-theory $T_{\mathcal{K}}$, such that $T_{\mathcal{K}}$ has QE and infinite models and $\mathcal{K}$ is the age of any model of $T_{\mathcal{K}}$.

The unique countable model of $T_{\mathcal{K}}$ is called the **Fraïssé limit** *of $\mathcal{K}$.*

(ii) Conversely, if $\mathcal{M}$ is an infinite $\mathcal{L}$-structure with QE, then age($\mathcal{M}$) *is a Fraïssé class.*

Example 8.40.

- The class of finite linear orders is a Fraïssé class with Fraïssé limit.
- The class of finite graphs is a Fraïssé class with Fraïssé limit the countable random graph.

Proof. For $\mathcal{A}$ a finite $\mathcal{L}$-structure: say $\mathcal{A} = \{a_1, \ldots, a_n\}$, let

$$\phi_{\mathcal{A},\overline{a}}(\overline{x}) := \bigwedge\{\phi(\overline{x}) : \phi(\overline{x}) \text{ basic; } \mathcal{A} \vDash \phi(\overline{a})\}.$$

So for $\mathcal{M}$ an $\mathcal{L}$-structure and $\overline{a}' \in \mathcal{M}^n$, we have $\mathcal{M} \vDash \phi_{\mathcal{A},\overline{a}}(\overline{a}')$ iff $\overline{a}' \mapsto \overline{a}$ defines an isomorphism $\langle \overline{a}' \rangle^{\mathcal{M}} \xrightarrow{\cong} \mathcal{A}$.

(ii) Suppose $\mathcal{M}$ is an infinite $\mathcal{L}$-structure with QE. Set $\mathcal{K} :=$ age($\mathcal{M}$).

Claim. *Suppose $g : \mathcal{A} \hookrightarrow \mathcal{B} \in \mathcal{K}$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{g(a_1), \ldots, g(a_n), b_1, \ldots, b_m\}$. Then*

$$\mathcal{M} \vDash \theta_{\mathcal{A},\mathcal{B},g,\overline{a},\overline{b}} := \forall \overline{x}.\ (\phi_{\mathcal{A},\overline{a}}(\overline{x}) \to \exists \overline{y}.\ \phi_{\mathcal{B},g(\overline{a})\overline{b}}(\overline{x}, \overline{y})).$$

Proof. $\mathcal{B} \in \mathcal{K} = \text{age}(\mathcal{M})$, so let $\overline{c}\overline{d} \in \mathcal{M}^{<\omega}$ with $\mathcal{M} \vDash \phi_{\mathcal{B},g(\overline{a})\overline{b}}(\overline{c},\overline{d})$. Then $\mathcal{M} \vDash \exists \overline{y}.\ \phi_{\mathcal{B},g(\overline{a})\overline{b}}(\overline{c},\overline{y})$. Now $\mathcal{M} \vDash \phi_{\mathcal{A},\overline{a}}(\overline{c})$, and by QE $\exists \overline{y}.\ \phi_{\mathcal{B},g(\overline{a})\overline{b}}(\overline{x},\overline{y})$ is equivalent to a qf formula, so it is implied by $\phi_{\mathcal{A},\overline{a}}(\overline{x})$. □

We show that $\mathcal{K}$ is a Fraïssé class.
(HP) holds by Lemma 8.34.
Let $g_i : \mathcal{A} \hookrightarrow \mathcal{B}_i$ be as in (AP). Composing with an isomorphism, we may assume $\mathcal{A} \leq \mathcal{M}$.
Say $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B}_i = \{g_i(a_1), \ldots, g_i(a_n), b_1^i, \ldots, b_{m_i}^i\}$.
Then by the Claim, $\mathcal{M} \vDash \theta_{\mathcal{A},\mathcal{B}_i,g_i,\overline{a},\overline{b}^i}$,
so $\mathcal{M} \vDash \bigwedge_{i=1,2} \exists \overline{y}.\ \phi_{\mathcal{B}_i,g_i(\overline{a})\overline{b}^i}(\overline{a},\overline{y}))$.
Let $\overline{c}^1$ and $\overline{c}^2$ be witnesses; then

$$f_i(g_i(\overline{a})) := \overline{a};\ f_i(\overline{b}^i) := \overline{c}^i$$

defines embeddings $f_i : \mathcal{B}_i \hookrightarrow \mathcal{M}$ with $f_1 \circ g_1 = f_2 \circ g_2$.
So we obtain (AP) by setting $\mathcal{C} := \langle f_1(\mathcal{B}_1) \cup f_2(\mathcal{B}_2)\rangle^{\mathcal{M}}$.

(i) Suppose $\mathcal{K}$ is a Fraïssé class.

For $n \in \omega$, let $\mathcal{K}_n \subseteq_{\text{fin}} \mathcal{K}$ be such that any $\mathcal{A} \in \mathcal{K}$ with $|\mathcal{A}| \leq n$ is isomorphic to some $\mathcal{A}' \in \mathcal{K}_n$, and let

$$\chi_n(x_1, \ldots, x_n) := \forall \overline{x}.\ \bigvee\{\phi_{\mathcal{A},\overline{a}}(\overline{x}) : \mathcal{A} \in \mathcal{K}_n;\ \mathcal{A} = \{a_1, \ldots, a_n\}\}).$$

Let $\Theta_{\mathcal{K}}$ be the class of triples $(\mathcal{A},\mathcal{B},\overline{a},\overline{b})$ such that $\mathcal{A} \leq \mathcal{B} \in \mathcal{K}$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$.
Let $\theta_{\mathcal{A},\mathcal{B},\overline{a},\overline{b}} := \theta_{\mathcal{A},\mathcal{B},\text{id}_{\mathcal{A}},\overline{a},\overline{b}}$.
Let

$$T_{\mathcal{K}} := \{\theta_{\mathcal{A},\mathcal{B},\overline{a},\overline{b}} : (\mathcal{A},\mathcal{B},\overline{a},\overline{b}) \in \Theta_{\mathcal{K}}\} \cup \{\chi_n : n \in \omega\}.$$

Any model of $T_{\mathcal{K}}$ has age $\mathcal{K}$, since it satisfies the and $\theta_{\emptyset,\mathcal{B},\emptyset,\overline{b}}$. By the above Claim, if $\mathcal{M}$ has QE and $\text{age}(\mathcal{M}) = \mathcal{K}$, then $\mathcal{M} \vDash T_{\mathcal{K}}$.

It remains to see that $T_{\mathcal{K}}$ has QE and infinite models; indeed, it is then complete (by Corollary 5.16), and the claimed uniqueness follows.

We verify QE via Theorem 5.15(iii).
Suppose $\mathcal{A} \leq \mathcal{M}_1, \mathcal{M}_2 \vDash T_{\mathcal{K}}$ is a finite common substructure of models of $T_{\mathcal{K}}$.
Say $\mathcal{A} = \{a_1, \ldots, a_n\}$.
Suppose $\exists y.\ \psi(\overline{x}, y)$ is a primitive existential $\mathcal{L}$-formula
and $\mathcal{M}_1 \vDash \psi(\overline{a}, b)$ for some $b \in \mathcal{M}_1$.
Let $\mathcal{B} = \langle \overline{a}b\rangle^{\mathcal{M}_1}$.
Since $\mathcal{M}_1 \vDash \chi_{n+1}$, we have $\mathcal{A}, \mathcal{B} \in \mathcal{K}$.
Then $\psi(\overline{x}, y)$ is implied by $\phi_{\mathcal{B},\overline{a}b}(\overline{x}, y)$.
So since $\mathcal{M}_2 \vDash \theta_{\mathcal{A},\mathcal{B},\overline{a},b}$ and $\mathcal{M}_2 \vDash \phi_{\mathcal{A},\overline{a}}(\overline{a})$,
also $\mathcal{M}_2 \vDash \exists y.\ \psi(\overline{a}, y)$, as required.

Finally, we construct an infinite model of $T_{\mathcal{K}}$.

Claim. *Let* $(\mathcal{A}, \mathcal{B}, \overline{a}, \overline{b}) \in \Theta$*, and suppose* $\mathcal{A} \leq \mathcal{D} \in \mathcal{K}$*. Then there exists* $\mathcal{D}' \in \mathcal{K}$ *such that* $\mathcal{D} \leq \mathcal{D}'$*, and there is an embedding* $f : \mathcal{B} \hookrightarrow \mathcal{D}'$ *such that* $f\restriction_{\mathcal{A}} = \mathrm{id}\restriction_{\mathcal{A}}$*.*

Proof. Immediate consequence of (AP) (and Lemma 2.2). □

Let $\xi : \omega \times \omega \to \omega$ be a bijection such that $\xi(i, j) \geq i$.

We construct a countable chain $\mathcal{D}_0 \leq \mathcal{D}_1 \leq \ldots$ with $\mathcal{D}_k \in \mathcal{K}$.
We simultaneously construct a sequence $(\mathcal{A}_i, \overline{a}_i)_{i\in\omega}$,
and for $k \in \omega$ some $m_k > k$,
such that if $\overline{a}$ is a tuple of distinct elements of $\mathcal{D}_k$,
then $\overline{a} = \overline{a}_i$ for some $i < m_k$
and for all $i < m_k$, we have $\mathcal{A}_i = \langle \overline{a}_i \rangle^{\mathcal{D}_k} \leq \mathcal{D}_k$.

For each $i \in \omega$ we take (using Remark 8.32) a sequence $(\mathcal{B}^i_j, \overline{b}^i_j)_{j\in\omega}$ with $(\mathcal{A}_i, \mathcal{B}^i_j, \overline{a}_i\overline{b}^i_j) \in \Theta$,
such that if $(\mathcal{A}_i, \mathcal{B}, \overline{a}_i\overline{b}) \in \Theta$, then

$$\overline{b} \mapsto \overline{b}^i_j;\ \overline{a}_i \mapsto \overline{a}_i$$

defines an isomorphism $\mathcal{B} \xrightarrow{\cong} \mathcal{B}^i_j$.
Let $\mathcal{D}_0 := \emptyset$ (and $\mathcal{A}_0 := \emptyset$ and $m_0 := 1$).
Given $\mathcal{D}_k$, say $k = \xi(i, j)$,
we have $i \leq \xi(i, j) = k < m_k$, so $\mathcal{A}_i \leq \mathcal{D}_k$.
By the Claim we find $\mathcal{D}_k \leq \mathcal{D}_{k+1} \in \mathcal{K}$,
such that $\mathcal{B}^i_j$ embeds in $\mathcal{D}_{k+1}$ over $\mathcal{A}_i$;
i.e. $\mathcal{D}_{k+1} \vDash \exists y.\ \phi_{\mathcal{B}^i_j,\overline{a}_i\overline{b}}(\overline{a}_i, y)$.
We may extend $(\mathcal{A}_i, \overline{a}_i)_i$ to include the substructures of $\mathcal{D}_{k+1}$, and set $m_{k+1} > m_k$ correspondingly.

So then $\mathcal{M}_{\mathcal{K}} := \bigcup_{k\in\omega} \mathcal{D}_k \vDash \theta_{\mathcal{A},\mathcal{B},\overline{a},\overline{b}}$ for all $(\mathcal{A}, \mathcal{B}, \overline{a}, \overline{b}) \in \Theta$.

Also $\mathcal{M} \vDash \chi_n$, since $\mathcal{D}_k \in \mathcal{K}$. So $\mathcal{M} \vDash T_{\mathcal{K}}$.

Finally, $\mathcal{M}$ is infinite, since $\mathcal{K}$ contains unboundedly large finite structures.

□

Remark 8.41. Analogues of Theorem 8.39 exist for arbitrary countable languages, with finitely generated substructures in place of finite substructures.

In non-relational languages, one must assume (JEP) as well as (AP), or equivalently assume (AP) and: $\langle\emptyset\rangle^{\mathcal{A}} \cong \langle\emptyset\rangle^{\mathcal{B}}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$.

In general $\aleph_0$-categoricity and QE are weakened to *ultrahomogeneity* of the Fraïssé limit: any partial automorphism with f.g. domain extends to an automorphism. The proof is essentially the same.

There are further generalisations, including "Hrushovski-Fraïssé constructions", in which (HP) is relaxed.

9 Cofinality and regularity

Definition 9.1. Let X be a linearly ordered set.

- A subset $A \subseteq X$ is **cofinal** if $\forall b \in X. \exists a \in A. a \geq b$.
- The **cofinality** of X, written $\mathrm{cof}(X)$, is the minimal cardinality of a cofinal subset.
- An infinite cardinal λ is **regular** if $\mathrm{cof}(\lambda) = \lambda$, otherwise it is **singular**.

Lemma 9.2. *Suppose λ is a regular cardinal and $(A_\alpha)_{\alpha\in\lambda}$ is an increasing chain of sets. Suppose $B \subseteq \bigcup_{\alpha\in\lambda} A_\alpha$ with $|B| < \lambda$. Then there exists an ordinal $\alpha \in \lambda$ such that $B \subseteq A_\alpha$.*

Proof. Otherwise, $\{\inf\{\alpha : b \in A_\alpha\} : b \in B\}$ is cofinal, so $\mathrm{cof}(\lambda) \leq |B| < \lambda$, contradicting regularity. □

Proposition 9.3. *Infinite successor cardinals are regular.*

Proof. Let κ be an infinite cardinal. Suppose $A \subseteq \kappa^+$ with $|A| < \kappa^+$. Then $|A| \leq \kappa$ and every $\alpha \in A$ has cardinality $|\alpha| \leq \kappa$, so $|\bigcup_{\alpha\in A} \alpha| \leq \kappa \cdot \kappa = \kappa$. But then $\sup_{\alpha\in A} \alpha = \bigcup_{\alpha\in A} \alpha < \kappa^+$, so A is not cofinal in κ^+.

So $\mathrm{cof}(\kappa^+) = \kappa^+$. □

10 Saturation

10.1 Existence

Definition 10.1. For $A \subseteq B \subseteq \mathcal{M}$, define

$$\restriction_A : S(B) \twoheadrightarrow S(A);\ \mathrm{tp}(a/B) \mapsto \mathrm{tp}(a/A)$$

(for $a \in \mathcal{M}' \succeq \mathcal{M}$).

Proposition 10.2. *Let T be a theory with infinite models.*

Let $\kappa \leq \lambda \geq |T|$ be infinite cardinals such that for any $\mathcal{M} \vDash T$ with $|\mathcal{M}| \leq \lambda$, there exists a subset $\Xi_{\mathcal{M}} \subseteq S_1(\mathcal{M})$ with $|\Xi_{\mathcal{M}}| \leq \lambda$ such that for any $A \subseteq \mathcal{M}$ with $|A| \leq \kappa$ we have $\restriction_A (\Xi_{\mathcal{M}}) = S_1(A)$.

Then T has a κ^+-saturated model of cardinality $\lambda + \kappa^+$.

Corollary 10.3. *Let T be a theory with infinite models.*

For any infinite cardinal $\kappa \geq |T|$, T has a κ^+-saturated model of cardinality 2^κ.

In particular, T has a μ-saturated model for any infinite cardinal μ.

Proof. If $\mathcal{M} \vDash T$ with $|\mathcal{M}| \leq \lambda$ und $A \subseteq \mathcal{M}$ with $|A| \leq \kappa$, then $|S_1(A)| \leq 2^{|T_A|} = 2^{|T|+|A|} \leq 2^\kappa$. Meanwhile

$$|A \subseteq \mathcal{M} : |A| \leq \kappa| \leq |\mathcal{M}|^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} = 2^\kappa.$$

Then

$$\left| \bigcup_{A\subseteq\mathcal{M};\ |A|\leq\kappa} S_1(A) \right| \leq 2^\kappa \cdot 2^\kappa = 2^\kappa = \lambda.$$

So we can find $|\Xi_{\mathcal{M}}| \leq \lambda$ as required.

Finally, $2^\kappa + \kappa^+ = 2^\kappa$. □

Bemerkung 10.4. It follows that if $\lambda^+ = 2^\lambda$, e.g. if we assume GCH, then T has a saturated model of cardinality λ^+. But (under a "reasonable" large cardinal assumption) it is consistent with ZFC that no infinite λ exists with $2^\lambda = \lambda^+$.

Proof of Proposition 10.2. We build an elementary chain $(\mathcal{M}_\alpha)_{\alpha\in\kappa^+}$ of models of cardinality λ.

Let (by Löwenheim-Skolem) $\mathcal{M}_0 \vDash T$ with $|\mathcal{M}_0| = \lambda$. For $\eta \in \kappa^+$ a limit ordinal, set $\mathcal{M}_\eta := \bigcup_{\alpha<\eta} \mathcal{M}_\alpha$. By Lemma 8.1, $\mathcal{M}_\eta \succeq \mathcal{M}_\alpha$ for $\alpha < \eta$. Since $|\eta| \leq \kappa \leq \lambda$, we have $|\mathcal{M}_\eta| = \lambda$.

Given $\mathcal{M}_\alpha$, By Lemma 7.16 and Löwenheim-Skolem there is $\mathcal{M}_{\alpha+1} \succeq \mathcal{M}_\alpha$ with $|\mathcal{M}_{\alpha+1}| = \lambda$ which realises every type in $\Xi_{\mathcal{M}_\alpha}$, and hence every type in $S_1(A)$ for $A \subseteq \mathcal{M}_\alpha$ with $|A| \leq \kappa$.

Now let $\mathcal{M} := \bigcup_{\alpha\in\kappa^+} \mathcal{M}_\alpha$. Then $\lambda \leq |\mathcal{M}| \leq \lambda \cdot \kappa^+ = \lambda + \kappa^+$.

If $A \subseteq \mathcal{M}$ with $|A| < \kappa^+$, then since κ^+ is a successor cardinal and hence is regular, already $A \subseteq \mathcal{M}_\alpha$ for some $\alpha \in \kappa^+$. So any $p \in S(A)$ is realised in $\mathcal{M}_{\alpha+1} \preceq \mathcal{M}$ and hence in $\mathcal{M}$. Hence $\mathcal{M}$ is κ^+-saturated.

It follows that $|\mathcal{M}| \geq \kappa^+$, so $|\mathcal{M}| = \lambda + \kappa^+$. □

10.2 Stability

Definition 10.5. Let κ be an infinite cardinal.

A structure $\mathcal{M}$ is **κ-stable** if $|S_1(A)| \leq \kappa$ for all $A \subseteq \mathcal{M}$ with $|A| \leq \kappa$.

A theory T is **κ-stable** if it has infinite models and every $\mathcal{M} \vDash T$ is κ-stable.

A model or theory is **stable** if it is κ-stable for some infinite cardinal κ.

We often write ω-stable for $\aleph_0$-stable.

Example 10.6. T_∞ and $T_{(\mathbb{Q};+)}$ are ω-stable.

DLO is not stable.

Corollary 10.7 (of Proposition 10.2)**.** *Let T be λ-stable, where $\lambda \geq |T|$.*

(i) T has a saturated model of cardinality λ^+.

(ii) Let $\kappa < \lambda$. Then T has a κ^+-saturated model of cardinality λ.

Proof. Let $\kappa \leq \lambda$. By Proposition 10.2, with $\Xi_{\mathcal{M}} := S_1(\mathcal{M})$, there is a κ^+-saturated model of cardinality $\lambda + \kappa^+$.

Then (i) follows by taking $\kappa := \lambda$, and (ii) with $\kappa < \lambda$. □

10.3 QE and saturation

Notation 10.8.

- Let $\mathcal{M}$ an $\mathcal{L}$-structure. For $\bar{b} \in \mathcal{M}^{<\omega}$,

$$\mathrm{qftp}^{\mathcal{M}}(\bar{b}/A) := \{\phi(\bar{x}) : \phi \text{ qf } \mathcal{L}(A)\text{-formula; } \mathcal{M} \vDash \phi(\bar{b})\}.$$

- Let $\mathcal{M}_1, \mathcal{M}_2$ be $\mathcal{L}$-structures. For $\bar{b}_i \in \mathcal{M}_i^{<\omega}$ we write

$$\bar{b}_1 \equiv \bar{b}_2$$

when $\mathrm{tp}^{\mathcal{M}_1}(\bar{b}_1) = \mathrm{tp}^{\mathcal{M}_2}(\bar{b}_2)$, and we write

$$\bar{b}_1 \equiv^{\mathrm{qf}} \bar{b}_2$$

when $\mathrm{qftp}^{\mathcal{M}_1}(\bar{b}_1) = \mathrm{qftp}^{\mathcal{M}_2}(\bar{b}_2)$.

- If $A \subseteq \mathcal{M}_1, \mathcal{M}_2$ is a common subset, we write

$$\bar{b}_1 \equiv_A \bar{b}_2$$

when $\mathrm{tp}^{\mathcal{M}_1}(\bar{b}_1/A) = \mathrm{tp}^{\mathcal{M}_2}(\bar{b}_2/A)$; ähnlich für $\bar{b}_1 \equiv_A^{\mathrm{qf}} \bar{b}_2$.

Remark 10.9.

- $\bar{b}_1 \equiv \bar{b}_2$ iff $\bar{b}_1 \mapsto \bar{b}_2$ defines a partial elementary map $\mathcal{M}_1 \dashrightarrow \mathcal{M}_2$.
- $\bar{b}_1 \equiv^{\mathrm{qf}} \bar{b}_2$ iff $\bar{b}_1 \mapsto \bar{b}_2$ defines a partial isomorphism $\mathcal{M}_1 \dashrightarrow \mathcal{M}_2$.

Proposition 10.10. *Let T be a theory. Let κ be an infinite cardinal. TFAE:*

(i) T has QE;

(ii) Let $\mathcal{M}_1, \mathcal{M}_2 \vDash T$ be κ-saturated models, $\bar{a}_i \in \mathcal{M}_i^{<\omega}$ with $\bar{a}_1 \equiv^{\mathrm{qf}} \bar{a}_2$, and $b_1 \in \mathcal{M}_1$.

Then there exists $b_2 \in \mathcal{M}_2$ with $\bar{a}_1 b_1 \equiv^{\mathrm{qf}} \bar{a}_2 b_2$.

Proof.

(i) ⇒ (ii) Let $\theta : \bar{a}_1 \mapsto \bar{a}_2$. This is a partial isomorphism, so by QE it is partial elementary. Hence (by Lemma 7.20) $\mathrm{tp}(b_1/\bar{a}_1)^\theta$ a type in $S^{\mathcal{M}_2}(\bar{a}_2)$. So by $\aleph_0$-saturation of $\mathcal{M}_2$, it is realised in $\mathcal{M}_2$, say by $b_2 \in \mathcal{M}_2$. Then $\bar{a}_1 b_1 \equiv \bar{a}_2 b_2$.

(ii) ⇒ (i) We verify QE via Theorem 5.15(iii).

Let $\mathcal{A} = \langle \bar{a} \rangle \leq \mathcal{M}_1, \mathcal{M}_2 \vDash T$ be a common finitely generated substructure of two models of T, and $\exists y.\, \psi(\bar{x}, y)$ a primitive existential formula, and $b_1 \in \mathcal{M}_1$ with $\mathcal{M}_1 \vDash \psi(\bar{a}, b_1)$.

By Corollary 10.3, we can find κ-saturated elementary extensions $\mathcal{M}_i' \succeq \mathcal{M}_i$. By (ii), there is $b_2 \in \mathcal{M}_2'$ with $\bar{a} b_1 \equiv^{\mathrm{qf}} \bar{a} b_2$, hence $\mathcal{M}_2 \vDash \exists y.\, \psi(\bar{a}, y)$, as required.

□

Example 10.11. An *ordered $\mathbb{Q}$-vector space* is a $\mathbb{Q}$-vector space with a linear ordering $<$ such that

$$\forall x, y, z.\, (x < y \to x + z < y + z).$$

Let $T_{\mathrm{o}\mathbb{Q}-\mathrm{VS}}$ be the $\{0, +, (q\cdot)_{q \in \mathbb{Q}}, <\}$-theory consisting of this axiom along with axioms for non-trivial $\mathbb{Q}$-vector spaces.

Claim. *$T_{\mathrm{o}\mathbb{Q}-\mathrm{VS}}$ has QE and is complete.*

Proof. Completeness follows from QE via Corollary 5.16, since $\langle \emptyset \rangle^{\mathcal{M}} = \{0\}$ for any $\mathcal{M} \vDash T_{\mathrm{o}\mathbb{Q}-\mathrm{VS}}$.

Let $\mathcal{M}_1, \mathcal{M}_2 \vDash T_{\mathrm{o}\mathbb{Q}-\mathrm{VS}}$ be $\aleph_1$-saturated. Let $\bar{a}_i \in \mathcal{M}_i^{<\omega}$ with $\bar{a}_1 \equiv^{\mathrm{qf}} \bar{a}_2$. Let $b_1 \in \mathcal{M}_1$.

A qf-type $\mathrm{qftp}(b/\bar{a})$ is determined by the formulas of form $x = \sum q_i \cdot a_i$ or $x < \sum q_i \cdot a_i$ or $x > \sum q_i \cdot a_i$.

Let $\mathcal{A}_i := \langle \overline{a}_i \rangle^{\mathcal{M}_i}$ Then $\overline{a}_1 \mapsto \overline{a}_2$ generates an order isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$. Note that the reduct $\mathcal{M}_i \restriction_<$ of $\mathcal{M}_i$ to an order is a model of DLO. So by QE for DLO, θ is a partial elementary map of linear orders $\theta : \mathcal{M}_1 \restriction_< \dashrightarrow \mathcal{M}_2 \restriction_<$, so $(\mathrm{tp}^{\mathcal{M}_1\restriction_<}(b_1/\mathcal{A}_1))^\theta$ is a type in $S_1^{\mathcal{M}_2\restriction_<}(\mathcal{A}_2)$. By $\aleph_1$-saturation of $\mathcal{M}_2$, say $b_2 \in \mathcal{M}_2$ realises this type.

Then $\overline{a}_1 b_1 \equiv^{\mathrm{qf}} \overline{a}_2 b_2$. □

One can deduce that the $\{0, +, <\}$-theory DOAG of linearly ordered divisible abelian groups (e.g. $(\mathbb{Q}; 0, +, <)$) is complete and has QE.

10.4 Bonus: Monsters

Definition 10.12. A structure $\mathcal{M}$ is **strongly κ-homogeneous**, if any partial elementary map $\theta : \mathcal{M} \dashrightarrow \mathcal{M}$ with $|\mathrm{dom}\,\theta| < \kappa$ extends to an automorphism of $\mathcal{M}$.

$\mathcal{M}$ is **strongly κ-saturated** if it is κ-saturated and strongly κ-homogeneous.

Proposition 10.13. *If a structure $\mathcal{M}$ is saturated, it is strongly $|\mathcal{M}|$-saturated.*

Proof. As in Exercise 7.2(a).

Briefly: If $\theta : \mathcal{M} \dashrightarrow \mathcal{M}$ is partial elementary and $|\mathrm{dom}\,\theta| < |\mathcal{M}|$, then θ induces an elementary equivalence between the expansions by constants $\mathcal{M}_{\mathrm{dom}\,\theta} \equiv \mathcal{M}_{\mathrm{im}\,\theta}$; but these structures are also saturated, so by Lemma 7.21 they are isomorphic. An isomorphism between these structures is an automorphism of $\mathcal{M}$ extending θ. □

Theorem 10.14. *Let T be a theory with infinite models. Let λ be an infinite cardinal. Then T has a strongly λ-saturated model.*

Proof. Increasing λ, we may assume $\lambda \geq |T|$ and λ is regular.

Let $\mathcal{M}_0 \vDash T$ with $|\mathcal{M}_0| > \lambda$. By Corollary 10.3 we may extend to an elementary chain $(\mathcal{M}_\alpha)_{\alpha\in\lambda}$ such that each $\mathcal{M}_{\alpha+1}$ is $|\mathcal{M}_\alpha|^+$-saturated, and for η a limit ordinal $\mathcal{M}_\eta = \bigcup_{\alpha<\eta} \mathcal{M}_\alpha$.

Let $\mathcal{M} := \bigcup_{\alpha\in\lambda} \mathcal{M}_\alpha$. Then $\mathcal{M}$ is λ-saturated since each $\mathcal{M}_{\alpha+1}$ is and λ is regular.

Now let $\theta : \mathcal{M} \dashrightarrow \mathcal{M}$ be partial elementary with $|\mathrm{dom}\,\theta| < \lambda$. Since λ is regular, for some $\alpha \in \lambda$ we have $\theta : \mathcal{M}_\alpha \dashrightarrow \mathcal{M}_\alpha$, i.e. $\mathrm{dom}\,\theta \cup \mathrm{im}\,\theta \subseteq \mathcal{M}_\alpha$. Let $\theta_\alpha := \theta$.

Now $\mathcal{M}_{\alpha+1}$ is $|\mathcal{M}_\alpha|$-saturated, so as in the proof of Lemma 7.23, θ_α extends to a partial elementary map $\theta_{\alpha+1} : M_{\alpha+1} \dashrightarrow M_{\alpha+1}$ with $M_\alpha \subseteq \mathrm{dom}\,\theta_{\alpha+1}$ and also $M_\alpha \subseteq \mathrm{im}\,\theta_{\alpha+1}$.

We obtain in this way, taking unions at limit ordinals, an increasing chain $(\theta_\beta)_{\alpha\leq\beta\in\lambda}$ of partial elementary maps $\theta_\beta : \mathcal{M}_\beta \dashrightarrow \mathcal{M}_\beta$ with each $\mathcal{M}_\beta \subseteq \mathrm{dom}\,\theta_{\beta+1} \cap \mathrm{im}\,\theta_{\beta+1}$.

Then $\sigma := \bigcup_\beta \theta_\beta \in \mathrm{Aut}(\mathcal{M})$ as required. □

11 ω-stability

Let T be a complete $\mathcal{L}$-theory with infinite models.

Lemma 11.1. *T is κ-stable iff for all $A \subseteq \mathcal{M} \vDash T$ with $|A| \leq \kappa$, we have $|S(A)| \leq \kappa$.*

In particular, T is ω-stable iff $\mathrm{Th}(\mathcal{M}_A)$ is small for all $A \subseteq \mathcal{M} \vDash T$ with $|A| = \aleph_0$.

Proof. We show by induction that $|S_n(A)| \leq \kappa$ for all n. For $n = 1$, this is the definition of κ-stability.

Suppose $|S_n(A)| \leq \kappa$.
Let $\mathcal{N} \succeq \mathcal{M}$ be κ^+-saturated.
Let $(\bar{b}_i \in \mathcal{N}^n)_{i\in\kappa}$ be such that $\{\mathrm{tp}(\bar{b}_i/A) : i \in \kappa\} = S_n(A)$.
Let $\bar{b}c \in \mathcal{N}^{n+1}$.
Then $\bar{b}c \equiv_A \bar{b}_i c'$ for some $i \in \kappa$ and $c' \in \mathcal{N}$.
For each $i \in \kappa$ there are at most $|S_1(A \cup \bar{b}_i)| \leq \kappa$ possibilities for $\mathrm{tp}(c'/\bar{b}_i)$.
So $|S_{n+1}(A)| \leq \kappa \cdot \kappa = \kappa$. □

Definition 11.2. T is **totally transcendental** if for any $\mathcal{M} \vDash T$, there is no binary tree $(\phi_s)_{s\in 2^{<\omega}}$ of formulas for $\mathrm{Th}(\mathcal{M}_{\mathcal{M}})$.

Theorem 11.3.

(a) If T is ω-stable then T is totally transcendental.

(b) If T is totally transcendental then T is κ-stable for all $\kappa \geq |T|$.

Corollary 11.4. *Suppose $|T| = \aleph_0$. TFAE:*

(i) T is ω-stable;

(ii) T is totally transcendental.

(iii) T is κ-stable for all $\kappa \geq \aleph_0$.

Proof of Theorem 11.3.

(a) Suppose $\mathcal{M} \vDash T$ and $\mathrm{Th}(\mathcal{M}_{\mathcal{M}})$ has a binary tree of formulas $(\phi_s)_{s\in 2^{<\omega}}$. Since $|2^{<\omega}| = \aleph_0$, there exists $A \subseteq \mathcal{M}$ with $|A| \leq \aleph_0$ such that each ϕ_s is a $\mathcal{L}(A)$-formula, and so $(\phi_s)_s$ is also a binary tree of formulas for $\mathrm{Th}(\mathcal{M}_A)$. Then by Lemma 8.20, $|S^{\mathcal{M}}(A)| = |S(\mathrm{Th}(\mathcal{M}_A))| = 2^{\aleph_0} > \aleph_0$, so T is not ω-stable.

(b) (cf. Exercise 6.1(a))

Let $A \subseteq \mathcal{M} \vDash T$ with $|A| \leq \kappa$. For $\phi(x)$ an $\mathcal{L}(A)$-formula, define $[\phi(x)] := \{p \in S_1(A) : \phi(x) \in p(x)\}$.

Claim. *If $|[\phi(x)]| > \kappa$, then there is an $\mathcal{L}(A)$-formula $\psi(x)$, such that $|[\phi(x) \wedge \psi(x)]| > \kappa < |[\phi(x) \wedge \neg\psi(x)]|$.*

Proof. Say $p \in S_1(A)$ is *small* if $\exists \psi \in p.\ |[\psi]| \leq \kappa$. There are at most $|\mathrm{Th}(\mathcal{M}_A)| \cdot \kappa \leq \kappa \cdot \kappa = \kappa$ small types in $S_1(A)$. Hence there are distinct non-small types $p_1, p_2 \in [\phi(x)]$. Let $\psi(x) \in p_1(x) \backslash p_2(x)$; then $\phi(x) \wedge \psi(x) \in p_1(x)$ and $\phi(x) \wedge \neg\psi(x) \in p_2(x)$, so ψ is as required. □

Now if $|[x \doteq x]| = |S_1(A)| > \kappa$, we build a binary tree with $|[\phi_s]| > \kappa$ for all $s \in 2^{<\omega}$: set $\phi_\emptyset := x \doteq x$; given ϕ_s, let $\phi_{s0} := \phi_s \wedge \psi$ and $\phi_{s1} := \phi_s \wedge \neg\psi$, where ψ is as in the claim.

□

11.1 Constructibility

Notation 11.5. If $A \subseteq \mathcal{M} \vDash T$ and $\mathcal{N} \vDash T$, we call a map $\theta : A \to \mathcal{N}$ *partial elementary* (*p.e.*) and write $\theta : A \xrightarrow{\equiv} \mathcal{N}$, if the corresponding partial map $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ is p.e..

Remark 11.6. $\theta : A \to \mathcal{N} \vDash T$ is p.e. iff $\mathcal{N}_{A,\theta} \equiv \mathcal{M}_A$, where $\mathcal{N}_{A,\theta}$ is the $\mathcal{L}(A)$-structure defined by $a^{\mathcal{N}_\theta(A)} := \theta(a)$.

Definition 11.7. Let $A \subseteq \mathcal{M} \vDash T$. Then $\mathcal{M}$ is **prime over** A if every p.e. map $A \xrightarrow{\equiv} \mathcal{N} \vDash T$ extends to an elementary embedding $\mathcal{M} \xhookrightarrow{\preceq} \mathcal{N}$.

Remark 11.8. Let $A \subseteq \mathcal{M} \vDash T$. Applying Remark 11.6, we find that $\mathcal{M}$ is prime over A iff $\mathcal{M}_A$ is a prime model of $\mathrm{Th}(\mathcal{M}_A)$.

Remark 11.9. By Proposition 8.23 and Proposition 8.15, a countable ω-stable theory has a unique prime model over any *countable* set $A \subseteq \mathcal{M} \vDash T$.

In fact this holds also for uncountable A. We prove existence in this section. Uniqueness requires further work.

Definition 11.10. If $A \subseteq B \subseteq \mathcal{M} \vDash T$, then B is **constructible** over A if B can be enumerated as $(b_\alpha)_{\alpha<\gamma}$ for some ordinal γ, such that for all $\alpha < \gamma$ the type $\mathrm{tp}(b_\alpha/A \cup b_{<\alpha})$ is isolated, where $b_{<\alpha} := \{b_\beta : \beta < \alpha\}$.

Lemma 11.11. *If $\mathcal{M} \vDash T$ is constructible over $A \subseteq \mathcal{M}$, then $\mathcal{M}$ is prime over A.*

Proof. Suppose $\theta_0 : A \xrightarrow{\equiv} \mathcal{N} \vDash T$. We construct a chain of p.e. maps $\theta_\alpha : A \cup b_{<\alpha} \xrightarrow{\equiv} \mathcal{N} \vDash T$ for $\alpha \leq \gamma$. Then $\theta_\gamma : \mathcal{M} \xhookrightarrow{\preceq} \mathcal{N}$ is an elementary embedding.

For $\eta \in \gamma$ a limit ordinal, $\theta_\eta := \bigcup_{\alpha\in\eta} \theta_\alpha$.

Given $\alpha < \gamma$ and θ_α, let $\theta_{\alpha+1}(b_\alpha)$ be a realisation in $\mathcal{N}$ of the isolated type $\mathrm{tp}(b_\alpha/A \cup b_{<\alpha})^{\theta_\alpha}$. □

Lemma 11.12. *If T is totally transcendental and $A \subseteq \mathcal{M} \vDash T$, then the isolated types are dense in $S(A)$.*

Proof. Immediate consequence of Lemma 8.22(i). □

Theorem 11.13. *If T is totally transcendental and $A \subseteq \mathcal{N} \vDash T$, then T has a constructible prime model $A \subseteq \mathcal{M} \preceq \mathcal{N}$ over A.*

Proof. A *construction sequence* over A is a sequence $(b_\alpha)_{\alpha<\gamma}$with each $\mathrm{tp}(b_\alpha/A\cup b_{<\alpha})$ isolated.

By Zorn, there is a maximal construction sequence $(b_\alpha)_{\alpha<\gamma}$ in $\mathcal{N}$. Let $M := b_{\leq\gamma} \subseteq \mathcal{N}$. We show by the Tarski Test that M is the domain of an elementary substructure $\mathcal{M} \preceq \mathcal{N}$, which is constructible over A and hence by Lemma 11.11 prime over A, as required.

So let $\phi(x)$ be an $\mathcal{L}(B)$-formula such that $\mathcal{N} \vDash \exists y.\phi(x)$. By Lemma 11.12, let $p(x) \in S(B)$ be isolated with $p(x) \vdash \phi(x)$. Let $c \in \mathcal{N}$ realise p. If $c \notin B$ then we could extend the construction sequence by setting $b_\gamma := c$, contradicting maximality. So $c \in B$, and $\mathcal{N} \vDash \phi(c)$, as required. □

Lemma 11.14. *Let $A \subseteq B \subseteq C \subseteq \mathcal{M} \vDash T$. Suppose C is constructible over B and B is constructible over A. Then C is constructible over A.*

Proof. Exercise. □

Definition. If $A \subseteq B \subseteq \mathcal{M} \models T$, then B **atomic over** A if $\mathrm{tp}(\overline{b}/A)$ is isolated for all $\overline{b} \in \mathcal{B}^{<\omega}$.

Lemma 11.15. *If $\mathcal{M} \models T$ is constructible over $A \subseteq \mathcal{M}$, then $\mathcal{M}$ is atomic over A.*

Proof. Say $\mathcal{M} = (b_\alpha)_{\alpha<\gamma}$ with $\mathrm{tp}(b_\alpha/A \cup b_{<\alpha})$ isolated. Let $\overline{b} \in \mathcal{M}^{<\omega}$. Permuting, we may assume $\overline{b} = b_\alpha\overline{b}'$, where $\overline{b}' \in b_{<\alpha}^{<\omega}$. Inductively, we may assume that $b_{<\alpha}$ is atomic over A.

Now $\mathrm{tp}(\overline{b}_\alpha/A \cup b_{<\alpha})$ isolated by some $\mathcal{L}(A \cup \overline{c})$-formula, where $\overline{c} \in b_{<\alpha}^{<\omega}$. Then $\mathrm{tp}(\overline{b}_\alpha/A \cup \overline{b}\overline{c})$ isolated. By atomicity, $\mathrm{tp}(\overline{b}'\overline{c}/A)$ is isolated. So by Lemma 8.12 (for $\mathcal{M}_A$) $\mathrm{tp}(\overline{b}_\alpha\overline{b}'\overline{c}/A)$ and hence $\mathrm{tp}(\overline{b}_\alpha\overline{b}'/A)$ are isolated. □

Fact. *The converse holds for countable $\mathcal{M}$: If $A \subseteq \mathcal{M} \models T$ and $\mathcal{M}$ is countable, then as in Lemma 8.13 if $\mathcal{M}$ is atomic over A then it is constructible and hence prime over A.*

But for uncountable $\mathcal{M}$, atomicity over A does not imply primeness over A, even if T is ω-stable and $|A| = |\mathcal{M}|$.

12 Strong minimality

Let T be a complete $\mathcal{L}$-theory with infinite models.

12.1 Algebraicity

Notation 12.1. Some abbreviations:

$$\overline{x} \doteq \overline{y} := \bigwedge_i x_i \doteq y_i$$

$$\exists^{\geq n}\overline{x}.\ \phi(\overline{x}) := \exists \overline{x}_1, \ldots, \overline{x}_n.\ (\bigwedge_i \phi(\overline{x}_i) \wedge \bigwedge_{i<j} \overline{x}_i \not\doteq \overline{x}_j)$$

$$\exists^{\leq n}\overline{x}.\ \phi(\overline{x}) := \neg\exists^{\geq n+1}\overline{x}.\ \phi(\overline{x})$$

$$\exists^{=n}\overline{x}.\ \phi(\overline{x}) := (\exists^{\geq n}\overline{x}.\ \phi(\overline{x}) \wedge \exists^{\leq n}\overline{x}.\ \phi(\overline{x})).$$

Definition 12.2. Let $\mathcal{M} \models T$.

- An $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$ is **algebraic** if $|\phi(\mathcal{M})| < \aleph_0$.
- A tuple $\overline{b} \in \mathcal{M}^{<\omega}$ is **algebraic over** a subset $A \subseteq \mathcal{M}$, and $\mathrm{tp}(\overline{b}/A)$ is an **algebraic type**, if $\mathrm{tp}(\overline{b}/A)$ contains an algebraic formula.
- The **algebraic closure** of a subset $A \subseteq \mathcal{M}$ in $\mathcal{M}$ is

$$\mathrm{acl}^{\mathcal{M}}(A) := \{b \in \mathcal{M} : \mathrm{tp}(b/A) \text{ is algebraic}\}.$$

Lemma 12.3. *(i) Let $\phi(\overline{x}, \overline{y})$ be an $\mathcal{L}$-formula. Let $\mathcal{M} \models T$. For $\overline{a} \in \mathcal{M}^{|\overline{y}|}$, whether $\phi(\overline{x}, \overline{a})$ is algebraic depends only on $\mathrm{tp}^{\mathcal{M}}(\overline{a})$.*

(ii) Let $A \subseteq \mathcal{M} \preceq \mathcal{N} \models T$. Then $\mathrm{acl}^{\mathcal{M}}(A) = \mathrm{acl}^{\mathcal{N}}(A)$. We usually write just $\mathrm{acl}(A)$.

Proof. (i) $\phi(\overline{x}, \overline{a})$ is algebraic iff for some $n \in \omega$

$$\mathcal{M} \vDash \exists^{=n}\overline{x}.\ \phi(\overline{x}, \overline{a}).$$

(ii) Let $\phi(x)$ be an algebraic $\mathcal{L}(A)$-formula. Let $n := |\phi(\mathcal{N})|$. Then $\mathcal{M} \vDash \exists^{=n}x.\ \phi(x)$, so $\phi(\mathcal{M}) = \phi(\mathcal{N})$.

Then $\mathrm{acl}^{\mathcal{N}}(A) = \bigcup_{\phi \text{ alg. } \mathcal{L}(A)\text{-formula}} \phi(\mathcal{N}) = \bigcup_{\phi \text{ alg. } \mathcal{L}(A)\text{-formula}} \phi(\mathcal{M}) = \mathrm{acl}^{\mathcal{M}}(A)$.

□

Examples 12.4. In a k-vector space, $\mathrm{acl}(A) = \langle A \rangle_k$.

In an algebraically closed field, acl is field theoretic algebraic closure: for $A \subseteq K \vDash \mathrm{ACF}$, let $k = \mathbb{Q}(A) \leq K$ be the subfield generated by A; then

$$\mathrm{acl}(A) = \{b \in K : \exists f \in k[X].\ f(b) = 0\}.$$

Lemma 12.5. *(i) Any algebraic type is isolated by an algebraic formula.*

(ii) Let $A \subseteq \mathcal{M} \vDash T$. Then $\mathrm{acl}(A)$ *is constructible over A.*

Proof. (i) (Exercise 5.3) If $\phi(x) \in \mathrm{tp}(b/A)$ is algebraic with $|\phi(\mathcal{M})|$ minimal, then $\phi(x)$ is isolated.

(ii) Enumerate $\mathrm{acl}(A)$ as $(b_\alpha)_{\alpha\in\gamma}$. Then $\mathrm{tp}(\overline{b}_\alpha/Ab_{<\alpha})$ is algebraic since $\mathrm{tp}(\overline{b}_\alpha/A)$ is, so is isolated by (i).

□

Lemma 12.6. *Let $A \subseteq \mathcal{M} \vDash T$. Then* $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

Proof. Let $c \in \mathrm{acl}(\mathrm{acl}(A))$. Say $\mathcal{M} \vDash \phi(c, \overline{b})$, where $\overline{b} \in \mathrm{acl}(A)^{<\omega}$ and $\phi(x, \overline{y})$ is an algebraic $\mathcal{L}$-formula. Then $\overline{b}$ is algebraic over A since each b_i is. So say $\psi(\overline{y}) \in \mathrm{tp}(\overline{b}/A)$ is an algebraic $\mathcal{L}(A)$-formula isolating $\mathrm{tp}(\overline{b}/A)$.

Then $\theta(x) := \exists\overline{y}.\ (\psi(\overline{y}) \wedge \phi(x, \overline{y})) \in \mathrm{tp}(c/A)$ is algebraic; indeed, $\theta(\mathcal{M}) = \bigcup_{\overline{b}' \in\psi(\mathcal{M})} \phi(\mathcal{M}, \overline{b}')$ is finite, since for each of the finitely many $\overline{b}' \in \psi(\mathcal{M})$ we have $\overline{b}' \equiv \overline{b}$ and hence $\phi(\mathcal{M}, \overline{b}')$ is finite. □

12.2 Minimal and strongly minimal formulas

Definition 12.7. Let $\mathcal{M} \vDash T$.

- An $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$ is **minimal in** $\mathcal{M}$ if $\phi(\overline{x})$ is not algebraic but for every $\mathcal{L}(\mathcal{M})$-formula $\psi(\overline{x})$, either $\phi(\overline{x}) \wedge \psi(\overline{x})$ or $\phi(\overline{x}) \wedge \neg\psi(\overline{x})$ is algebraic.
- An $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$ is **strongly minimal** if it is minimal in every $\mathcal{N} \succeq \mathcal{M}$.

Example 12.8. Let $\mathcal{M} := (\{(i, j) : i < j < \omega\}; E)$, where $(i, j)E(i', j')$ iff $j = j'$. Then $x \doteq x$ is minimal in $\mathcal{M}$. (To see this, note $\mathrm{Th}(\mathcal{M})$ has QE once we add for each $n \in \omega$ a predicate for $\{(i, j) : j > n\}$.)

But $x \doteq x$ isn't strongly minimal. Indeed, let $b \in \mathcal{N} \succeq \mathcal{M}$ realise the partial type $\{\exists^{\geq n}y.\ yEx : n \in \omega\}$; then neither xEb nor $\neg xEb$ is algebraic.

Lemma 12.9. *Let $\phi(\overline{x},\overline{y})$ be an $\mathcal{L}$-formula. Let $\mathcal{M} \vDash T$. For $\overline{a} \in \mathcal{M}^{|\overline{y}|}$, $\phi(\overline{x},\overline{a})$ is strongly minimal iff it is non-algebraic and for every $\mathcal{L}$-formula $\psi(\overline{x},\overline{y})$ there is $n_\psi \in \omega$ such that*

$$\mathcal{M} \vDash \forall \overline{y}.\ \exists^{\leq n_\psi} \overline{x}.\ (\phi(\overline{x},\overline{a}) \wedge \psi(\overline{x},\overline{y})) \vee \forall \overline{y}.\ \exists^{\leq n_\psi} \overline{x}.\ (\phi(\overline{x},\overline{a}) \wedge \neg\psi(\overline{x},\overline{y})).$$

In particular, strong minimality of $\phi(\overline{x},\overline{a})$ depends only on $\mathrm{tp}^{\mathcal{M}}(\overline{a})$.

Proof. $\Leftarrow$ Immediate.

$\Rightarrow$ Suppose no such n_ψ exists. Then

$$\pi(\overline{y}) := \{\exists^{\geq n} \overline{x}.\ (\phi(\overline{x},\overline{a}) \wedge \psi(\overline{x},\overline{y})) \wedge \exists^{\geq n} \overline{x}.\ (\phi(\overline{x},\overline{a}) \wedge \neg\psi(\overline{x},\overline{y})) : n \in \omega\}$$

is a partial type, so say $\overline{b} \in \mathcal{N} \succeq \mathcal{M}$ realises $\pi(\overline{y})$. Then neither $\phi(\overline{x},\overline{a}) \wedge \psi(\overline{x},\overline{b})$ nor $\phi(\overline{x},\overline{a}) \wedge \neg\psi(\overline{x},\overline{b})$ is algebraic, so $\phi(\overline{x},\overline{a})$ is not minimal in $\mathcal{N}$, contradicting strong minimality.

□

Definition 12.10.
- T is **strongly minimal** if $x \doteq x$ is strongly minimal, i.e. any definable subset of any $\mathcal{M} \vDash T$ is finite or cofinite. ($X \subseteq Y$ is *cofinite* if $Y \setminus X$ is finite.)
- A structure $\mathcal{M}$ is **strongly minimal** if $\mathcal{M}$ is infinite and $\mathrm{Th}(\mathcal{M})$ is strongly minimal.

Example. T_∞, $\mathrm{Th}((\mathbb{Z}; S))$, $T_{k-\mathrm{VS}}$ and the completions of ACF are all strongly minimal.

12.3 Existence of (strongly) minimal formulas in ω-stable theories

Lemma 12.11. *Suppose T is totally transcendental. Let $\mathcal{M} \vDash T$. Let $|\overline{x}| > 0$. Then there exists an $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$ which is minimal in $\mathcal{M}$.*

Proof. Suppose not. Then if $\phi_s(\overline{x})$ is a non-algebraic $\mathcal{L}(\mathcal{M})$-formula, there is $\psi(\overline{x})$ such that $\phi_{s0} := \phi_s(\overline{x}) \wedge \psi(\overline{x})$ and $\phi_{s1} := \phi_s(\overline{x}) \wedge \neg\psi(\overline{x})$ are non-algebraic $\mathcal{L}(\mathcal{M})$-formulas. We obtain a binary tree $(\phi_s)_{s \in 2^{<\omega}}$, contradicting total transcendence. □

Lemma 12.12. *Let $\mathcal{M} \vDash T$ be $\aleph_0$-saturated. Let $\phi(\overline{x})$ be an $\mathcal{L}(\mathcal{M})$-formula which is minimal in $\mathcal{M}$. Then $\phi(\overline{x})$ is strongly minimal.*

Proof. Otherwise, the type in the proof of Lemma 12.9 is realised in $\mathcal{M}$ by $\aleph_0$-saturation, contradicting minimality. □

Definition 12.13. T **eliminates** $\exists^\infty$ if for every $\mathcal{L}$-formula $\phi(\overline{x},\overline{y})$ there is $n_\phi \in \omega$ such that for all $\mathcal{M} \vDash T$ and $\overline{b} \in \mathcal{M}^{|\overline{y}|}$ we have

$$|\phi(\mathcal{M},\overline{b})| < \aleph_0 \Rightarrow |\phi(\mathcal{M},\overline{b})| \leq n_\phi.$$

(So "$\exists^\infty \overline{x}.\ \phi(\overline{x},\overline{y})$" is expressed by $\exists^{>n_\phi} \overline{x}.\ \phi(\overline{x},\overline{y})$.)

Lemma 12.14. *If T eliminates $\exists^\infty$ then any minimal ϕ is strongly minimal.*

Proof. Lemma 12.9. □

Corollary 12.15. *Let $|\overline{x}| > 0$. If T is totally transcendental, then for every $\aleph_0$-saturated $\mathcal{M} \vDash T$ there is a strongly minimal $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$.*

If furthermore T eliminates $\exists^\infty$, then for every $\mathcal{M} \vDash T$ there is a strongly minimal $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$.

12.4 Strong minimality and stability

Let $\phi(\overline{x})$ be a strongly minimal $\mathcal{L}$-formula. By Lemma 12.9, this precisely means that $\phi(\overline{x})$ is minimal in every $\mathcal{M} \vDash T$.

Lemma 12.16. *Let $A \subseteq \mathcal{M} \vDash T$.*

(i) There is a unique non-algebraic type $\mathfrak{p}_A(\overline{x}) \in S(A)$ with $\phi(\overline{x}) \in \mathfrak{p}_A(\overline{x})$. This type $\mathfrak{p}_A(\overline{x})$ is the **generic** *type of ϕ over A.*

(ii) For any $n \in \omega$, there is a unique type $\mathfrak{p}_A^{(n)}(\overline{x}_1, \ldots, \overline{x}_n)$ of a sequence $\overline{a}_1, \ldots, \overline{a}_n \in \phi(\mathcal{M})$ with $\overline{a}_i \notin \operatorname{acl}(A \cup \overline{a}_{<i})^{<\omega}$ for $1 \leq i \leq n$. Such a sequence is also called **generic** *over A.*

Proof. (i) A non-algebraic type exists, since

$$\{\phi(\overline{x}) \wedge \neg\psi(\overline{x}) : \psi(\overline{x}) \text{ an algebraic } \mathcal{L}(A)\text{-formula}\}$$

is finitely consistent, since ϕ is not algebraic.

If two distinct such types exist, some $\mathcal{L}(A)$-formula ψ separates them, and then neither $\phi \wedge \psi$ nor $\phi \wedge \neg\psi$ is algebraic, contradicting minimality of ϕ in $\mathcal{M}$.

(ii) Suppose inductively this holds for $n \in \omega$, and consider two such sequences $\overline{a}_1, \ldots, \overline{a}_{n+1}$ and $\overline{b}_1, \ldots, \overline{b}_{n+1}$. Then

$$(\overline{a}_1, \ldots, \overline{a}_n) \equiv (\overline{b}_1, \ldots, \overline{b}_n),$$

so there is $\overline{c} \in \phi(\mathcal{N})$ for some $\mathcal{N} \succeq \mathcal{M}$ such that

$$(\overline{a}_1, \ldots, \overline{a}_n, \overline{a}_{n+1}) \equiv_A (\overline{b}_1, \ldots, \overline{b}_n, \overline{c})$$

(namely, a realisation of $\operatorname{tp}(\overline{a}_{n+1}/\overline{a}_{<n+1})^{\mathrm{id}_A \cup \bigcup_i \overline{a}_i \mapsto \overline{b}_i}$).

Then $\overline{c}, \overline{b}_{n+1} \vDash \mathfrak{p}_{A \cup \overline{b}_{<n+1}}$, so

$$(\overline{a}_1, \ldots, \overline{a}_{n+1}) \equiv_A (\overline{b}_1, \ldots, \overline{b}_n, \overline{c}) \equiv_A (\overline{b}_1, \ldots, \overline{b}_{n+1}).$$

□

Lemma 12.17. *Countable strongly minimal theories are ω-stable.*

Proof. Let $A \subseteq \mathcal{M} \vDash T$ with $|A| \leq \aleph_0$. By Lemma 12.16 for the strongly minimal formula $x \doteq x$, if $\mathcal{N} \succeq \mathcal{M}$ and $b \in \mathcal{N} \setminus \operatorname{acl}(A)$ then $b \vDash \mathfrak{p}_A(x)$. So $|S_1(A)| \leq |\operatorname{acl}(A)| + 1 \leq |T| + |A| = \aleph_0$. □

Remark 12.18. In fact, any strongly minimal theory is totally transcendental.

Lemma 12.19. *Let $A \subseteq \mathcal{M} \vDash T$. If $(\overline{a}, \overline{b}) \vDash \mathfrak{p}_A^{(2)}$ then $(\overline{b}, \overline{a}) \vDash \mathfrak{p}_A^{(2)}$.*

Proof. It suffices to show that for *some* $(\overline{a}, \overline{b}) \vDash \mathfrak{p}_A^{(2)}$ we have $(\overline{b}, \overline{a}) \vDash \mathfrak{p}_A^{(2)}$.

Let $\overline{a}_i \vDash \mathfrak{p}_A$ for $i \in \omega$ be distinct realisations, and let $\overline{b} \vDash \mathfrak{p}_{A \cup \bigcup_i \overline{a}_i}$, all realised in some $\mathcal{N} \succeq \mathcal{M}$.

Then $(\overline{a}_i, \overline{b}) \vDash \mathfrak{p}_A^{(2)}$ for all $i \in \omega$, so $\overline{a}_i \equiv_{A \cup \{\overline{b}\}} \overline{a}_j$ for all $i, j \in \omega$. It follows that $\overline{a}_0 \notin \operatorname{acl}(A \cup \{\overline{b}\})^{<\omega}$, so $(\overline{b}, \overline{a}_0) \vDash \mathfrak{p}_A^{(2)}$. □

12.5 Pregeometries

Definition 12.20. A pair (S, cl), where S is a set and $\operatorname{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$, is a **pregeometry** if for $A, B \subseteq S$ and $b, c \in S$:

(PG1) $A \subseteq B \Rightarrow A \subseteq \operatorname{cl}(A) \subseteq \operatorname{cl}(B)$

(PG2) $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$

(PG3) $\operatorname{cl}(A) = \bigcup_{A_0 \subseteq_{\text{fin}} A} \operatorname{cl}(A_0)$

(PG4) "Exchange": If $b \in \operatorname{cl}(A \cup \{c\}) \setminus \operatorname{cl}(A)$ then $c \in \operatorname{cl}(A \cup \{b\})$.

Remark. A finite pregeometry is also known as a *matroid.*

Let $\phi(\overline{x})$ be a strongly minimal $\mathcal{L}$-formula.

Lemma 12.21. *Let $\mathcal{M} \vDash T$. Define $\operatorname{acl}^{\mathcal{M}} \restriction_{\phi(\mathcal{M})}: \mathcal{P}(\phi(\mathcal{M})) \to \mathcal{P}(\phi(\mathcal{M}))$ by $\operatorname{acl}^{\mathcal{M}} \restriction_{\phi(\mathcal{M})} (A) := \operatorname{acl}^{\mathcal{M}}(A) \cap \phi(\mathcal{M})$. Then $(\phi(\mathcal{M}), \operatorname{acl}^{\mathcal{M}} \restriction_{\phi(\mathcal{M})})$ is a pregeometry.*

When no ambiguity can result, we write just acl *or* $\operatorname{acl}^{\mathcal{M}}$ *for* $\operatorname{acl}^{\mathcal{M}} \restriction_{\phi(\mathcal{M})}$.

Proof.

(PG1) Clear.

(PG2) Lemma 12.6.

(PG3) An algebraic formula uses only finitely many parameters.

(PG4) Lemma 12.19.

□

Definition 12.22. Let (S, cl) be a pregeometry. A subset $A \subseteq S$ is cl-**independent** if $a \notin \operatorname{cl}(A \setminus \{a\})$ for all $a \in A$. A cl-**basis** for S is a maximal cl-independent subset.

Lemma 12.23. *Let (S, cl) be a pregeometry.*

(i) S has a basis.

(ii) If $B \subseteq S$ is a basis, then $\operatorname{cl}(B) = S$.

Proof. (i) By (PG3), the union of a chain of independent sets is independent. We conclude by Zorn. .

(ii) Suppose $c \in S \setminus \mathrm{cl}(B)$. For any $b \in B$, we have $b \notin \mathrm{cl}(B \setminus \{b\})$, so by (PG4), $b \notin \mathrm{cl}(B \setminus \{b\}) \cup \{c\}$. But then $B \cup \{c\}$ is independent, contradicting maximality.

□

Proposition 12.24. *Let (S, cl) be a pregeometry. Then all bases have the same cardinality. This cardinality is the* **dimension** $\dim((S, \mathrm{cl}))$ *of the pregeometry.*

Proof. Let $X, Y \subseteq S$ with $\mathrm{cl}(X) = S$ and Y independent. We show $|Y| \leq |X|$.

First we prove this for X finite. Suppose $n = |X| \geq 0$, and assume the result for $|X| = n - 1$. Enumerate X as $\{x_1, \ldots, x_n\}$. If $Y = \emptyset$, we are done. Else, let $y \in Y$. Then $y \notin \mathrm{cl}(\emptyset)$ by independence, and $y \in \mathrm{cl}(X)$. So there is $i \geq 1$ be such that $y \in \mathrm{cl}(x_{\leq i}) \setminus cl(x_{<i})$. Then by (PG4), $x_i \in \mathrm{cl}(\{x_1, \ldots, x_{i-1}, y\})$.

Now consider the pregeometry (S, cl_y), where $\mathrm{cl}_y(A) := \mathrm{cl}(A \cup \{y\})$. Then $\mathrm{cl}_y(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}) = S$, and $Y \setminus \{y\}$ is cl_y-independent, so by the inductive hypothesis,

$$|Y| - 1 = |Y \setminus \{y\}| \leq |X \setminus \{x_i\}| = |X| - 1,$$

so $|Y| \leq |X|$, as required.

Now suppose $|X| \geq \aleph_0$. For $X_0 \subseteq_{\mathrm{fin}} X$, by the finite case above (applied to $(\mathrm{cl}(X_0), \mathrm{cl}\restriction_{\mathrm{cl}(X_0)})$), $|Y \cap \mathrm{cl}(X_0)| \leq |X_0| < \aleph_0$. Now by (PG3), $Y = \bigcup_{X_0 \subseteq_{\mathrm{fin}} X} (Y \cap \mathrm{cl}(X_0))$, so since $|\{X_0 : X_0 \subseteq_{\mathrm{fin}} X\}| = |X|$, we have $|Y| \leq |X|$. □

Definition 12.25. For $\mathcal{M} \vDash T$, define $\dim^{\phi}(\mathcal{M}) := \dim((\phi(\mathcal{M}), \mathrm{acl}))$.

If T is strongly minimal, $\dim(\mathcal{M}) := \dim^{x \doteq x}(\mathcal{M})$.

Remark 12.26. The acl-dimension of an algebraically closed field is also known as its "transcendence degree".

Lemma 12.27. *(i) The independent sets of size n in $(\phi(\mathcal{M}), \mathrm{acl})$ are precisely the realisations of $\mathfrak{p}_{\emptyset}^{(n)}$.*

(ii) More generally, $A \subseteq \phi(\mathcal{M})$ is independent iff for every $n \in \omega$ every tuple of n distinct elements of A realises $\mathfrak{p}_{\emptyset}^{(n)}$.

In particular, if $\mathcal{M}_1, \mathcal{M}_2 \vDash T$ and $A_i \subseteq \phi(\mathcal{M}_i)$ are independent sets with $|A_1| = |A_2|$, then any bijection $\theta : A_1 \to A_2$ is partial elementary.

Proof. (i) Let $\mathcal{M} \vDash T$ be $\aleph_0$-saturated and take an acl-basis B for $\phi(\mathcal{M})$. Then B is infinite. Take a subset $\{\bar{b}_1, \ldots, \bar{b}_n\} \subseteq B$ of size n. Then $(\bar{b}_1, \ldots, \bar{b}_n) \vDash \mathfrak{p}_{\emptyset}^{(n)}$ and $\{\bar{b}_1, \ldots, \bar{b}_n\}$ is independent, so this holds of any realisation of $\mathfrak{p}_{\emptyset}^{(n)}$.

(ii) This follows from (i).

□

12.6 Minimal subsets

The following notion of minimality of a subset is entirely separate from the notion of a formula being minimal.

Definition 12.28. Let $A \subseteq B \subseteq \mathcal{M} \vDash T$. Then B is **minimal over** A (in $\mathcal{M}$) if for any $\mathcal{N} \preceq \mathcal{M}$ with $A \subseteq \mathcal{N}$, also $B \subseteq \mathcal{N}$.

Lemma 12.29. *Let $A \subseteq B \subseteq \mathrm{acl}^{\mathcal{M}}(A) \subseteq \mathcal{M} \vDash T$. Then B is constructible and minimal over A in $\mathcal{M}$.*

Proof. Constructibility follows as in Lemma 12.5. For minimality: If $A \subseteq \mathcal{N} \preceq \mathcal{M}$, then $B \subseteq \mathrm{acl}^{\mathcal{M}}(A) = \mathrm{acl}^{\mathcal{N}}(A) \subseteq \mathcal{N}$ (by Lemma 12.3(ii)). □

Lemma 12.30. *Let $A \subseteq \mathcal{M} \vDash T$ and $A' \subseteq \mathcal{M}' \vDash T$. Suppose $\mathcal{M}$ is constructible over A and $\mathcal{M}'$ is minimal over A'. Suppose $\theta : A \xrightarrow{\equiv} A'$ is a p.e. bijection. Then θ extends to an isomorphism $\mathcal{M} \xrightarrow{\cong} \mathcal{M}'$.*

Proof. By Lemma 11.11, $\mathcal{M}$ is prime over A, so θ extends to an elementary embedding $\theta' : \mathcal{M} \xhookrightarrow{\preceq} \mathcal{M}'$. Then $A' \subseteq \mathrm{im}(\theta') \subseteq \mathcal{M}'$, so by minimality $\mathrm{im}(\theta') = \mathcal{M}'$. □

12.7 Classifying the models of a strongly minimal theory

Suppose T is strongly minimal.

Theorem 12.31. *Let $\mathcal{M}_1, \mathcal{M}_2 \vDash T$. Then*

$$\mathcal{M}_1 \cong \mathcal{M}_2 \iff \dim(\mathcal{M}_1) = \dim(\mathcal{M}_2).$$

In particular, T is κ-categorical for all $\kappa > |T|$.

Proof. Suppose $\dim(\mathcal{M}_1) = \dim(\mathcal{M}_2)$. Let B_i be an acl-basis of $\mathcal{M}_i$. By Lemma 12.27(ii), any bijection $\theta : B_1 \to B_2$ is partial elementary. By Lemma 12.23(ii) $\mathrm{acl}^{\mathcal{M}_i}(B_i) = \mathcal{M}_i$. By Lemma 12.29 and Lemma 12.30, θ extends to an isomorphism $\mathcal{M}_1 \xrightarrow{\cong} \mathcal{M}_2$.

The converse implication is clear.

For the "in particular" clause: it suffices to observe that $\dim(\mathcal{M}) \leq |\mathcal{M}| \leq |T| + \dim(\mathcal{M})$ (for all $\mathcal{M} \vDash T$). □

Lemma 12.32. *(i) Let $A \subseteq \mathcal{M} \vDash T$. Suppose $A = \mathrm{acl}^{\mathcal{M}}(A)$ and $|A| \geq \aleph_0$. Then A is the domain of an elementary substructure of $\mathcal{M}$.*

(ii) For some cardinal $0 \leq \lambda \leq \aleph_0$,

$$\{\dim(\mathcal{M}) : \mathcal{M} \vDash T\} = [\lambda, \infty) = \{\kappa \in \mathrm{Card} : \lambda \leq \kappa\}.$$

In particular, a countable strongly minimal theory is $\aleph_0$-categorical iff it has no finite-dimensional models.

Proof. Exercise; (i) is an easy application of the Tarski Test, and (ii) follows. □

Example 12.33.

- $T_{\mathbb{F}_q-\mathrm{VS}}$ is κ-categorical for all infinite κ.
- For k an infinite field, $T_{k-\mathrm{VS}}$ has models in dimensions $[1, \infty)$.
- ACF_p ($p = 0$ or p prime) has models in dimensions $[0, \infty)$.

12.8 Building uncountably categorical theories

Definition 12.34. A countable theory is **uncountably categorical** if it is κ-categorical for some $\kappa > \aleph_0$.

By Theorem 12.31, countable strongly minimal theories are uncountably categorical.

We now begin to address the following questions:

- In *which* uncountable cardinals are uncountably categorical theories categorical? e.g. can a countable T be $\aleph_2$ categorical but not $\aleph_1$-categorical, or vice-versa?
- Which theories are uncountably categorical? Do they have to be strongly minimal, or somehow associated to a strongly minimal theory?

Example 12.35. Let X be an infinite set. Let $\mathrm{Cart}^n(X) := (X^n \dot\cup X; P, \pi_1, \ldots, \pi_n)$, where $P(\mathrm{Cart}^n(X)) := X$ and $\pi_i : (x_1, \ldots, x_n) \mapsto x_i$ (and $\pi_i\restriction_X := \mathrm{id}\restriction_X$).

Let $T := \mathrm{Th}(\mathrm{Cart}^n(X))$. It is not hard to see that the models of T are precisely $\{\mathrm{Cart}^n(Y) : |Y| \geq \aleph_0\}$, so T is κ-categorical for all $\kappa \geq \aleph_0$.

Now T is not strongly minimal, but $P(x)$ is strongly minimal. Let $\mathcal{N} \vDash T$. Then $\mathcal{N} \subseteq \mathrm{acl}(P(\mathcal{N}))$, so by Lemma 12.29, $\mathcal{N}$ is constructible and minimal over $P(\mathcal{N})$, and the categoricity follows (since any bijection $P(\mathcal{M}_1) \to P(\mathcal{M}_2)$ is elementary).

Example 12.36. Let $n \in \omega$. Let V be an n-dimensional $\mathbb{C}$-vector space. Let $\mathrm{VS}^n(\mathbb{C}) := (V \dot\cup \mathbb{C}; P, +, \cdot, *)$, where $P(\mathrm{VS}^n(\mathbb{C})) := \mathbb{C}$, $+, \cdot$ are the ring operations on $\mathbb{C}$, and $*$ is scalar multiplication $\mathbb{C} \times V \to V$ (making these into total functions on $V \dot\cup \mathbb{C}$ by setting the value to $0 \in \mathbb{C}$ when it would otherwise be undefined).

Let $T := \mathrm{Th}(\mathrm{VS}^n(\mathbb{C}))$. It is not hard to see that the models of T are precisely $\{\mathrm{VS}^n(K) : K \vDash \mathrm{ACF}_0\}$, and so T is κ-categorical for all $\kappa > \aleph_0$.

Now $P(x)$ is strongly minimal, but $\mathrm{VS}^n(K) \not\subseteq \mathrm{acl}(K)$. However, if we pick a K-basis $B = \{b_1, \ldots, b_n\}$ for the vector space, then $\mathrm{VS}^n(K) \subseteq \mathrm{acl}(B \cup K)$. This suffices to explain categoricity, by the following Proposition.

Proposition 12.37. *Let T be a complete countable theory.*

Let $\mathcal{M}_0 \vDash T$ be prime. Let $\phi(x)$ be a strongly minimal $\mathcal{L}(\mathcal{M}_0)$-formula. Suppose that any $\mathcal{M} \succeq \mathcal{M}_0$ is constructible and minimal over $\mathcal{M}_0 \cup \phi(\mathcal{M})$.

Then T is κ-categorical for all $\kappa > \aleph_0$, and T has $\leq \aleph_0$ countable models up to isomorphism.

Proof.

Claim. *Suppose $\mathcal{M}_0 \preceq \mathcal{M} \vDash T$. Let B be an $\mathrm{acl}^{\mathcal{M}_{\mathcal{M}_0}}$-basis for $\phi(\mathcal{M})$. Then $\mathcal{M}_{\mathcal{M}_0}$ is constructible and minimal over B.*

Proof. We have $B \subseteq \phi(\mathcal{M}) \subseteq \mathcal{M}_{\mathcal{M}_0}$. By Lemma 12.29, $\phi(\mathcal{M})$ is constructible and minimal over B in $\mathcal{M}_{\mathcal{M}_0}$. Since $\mathcal{M}$ is constructible and minimal over $\mathcal{M}_0 \cup \phi(\mathcal{M})$, also $\mathcal{M}_{\mathcal{M}_0}$ is constructible and minimal over $\phi(\mathcal{M})$. Now the claim follows from Lemma 11.14 and its (easily verified) analogue for minimality. □

Let $\mathcal{M}_1, \mathcal{M}_2 \vDash T$ with $|\mathcal{M}_i| = \kappa > \aleph_0$; we show $\mathcal{M}_1 \cong \mathcal{M}_2$. WLOG $\mathcal{M}_0 \preceq \mathcal{M}_i$.

$\mathcal{M}_i$ is constructible and hence prime over $\phi(\mathcal{M}_i) \cup \mathcal{M}_0$, hence nach Löwenheim-Skolem, $\mathcal{M}_i$ embeds in a model of cardinality $\aleph_0 + |\phi(\mathcal{M}_i)|$, and it follows that $|\phi(\mathcal{M}_i)| = \kappa$.

Now by the Claim and Lemma 12.30, a bijection of bases extends to an isomorphism $(\mathcal{M}_1)_{\mathcal{M}_0} \cong (\mathcal{M}_2)_{\mathcal{M}_0}$; in particular $\mathcal{M}_1 \cong \mathcal{M}_2$.

For the case $\kappa = \aleph_0$: by the same argument countable models $\mathcal{M}_i$ are isomorphic if $\dim^\phi(\mathcal{M}_1) = \dim^\phi(\mathcal{M}_2)$. and $\dim^\phi(\mathcal{M}_i) \leq \aleph_0$. □

In all the examples of uncountably categorical theories we have seen so far, the hypotheses of Proposition 12.37 hold with "constructible and minimal" strengthened to "algebraic". Such a theory is called *almost strongly minimal*. The following is a natural example of an uncountably categorical theory which is not almost strongly minimal.

Example 12.38. $T := \mathrm{Th}((\mathbb{Z}/4\mathbb{Z})^\omega; 0, +)$.

Exercise: T has QE and is axiomatised by

$$[\text{axioms for infinite abelian groups}] \cup \{\forall x.(2x = 0 \leftrightarrow \exists y.2y = x)\}.$$

Let $G \vDash T$. Let $\lambda := |G|$.

Claim. $G \cong \bigoplus_{i<\lambda} \mathbb{Z}/4\mathbb{Z}$.

Proof. Let $[2] : G \to G;\ x \mapsto 2x$. Then $\ker([2]) = \mathrm{im}([2]) = 2G$. So $2G$ is an $\mathbb{F}_2$-vector space, and $|2G| = |G| = \lambda$.

Let $(b_i)_\lambda$ be an $\mathbb{F}_2$-basis for $2G$. Let $e_i \in G$ such that $2e_i = b_i$. So $\mathrm{ord}(e_i) = 4$.

Now if $g \in G$ then $2g = b_{i_1} + \ldots + b_{i_m}$ say; let $g' := e_{i_1} + \ldots + e_{i_m}$, then $2(g - g') = 0$ so $g - g' \in 2G$. Hence G is generated by $(e_i)_i$.

Suppose $\sum_{i=1}^k n_{j_i} e_{j_i} = 0$ with $n_{j_i} \in \mathbb{Z}$. Then $\sum_{i=1}^k n_{j_i} b_{j_i} = 2 \cdot 0 = 0$, so $2 | n_{j_i}$. Then $\sum_{i=1}^k \frac{n_{j_i}}{2} b_{j_i} = 0$, so $2 | \frac{n_{j_i}}{2}$. Hence $4 | n_{j_i}$.

So we conclude that $G = \bigoplus_{i<\lambda} (\mathbb{Z}/4\mathbb{Z}) e_i$. □

It follows that T is κ-categorical for all $\kappa \geq \aleph_0$.

Let $\phi(x) := \exists y.\, x = y + y$, so $\phi(G) = 2G$. By the QE, ϕ is strongly minimal.

Claim. G is constructible and minimal over $2G$.

Proof.

Minimality: If $2G \subseteq G' \preceq G$ and $g \in G \setminus 2G$, then $2g' = 2g$ for some $g' \in G'$, but then $g' - g \in 2G \subseteq G'$, so also $g \in G'$. So $G' = G$.

Constructibility: Let $(e_i)_{i \in \lambda}$ be as above. It follows from the QE (or by considering automorphisms and the proof of the previous claim) that for $j \in \lambda$, $\mathrm{tp}(e_j / 2G \cup \bigoplus_{i<j} (\mathbb{Z}/4\mathbb{Z}) e_i)$ is isolated by $x + x = b_j$, and then $2G \cup \bigoplus_{i \leq j} (\mathbb{Z}/4\mathbb{Z}) e_i$ is algebraic and hence constructible over $2G \cup \bigoplus_{i<j} (\mathbb{Z}/4\mathbb{Z}) e_i \cup \{e_j\}$.

So we obtain a construction sequence for G over $2G$ in this way.

□

13 Indiscernible sequences

13.1 Ramsey theory

If A is a set and $n \in \omega$, we write $[A]^n$ for the set of n-element subsets of A:

$$[A]^n := \{A_0 \subseteq A : |A_0| = n\} \subseteq \mathcal{P}(A).$$

Theorem 13.1 (Infinite Ramsey Theorem). *Let A be an infinite set. Let $n \in \omega$, and let C be a finite set. Let $f : [A]^n \to C$ be a function (a "colouring" of the n-element subsets). Then there exists an infinite subset $B \subseteq A$ which is "f-monochromatic", i.e. such that f is constant on $[B]^n$.*

Remark. With $n = 2 = |C|$, this gives that any infinite graph has an infinite clique or an infinite anticlique.

Proof. The case $n = 0$ is clear. Suppose $n > 0$ and the result holds for $n - 1$.

For $a \in A$, define $f_a : [A \setminus \{a\}]^{n-1} \to C; f_a(A') := f(A' \cup \{a\})$. Recursively construct a sequence of infinite sets $A =: B_0 \supseteq B_1 \supseteq \ldots$ and elements $a_i \in B_i \setminus B_{i+1}$ and $c_i \in C$ as follows: given B_i, let $a_i \in B_i$, and let $B_{i+1} \subseteq B_i \setminus a_i$ be an infinite f_{a_i}-monochromatic subset, which exists by the induction hypothesis since B_i is infinite, and let c_i be such that $f_{a_i}([B_{i+1}]^{n-1}) = \{c_i\}$. By the pigeonhole principle, let $c \in C$ be such that $c_i = c$ for infinitely many $i \in \omega$, and let $B := \{a_i : c_i = c\}$.

Then B is f-monochromatic. Indeed, if $\{a_{i_1}, \ldots, a_{i_n}\} \subseteq B$ is a subset with $i_1 < \ldots < i_n$, then $a_{i_2}, \ldots, a_{i_n} \in B_{i_1+1}$, so

$$f(\{a_{i_1}, \ldots, a_{i_n}\}) = f_{a_{i_1}}(\{a_{i_2}, \ldots, a_{i_n}\}) = c_{i_1} = c.$$

□

13.2 Indiscernible sequences

Notation 13.2. If I is a linear order and $n \in \omega$, we write $I^{\overrightarrow{n}}$ for the set of *I-ordered n-tuples* of I,

$$I^{\overrightarrow{n}} := \{(i_1, \ldots, i_n) \in I^n : i_1 < \ldots < i_n\}.$$

If $(a_i)_{i \in I}$ is a sequence and $\bar{i} \in I^{\overrightarrow{n}}$, let $a_{\bar{i}} := (a_{i_1}, \ldots, a_{i_n})$.

Definition 13.3. Let I be a linear order. A sequence $(a_i)_{i \in I}$ of elements of a structure $\mathcal{M}$ is **indiscernible** if for any $n \in \omega$ and any $\bar{i}, \bar{j} \in I^{\overrightarrow{n}}$,

$$a_{\bar{i}} \equiv a_{\bar{j}}.$$

In other words, for any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n)$, for any $i_1 < \ldots < i_n \in I$ and $j_1 < \ldots < j_n \in I$,

$$\mathcal{M} \vDash \phi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \phi(a_{j_1}, \ldots, a_{j_n}).$$

Examples 13.4.
- Any constant sequence, $(a_i)_{i \in I}$ with $a_i = a \in \mathcal{M}$ for all i, is indiscernable.

- If $\phi(x)$ is strongly minimal in $\mathrm{Th}(\mathcal{M})$ and $A \subseteq \phi(\mathcal{M})$ is acl-independent, then Lemma 12.27(ii) shows that for any linear order I and any injection $I \to A; i \mapsto a_i$, the sequence $(a_i)_{i\in I}$ is indiscernible.
- Let $\mathcal{M} \vDash \mathrm{DLO}$. Then by QE, any strictly increasing sequence of elements $(a_i)_{i\in I}$ is indiscernible.

Definition 13.5. Let $(a_i)_{i\in I}$ be a sequence of elements of an $\mathcal{L}$-structure $\mathcal{M}$, where I is a linear order.

The **Ehrenfeucht-Mostowski type** (EM-type) of $(a_i)_{i\in I}$ in $\mathcal{M}$ is

$$\mathrm{EM}((a_i)_i) = \{\phi(\overline{x}) : \phi(\overline{x}) \text{ an } \mathcal{L}\text{-formula; } \forall \overline{i} \in I^{\overrightarrow{|\overline{x}|}}.\, \mathcal{M} \vDash \phi(a_{\overline{i}})\}.$$

Remark 13.6. $(a_i)_{i\in I}$ is indiscernible if and only if $\mathrm{EM}((a_i)_i)$ is complete in the sense that for every $\mathcal{L}$-formula ϕ either ϕ or $\neg\phi$ is in the EM-type.

Lemma 13.7. *Let I and J be infinite linear orders. Let $(a_i)_{i\in I}$ be a sequence of elements of a structure $\mathcal{M}$.*

Then there exists $\mathcal{M}' \equiv \mathcal{M}$ and an indiscernible sequence $(b_j)_{j\in J}$ of elements of $\mathcal{M}'$ such that $\mathrm{EM}((a_i)_{i\in I}) \subseteq \mathrm{EM}((b_j)_{j\in J})$.

Proof. Let $(c_j)_{j\in J}$ be new constants. It suffices to show consistency of

$$\begin{aligned} T := \mathrm{Th}(\mathcal{M}) &\cup \{\psi(c_{\overline{j}}) : \psi(\overline{x}) \in \mathrm{EM}((a_i)_i);\ \overline{j} \in J^{\overrightarrow{|\overline{x}|}}\} \\ &\cup \{\phi(c_{\overline{j}}) \leftrightarrow \phi(c_{\overline{j}'}) : \phi(\overline{x}) \text{ an } \mathcal{L}\text{-formula; } \overline{j}, \overline{j}' \in J^{\overrightarrow{|\overline{x}|}}\}. \end{aligned}$$

Let $T_0 \subseteq_{\mathrm{fin}} T$ be finite. By compactness, it suffices to show consistency of T_0. Let n be the maximum number of free variables in the formulas ϕ such that $(\phi(c_{\overline{j}}) \leftrightarrow \phi(c_{\overline{j}'}))$ appears in T_0, so we can write each such ϕ as $\phi(x_1, \ldots, x_n)$. Let Δ be the finite set of these $\phi(x_1, \ldots, x_n)$.

Define $f : I^{\overrightarrow{n}} \to 2^{\Delta}$ by

$$f(\overline{i}) := \left(\phi \mapsto \begin{cases} 1 & \mathcal{M} \vDash \phi(a_{\overline{i}}) \\ 0 & \mathcal{M} \vDash \neg\phi(a_{\overline{i}}) \end{cases}\right).$$

By Ramsey (applied via the obvious bijection $I^{\overrightarrow{n}} \xrightarrow{\equiv} [I]^n$, namely $\overline{i} \mapsto \{i_1, \ldots, i_n\}$), let $I' \subseteq I$ be an infinite f-monochromatic subset. Let $\overline{j} \in J^{\overrightarrow{m}}$ be such that $c_{\overline{j}}$ is the tuple of those constants which appear in T_0. Let $\overline{i} \in (I')^{\overrightarrow{m}}$. Then $(\mathcal{M}; a_{\overline{i}}) \vDash T_0$. □

Lemma 13.8. *Let T be a theory with infinite models. Let J be an infinite linear order. Then T has a model with a non-constant indiscernible sequence $(b_j)_{j\in J}$.*

Proof. Let $\mathcal{M}$ be an infinite model, and let $(a_i)_{i\in\omega}$ be a sequence of distinct elements of $\mathcal{M}$. Now apply Lemma 13.7 to obtain $(b_j)_{j\in J}$, and note $b_j \neq b_{j'}$ for $j \neq j'$, since $x_1 \neq x_2 \in \mathrm{EM}((a_i)_i) \subseteq \mathrm{EM}((b_j)_J)$. □

13.3 Uncountable categoricity ⇒ ω-stability

Let T be a countable complete theory with infinite models.

Lemma 13.9. *Let κ be an infinite cardinal. Then there is $\mathcal{M} \vDash T$ such that $|\mathcal{M}| = \kappa$ and if $B \subseteq \mathcal{M}$ with $|B| \leq \aleph_0$ then*

$$|\{\text{tp}(a/B) : a \in \mathcal{M}\}| \leq \aleph_0.$$

Proof. By Lemma 6.4, we may assume T has built-in Skolem functions.

By Lemma 13.8, we may find $\mathcal{N} \vDash T$ with a non-constant indiscernible sequence of elements $(a_i)_{i \in \kappa}$.

Let $\mathcal{M} := \langle \{a_i\}_i \rangle^{\mathcal{N}}$. By Lemma 6.3, $\mathcal{M} \preceq \mathcal{N}$. Since $|T| \leq \aleph_0$, we have $|\mathcal{M}| = \kappa$.

Let $B \subseteq \mathcal{M}$ be countable; we conclude by showing that $\mathcal{M}$ realises only countably many types over B. Say $B = \{f_k(a_{\bar{i}_k}) : k \in \omega\}$, where f_k is a term and $\bar{i}_k \in \kappa^{\overrightarrow{n_k}}$; let $I_B \subseteq \kappa$ be the indices appearing. Let $c \in \mathcal{M}$. Say $c = g(a_{\bar{j}})$, where g is a term and $\bar{j} \in \kappa^{\overrightarrow{n}}$. Then by indiscernability, $\text{tp}(c/B)$ depends only on the term g (for which there are countably many possibilities) and the quantifier-free 1-types $\text{qftp}^{(\kappa;<)}(j_i/I_B)$ for $1 \leq i \leq |\bar{j}|$. So we conclude by the following Claim and the countability of I_B.

Claim. *Let $J \subseteq \kappa$ infinite. Then $|\{\text{qftp}^{(\kappa;<)}(\alpha/J) : \alpha \in \kappa\}| = |J|$.*

Proof. Let $\alpha \in \kappa \setminus J$. Then $\text{qftp}(\alpha/J)$ is determined by the cut α makes in J, i.e. by $J_{>\alpha} := \{\gamma \in J : \gamma > \alpha\}$. If $J_{>\alpha}$ is non-empty, it is determined by $\min J_{>\alpha} \in J$, which exists by well-orderedness.

So there are $\leq |J|$ possibilities for $\text{qftp}(\alpha/J)$ with $\alpha \in \kappa \setminus J$, and clearly there are $|J|$ possibilities for $\text{qftp}(\alpha/J)$ with $\alpha \in J$. □

□

Proposition 13.10. *If T is an uncountably categorical theory then T totally transcendental.*

Proof. Say T is categorical in $\lambda > \aleph_0$ but not totally transcendental. By Theorem 11.3, T is not ω-stable. So say $A \subseteq \mathcal{M} \vDash T$ with $|A| \leq \aleph_0$ but $|S_1(A)| > \aleph_0$. Let $P \subseteq S_1(A)$ with $|P| = \aleph_1$. By Lemma 7.16 we find $\mathcal{M}' \succeq \mathcal{M}$ and $b_p \in \mathcal{M}'$ for $p \in P$ with $\text{tp}(b_p/A) = p$. By Löwenheim-Skolem we find an elementary extension or substructure $\mathcal{M}''$ of $\mathcal{M}'$ containing $A \cup \{b_p : p \in P\}$, with $|\mathcal{M}''| = \lambda$.

But by Lemma 13.9, there is a model of cardinality λ which realises only countably many types over countable sets, which therefore is not isomorphic to $\mathcal{M}''$, contradicting categoricity. □

Corollary 13.11. *Let $\lambda > \aleph_0$. Then T is λ-categorical iff every model of cardinality λ is saturated.*

Proof. $\Leftarrow$ Lemma 7.21.

$\Rightarrow$ By Proposition 13.10 and Corollary 11.4, T is λ-stable. For each $\kappa < \lambda$ by Corollary 10.7(ii) T has a κ^+-saturated model of cardinality λ. So by λ-categoricity, every model of cardinality λ is κ^+-saturated for all $\kappa < \lambda$, and hence λ-saturated.

□

14 Vaughtian pairs

Let T be a countable complete $\mathcal{L}$-theory with infinite models.

Definition 14.1.

- A **Vaughtian triple** (in T) is a triple $(\mathcal{N}, \mathcal{M}, \phi)$, where
  - $\mathcal{M} \vDash T$;
  - $\mathcal{N} \succeq \mathcal{M}$ is a proper elementary extension;
  - $\phi(x)$ is an non-algebraic $\mathcal{L}(\mathcal{M})$-formula;
  - $\phi(\mathcal{N}) = \phi(\mathcal{M})$.

  $(\mathcal{N}, \mathcal{M})$ is then called a **Vaughtian pair**.
- T **has a Vaughtian pair** if there is some Vaughtian pair, i.e. if there is some Vaughtian triple.

Examples 14.2.

- $((\{0,1\} \times \mathbb{Q}; <), (\{0\} \times \mathbb{Q}; <), x < (0,0))$ is a Vaughtian triple in DLO, where the order is the lexicographic order.
- $((\omega+\omega; \omega), ((\omega+\omega)\setminus\{0\}; \omega\setminus\{0\}), \neg P)$ is a Vaughtian triple in the language $\{P\}$, where P is a unary predicate.

Lemma 14.3. *If T has no Vaughtian pairs then T eliminates $\exists^\infty$.*

Proof. It suffices to consider the one variable case, since $\exists^\infty \overline{x}.\ \phi(\overline{x}, \overline{y})$ can be expressed by $\bigvee_i \exists^\infty x_i.\ \exists x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{|\overline{x}|}.\ \phi(\overline{x}, \overline{y})$.

So suppose for a contradiction that $\phi(x, \overline{y})$ is an $\mathcal{L}$-formula and $\overline{b}_i \in \mathcal{M}_i \vDash T$ are such that $i \mapsto |\phi(\mathcal{M}_i, \overline{b}_i)|$ is a strictly increasing function $\omega \to \omega$. Realising each $\operatorname{tp}(\overline{b}_i)$ in an ω-saturated model $\mathcal{M}$, we may assume $\mathcal{M}_i = \mathcal{M}$ for all $i \in \omega$.

Let $\mathcal{N} \succeq \mathcal{M}$ be a proper elementary extension. Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then $\mathcal{N}^{\mathcal{U}} \succeq \mathcal{M}^{\mathcal{U}}$ is also a proper elementary extension (indeed, if $c \in \mathcal{N} \setminus \mathcal{M}$, then (via the diagonal embedding) $c \in \mathcal{N}^{\mathcal{U}} \setminus \mathcal{M}^{\mathcal{U}}$). Let $\overline{b} = \lim_{i \to \mathcal{U}} \overline{b}_i \in \mathcal{M}^{\mathcal{U}}$. Then $\phi(\mathcal{M}^{\mathcal{U}}, \overline{b})$ is infinite, but $\phi(\mathcal{N}^U, \overline{b}) = \phi(\mathcal{M}^U, \overline{b})$, since $\phi(\mathcal{N}, \overline{b}_i) = \phi(\mathcal{M}, \overline{b}_i)$ for all i. So $(\mathcal{N}^{\mathcal{U}}, \mathcal{M}^{\mathcal{U}}, \phi(x, \overline{b}))$ is Vaughtian. □

Lemma 14.4. *If T has no Vaughtian pairs and $A \subseteq \mathcal{N} \vDash T$ and $\phi(x)$ is a non-algebraic $\mathcal{L}(A)$-formula, then $\mathcal{N}$ is minimal over $A \cup \phi(\mathcal{N})$.*

Proof. Otherwise, there is $\mathcal{M} \preceq \mathcal{N}$ with $A \cup \phi(\mathcal{N}) \subseteq \mathcal{M} \subsetneq \mathcal{N}$. Then $\phi(\mathcal{M}) = \phi(\mathcal{N})$ and ϕ is an $\mathcal{L}(\mathcal{M})$-formula. So $(\mathcal{N}, \mathcal{M}, \phi)$ is Vaughtian. □

Lemma 14.5. *Suppose T is ω-stable and $(\mathcal{N}_0, \mathcal{M}_0, \phi)$ is a Vaughtian triple. Let $\kappa > \aleph_0$. Then there exists a Vaughtian triple $(\mathcal{N}^\kappa, \mathcal{M}, \phi)$ with $|\mathcal{N}^\kappa| = \kappa$ and $|\mathcal{M}| = \aleph_0$.*

Proof. ϕ is an $\mathcal{L}(A)$-formula for some finite $A \subseteq \mathcal{M}_0$. Replacing T with $\operatorname{Th}((\mathcal{M}_0)_A)$, which is also ω-stable, we may assume ϕ is an $\mathcal{L}$-formula.

Claim. *There is a Vaughtian triple $(\mathcal{N}, \mathcal{M}, \phi)$ with $\mathcal{N}$ and $\mathcal{M}$ countable saturated models.*

Proof. Let $\mathcal{L}_P := \mathcal{L}\dot\cup\{P\}$, where P is a unary predicate symbol. Let

$$\begin{aligned} T_P := T \\ &\cup \{\forall \overline{x}.\, (\bigwedge_i P(x_i) \rightarrow (\exists y.\, \psi(\overline{x}, y) \rightarrow \exists y.\, (P(y) \wedge \psi(\overline{x}, y)))) : \psi(\overline{x}, y) \text{ an } \mathcal{L}\text{-formula}\} \\ &\cup \{\forall x.\, (\phi(x) \rightarrow P(x))\} \\ &\cup \{\exists x.\, \neg P(x)\}. \end{aligned}$$

For $\mathcal{N}$ an $\mathcal{L}$-structure and $A \subseteq \mathcal{N}$, let $(\mathcal{N}; A)$ be the $\mathcal{L}_P$-structure expanding $\mathcal{N}$ with $P((\mathcal{N}; A)) = A$. Then (using the Tarski Test) $(\mathcal{N}; \mathcal{M}) \vDash T_P$ if and only if $\mathcal{N} \vDash T$ and $(\mathcal{N}, \mathcal{M}, \phi)$ is Vaughtian.

T_P is consistent, since $(\mathcal{N}_0; \mathcal{M}_0) \vDash T_P$. By Löwenheim-Skolem, let $(\mathcal{N}'_0; \mathcal{M}'_0)$ be a countable model of T_P. We now proceed as in Theorem 8.4: Build an elementary chain of countable models $(\mathcal{N}'_0; \mathcal{M}'_0) \preceq (\mathcal{N}'_1; \mathcal{M}'_1) \preceq \ldots$ by taking $(\mathcal{N}'_{i+1}; \mathcal{M}'_{i+1})$ such that $\mathcal{N}'_{i+1}$ realises all $\mathcal{L}$-types over finite subsets of $\mathcal{N}'_i$ and $\mathcal{M}'_{i+1}$ realises all $\mathcal{L}$-types over finite subsets of $\mathcal{M}'_i$; this is possible since T is small since ω-stable, and any $\mathcal{L}$-type $p(x)$ over a subset of $\mathcal{M}'_i$ is consistent modulo T_P with $P(x)$.

Then $(\mathcal{N}; \mathcal{M}) := \bigcup_{i\in\omega}(\mathcal{N}'_i; \mathcal{M}'_i) \vDash T_P$ is countable, and both $\mathcal{N}$ and $\mathcal{M}$ are saturated as models of T. □

Claim. *There is $\mathcal{N}' \succeq \mathcal{M}$ with $|\mathcal{N}'| = \aleph_1$ such that $(\mathcal{N}', \mathcal{M}, \phi)$ is Vaughtian. In particular, $|\phi(\mathcal{N}')| = \aleph_0$.*

Proof. We build an elementary chain $(\mathcal{M}^\alpha)_{\alpha\in\aleph_1}$ of countable saturated models with $\phi(\mathcal{M}^\alpha) = \phi(\mathcal{M})$.

Let $\mathcal{M}^0 := \mathcal{M}$. Given $\mathcal{M}^\alpha$, let $\mathcal{M}^{\alpha+1} \succeq \mathcal{M}^\alpha$ be such that $(\mathcal{M}^{\alpha+1}, \mathcal{M}^\alpha) \cong (\mathcal{N}, \mathcal{M})$, which exists since $\mathcal{M}^\alpha \cong \mathcal{M}$ (by saturation and Lemma 7.21). Then $\phi(\mathcal{M}^{\alpha+1}) = \phi(\mathcal{M}^\alpha) = \phi(\mathcal{M})$, and $\mathcal{M}^{\alpha+1}$ is countable and saturated. For $\eta \in \aleph_1$ a limit ordinal, let $\mathcal{M}^\eta := \bigcup_{\alpha<\eta} \mathcal{M}^\alpha$. Then $\phi(\mathcal{M}^\eta) = \bigcup_{\alpha<\eta} \phi(\mathcal{M}^\alpha) = \phi(\mathcal{M})$, and $\mathcal{M}^\eta$ is countable and saturated.

Finally, let $\mathcal{N}' := \bigcup_{\alpha\in\aleph_1} \mathcal{M}^\alpha$. Then $\phi(\mathcal{N}') = \phi(\mathcal{M})$, and $|\mathcal{N}'| = \aleph_1$ since $\mathcal{M}^{\alpha+1} \supsetneq \mathcal{M}^\alpha$. □

Claim. *Suppose $\mathcal{A} \vDash T$ and $|\mathcal{A}| > \aleph_0$ but $|\phi(\mathcal{A})| = \aleph_0$. Then there is $\mathcal{B}$ such that $(\mathcal{B}, \mathcal{A}, \phi)$ is Vaughtian and $|\mathcal{B}| = |\mathcal{A}|$.*

Proof. Say an $\mathcal{L}(\mathcal{A})$-formula $\theta(x)$ is *countable* resp. *uncountable* if $\theta(\mathcal{A})$ is. Since $x \doteq x$ is uncountable and there is no binary tree of $\mathcal{L}(\mathcal{A})$-formulas, there exists an uncountable $\psi(x)$ such that for every $\mathcal{L}(A)$-formula $\theta(x)$ either $\psi(x) \wedge \theta(x)$ or $\psi(x) \wedge \neg\theta(x)$ is countable[5]. Then $p(x) := \{\theta(x) : \psi(x) \wedge \theta(x) \text{ is uncountable}\}$ is a type.

Let $b \vDash p(x)$ be a realisation in some elementary extension, $b \in \mathcal{A}' \succeq \mathcal{A}$. By Theorem 11.13, we can find $\mathcal{A} \cup \{b\} \subseteq \mathcal{B} \preceq \mathcal{A}'$ with $\mathcal{B}$ constructible over $\mathcal{A} \cup \{b\}$. By Löwenheim-Skolem and primeness, $|\mathcal{B}| = |\mathcal{A} \cup \{b\}| = |\mathcal{A}|$. By Lemma 11.15 (Exercise 10.2), $\mathcal{B}$ is atomic over $\mathcal{A} \cup \{b\}$.

Suppose $\phi(\mathcal{B}) \neq \phi(\mathcal{A})$, say $c \in \phi(\mathcal{B}) \setminus \phi(\mathcal{A})$. Say $\xi(x, b)$ is an $\mathcal{L}(\mathcal{A})$-formula isolating $\mathrm{tp}(c/\mathcal{A} \cup \{b\})$. Then

$$b \vDash \Theta(y) := \{\exists x.\, \xi(x, y)\} \cup \{\forall x.\, (\xi(x, y) \rightarrow (\phi(x) \wedge x \neq a)) : a \in \phi(\mathcal{A})\}.$$

[5] Such a formula ψ is called *quasiminimal*, by analogy with a minimal formula.

Now $\Theta(y) \subseteq p(y)$ consists of countably many formulas, so by the definition of p, $\psi(\mathcal{A}) \cap \bigcap\{\theta(\mathcal{A}) : \theta(y) \in \Theta(y)\}$ is an intersection of countably many cocountable subsets of the uncountable set $\psi(\mathcal{A})$, so is non-empty. So there is $b' \in \mathcal{A}$ such that $b' \vDash \Theta(y)$. Then there is c' such that $\mathcal{A} \vDash \xi(c', b')$, and then $\mathcal{A} \vDash \phi(c')$, but for each $a \in \phi(\mathcal{A})$ we have $\overline{c}' \neq a$, which is a contradiction. □

Using this last claim at successor stages and taking unions at limit stages and setting $\mathcal{N}^0 := \mathcal{N}'$, we build an elementary chain $(\mathcal{N}^\alpha)_{\alpha\in\kappa}$ such that for all $\alpha \in \kappa$:

- $\phi(\mathcal{N}^\alpha) = \phi(\mathcal{M})$;
- $\mathcal{N}^{\alpha+1}$ is a proper extension of $\mathcal{N}^\alpha$;
- $|\mathcal{N}^\alpha| = |\alpha| + \aleph_1$.

Set $\mathcal{N}^\kappa := \bigcup_{\alpha\in\kappa} \mathcal{N}^\alpha$. Then $|\mathcal{N}^\kappa| = \kappa$, and $(\mathcal{N}^\kappa, \mathcal{M}, \phi)$ is Vaughtian. □

Proposition 14.6. *Suppose T is uncountably categorical. Then T has no Vaughtian pairs.*

Proof. By Proposition 13.10, T is ω-stable. Say $\kappa > \aleph_0$ is such that T is κ-categorical, and let $\mathcal{M}$ be the model of cardinality κ. Suppose T has a Vaughtian pair. Then by Lemma 14.5, there is an $\mathcal{L}(\mathcal{M})$-formula $\phi(x)$ such that $|\phi(\mathcal{M})| = \aleph_0$. But then $\mathcal{M}$ is not saturated, since it omits $\{\phi(x)\} \cup \{x \neq a : a \in \phi(\mathcal{M})\}$. This contradicts Corollary 13.11. □

15 Baldwin-Lachlan

Lemma 15.1. *Let T be a totally transcendental theory. If $A \subseteq \mathcal{M} \vDash T$ and $\mathcal{M}$ is minimal over A, then $\mathcal{M}$ is constructible over A.*

Proof. By Theorem 11.13, there is $A \subseteq \mathcal{N} \preceq \mathcal{M}$ with $\mathcal{N}$ constructible over A. By the minimality, $\mathcal{N} = \mathcal{M}$. □

Theorem 15.2 (Baldwin-Lachlan)**.** *Let T be a countable complete theory with infinite models. TFAE:*

(i) T is κ-categorical for some $\kappa > \aleph_0$;

(ii) T is ω-stable and has no Vaughtian pairs;

(iii) T has a prime model $\mathcal{M}_0$ and a strongly minimal $\mathcal{L}(\mathcal{M}_0)$-formula $\phi(x)$ such that any $\mathcal{M} \succeq \mathcal{M}_0$ is constructible and minimal over $\mathcal{M}_0 \cup \phi(\mathcal{M})$;

(iv) T is κ-categorical for all $\kappa > \aleph_0$ and has $\leq \aleph_0$ countable models up to isomorphism.

Proof.

(i) ⇒ (ii) Proposition 13.10 and Proposition 14.6.

(ii) ⇒ (iii) By Theorem 11.13 (or Proposition 8.23), T has a prime model $\mathcal{M}_0$.

By Lemma 14.3 and Corollary 12.15, there is a strongly minimal $\mathcal{L}(\mathcal{M}_0)$-formula $\phi(x)$.

Let $\mathcal{M} \succeq \mathcal{M}_0$. By Lemma 14.4, $\mathcal{M}$ is minimal over $\mathcal{M}_0 \cup \phi(\mathcal{M})$. By Lemma 15.1, $\mathcal{M}$ is also constructible over $\mathcal{M}_0 \cup \phi(\mathcal{M})$.

(iii) ⇒ (iv) Proposition 12.37.

(iv) ⇒ (i) Immediate.

□

Corollary 15.3 (Morley's Theorem)**.** *A countable complete theory is categorical in some uncountable cardinal if and only if it is categorical in all uncountable cardinals.*

Fact 15.4 (Baldwin-Lachlan)**.** *(iv) can be improved to say "either 1 or $\aleph_0$ countable models".*

16 Morley rank

Definition 16.1. Let $\mathrm{On}^{\pm\infty} := \{-\infty\} \cup \mathrm{On} \cup \{+\infty\}$ be the well-order extending On, where $\forall \alpha \in \mathrm{On}.\ -\infty < \alpha < +\infty$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

- The **Morley rank in** $\mathcal{M}$ of an $\mathcal{L}(\mathcal{M})$-formula $\phi(\overline{x})$ is

$$\mathrm{MR}^{\mathcal{M}}(\phi) := \inf\{\alpha \in \mathrm{On}^{\pm\infty} : \mathrm{MR}^{\mathcal{M}}(\phi) \leq \alpha\} \in \mathrm{On}^{\pm\infty},$$

  where we recursively define $\mathrm{MR}^{\mathcal{M}}(\phi) \leq \alpha \in \mathrm{On}^{\pm\infty}$ by:

  - $\mathrm{MR}^{\mathcal{M}}(\phi) \leq -\infty$, if $\phi(\mathcal{M}) = \emptyset$;
  - $\mathrm{MR}^{\mathcal{M}}(\phi) \leq \alpha \in \mathrm{On}$, if for any $\mathcal{L}(\mathcal{M})$-formulas $(\psi_i(\overline{x}))_{i\in\omega}$ defining disjoint subsets of $\phi(\mathcal{M})$ (i.e. for all $i \neq j \in \omega$ we have $\psi_i(\mathcal{M}) \subseteq \phi(\mathcal{M})$ and $\psi_i(\mathcal{M}) \cap \psi_j(\mathcal{M}) = \emptyset$), there are $i \in \omega$ and $\beta \in \mathrm{On}^{\pm\infty}$ with $\beta < \alpha$, such that $\mathrm{MR}^{\mathcal{M}}(\psi_i) \leq \beta$.
  - $\mathrm{MR}^{\mathcal{M}}(\phi) \leq +\infty$ for any ϕ;

- Define the **Morley rank** of an $\mathcal{L}(\mathcal{M})$-formula by $\mathrm{MR}(\phi) := \mathrm{MR}^{\mathcal{N}}(\phi)$, where $\mathcal{N} \succeq \mathcal{M}$ is any $\aleph_0$-saturated elementary extension of $\mathcal{M}$. (We prove in Lemma 16.3(ii) that this is well-defined.)

Remark 16.2.

- $\mathrm{MR}^{\mathcal{M}}(\phi(\overline{x})) = 0 \Leftrightarrow 0 < |\phi(\mathcal{M})| \in \omega \Leftrightarrow \mathrm{MR}(\phi(\overline{x})) = 0$.
- If $\phi(\overline{x})$ is minimal in $\mathcal{M}$, then $\mathrm{MR}^{\mathcal{M}}(\phi) = 1$.
- If $\phi(\overline{x})$ is strongly minimal in $\mathcal{M}$, then $\mathrm{MR}(\phi) = 1$.

Lemma 16.3. *(i) Let $\mathcal{M}, \mathcal{M}'$ be $\aleph_0$-saturated $\mathcal{L}$-structures. Let $\phi(\overline{x}, \overline{y})$ be an $\mathcal{L}$-formula. Let $\overline{a} \in \mathcal{M}^{|\overline{y}|}$ and $\overline{a}' \in (\mathcal{M}')^{|\overline{y}|}$ with* $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}'}(\overline{a}')$. *Then* $\mathrm{MR}^{\mathcal{M}}(\phi(\overline{x}, \overline{a})) = \mathrm{MR}^{\mathcal{M}'}(\phi(\overline{x}, \overline{a}'))$.

(ii) Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\mathcal{N}, \mathcal{N}' \succeq \mathcal{M}$ be $\aleph_0$-saturated elementary extensions. Let ϕ be an $\mathcal{L}(\mathcal{M})$-formula. Then $\mathrm{MR}^{\mathcal{N}}(\phi) = \mathrm{MR}^{\mathcal{N}'}(\phi)$.

Hence $\mathrm{MR}(\phi)$ is well-defined.

Proof. (i) By the definition of $\mathrm{MR}^{\mathcal{M}}$, it suffices to show:

Claim. *Let $\alpha \in \mathrm{On}^{\pm\infty}$. Suppose $\mathrm{MR}^{\mathcal{M}}(\phi(\overline{x}, \overline{a})) \leq \alpha$. Then $\mathrm{MR}^{\mathcal{M}'}(\phi(\overline{x}, \overline{a}') \leq \alpha$.*

Proof. By induction on α. If $\alpha = -\infty$, then $\mathcal{M}' \vDash \neg\exists\overline{x}.\phi(\overline{x}, \overline{a}')$ since $\mathcal{M} \vDash \neg\exists\overline{x}.\phi(\overline{x}, \overline{a})$ and $\overline{a} \equiv \overline{a}'$.

Let $\alpha \in \mathrm{On}$. Suppose there are $\psi_i(\overline{x}, \overline{c}_i')$ for $i \in \omega$ with $\psi_i(\overline{x}, \overline{y}_i)$ an $\mathcal{L}$-formula and $\overline{c}_i' \in (\mathcal{M}')^{|\overline{y}_i|}$, such that $\psi_i(\mathcal{M}', \overline{c}_i')$ are disjoint subsets of $\phi(\mathcal{M}', \overline{a}')$. By $\aleph_0$-saturation and $\overline{a} \equiv \overline{a}'$, we can recursively find $\overline{c}_i \in \mathcal{M}^{|y_i|}$ such that $\overline{a}, \overline{c}_0, \ldots, \overline{c}_n \equiv \overline{a}', \overline{c}_0', \ldots, \overline{c}_n'$ (for all $n \in \omega$). Then $\psi_i(\mathcal{M}, \overline{c}_i)$ are disjoint subsets of $\phi(\mathcal{M}, \overline{a})$, so for some $i \in \omega$ and $\beta < \alpha$ we have $\mathrm{MR}^{\mathcal{M}}(\psi_i(\mathcal{M}, \overline{c}_i)) \leq \beta$. But $\overline{c}_i \equiv \overline{c}_i'$, so by the inductive hypothesis we have $\mathrm{MR}^{\mathcal{M}'}(\psi_i(\mathcal{M}, \overline{c}_i')) \leq \beta$. So $\mathrm{MR}^{\mathcal{M}'}(\phi(\overline{x}, \overline{a}') \leq \alpha$.

For $\alpha = +\infty$ the claim is clear. □

(ii) Say $\phi = \phi(\overline{x}, \overline{a})$, where $\phi(\overline{x}, \overline{y})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M}^{|\overline{y}|}$. Then $\mathrm{tp}^{\mathcal{N}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}'}(\overline{a})$, so we conclude by (i).

□

Lemma 16.4. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let ϕ, ϕ' be $\mathcal{L}(\mathcal{M})$-formulas.*

(i) $\phi(\mathcal{M}) \subseteq \phi'(\mathcal{M}) \Rightarrow \mathrm{MR}(\phi) \leq \mathrm{MR}(\phi')$.

(ii) $\mathrm{MR}(\phi \vee \phi') = \max\{\mathrm{MR}(\phi), \mathrm{MR}(\phi')\}$.

Proof. (i) This follows directly from the definitions.

(ii) By (i), it suffices to see $\leq$. We show this by induction on α.

So suppose $\mathrm{MR}(\phi), \mathrm{MR}(\phi') \leq \alpha \in \mathrm{On}$, and we show $\mathrm{MR}(\phi \vee \phi') \leq \alpha$. So let $\mathcal{N} \succeq \mathcal{M}$ be $\aleph_0$-saturated, and say $(\psi_i)_{i\in\omega}$ are $\mathcal{L}(\mathcal{N})$-formulas such that $(\psi_i(\mathcal{N}))_i$ are disjoint subsets of $\phi(\mathcal{N}) \cup \phi'(\mathcal{N})$. Then $\{i \in \omega : \mathrm{MR}(\phi \wedge \psi_i) \geq \mathrm{MR}(\phi)\}$ is finite, and similarly for ϕ'. So there is $i \in \omega$ such that $\mathrm{MR}(\phi \wedge \psi_i) < \mathrm{MR}(\phi) \leq \alpha$ and $\mathrm{MR}(\phi' \wedge \psi_i) < \mathrm{MR}(\phi') \leq \alpha$, so by the inductive hypothesis,

$$\mathrm{MR}(\psi_i) = \mathrm{MR}((\phi \wedge \psi_i) \vee (\phi' \wedge \psi_i)) < \alpha.$$

□

16.1 Morley degree

Let $\mathcal{M}$ be an $\aleph_0$-saturated $\mathcal{L}$-structure, and let $T := \mathrm{Th}(\mathcal{M})$.

Lemma 16.5. *Let ϕ be an $\mathcal{L}(\mathcal{M})$ formula. If $\mathrm{MR}(\phi) \geq (2^{|T|})^+$, then $\mathrm{MR}(\phi) = +\infty$.*

Proof. Let $\alpha \in \mathrm{On}$ be minimal such that no $\mathcal{L}(\mathcal{M})$-formula has rank α. Then by transfinite induction, no formula has ordinal rank $\geq \alpha$. So $|\alpha|$ is the number of ordinal ranks attained by $\mathcal{L}(\mathcal{M})$-formulas. Now by Lemma 16.3(i), $\mathrm{MR}(\phi(\overline{x},\overline{a}))$ depends only on the $\mathcal{L}$-formula $\phi(\overline{x},\overline{y})$ and $\mathrm{tp}(\overline{a})$. So $|\alpha| \leq |T|\cdot|S(\emptyset)| \leq 2^{|T|}$. □

Definition 16.6. For $\alpha \in \mathrm{On}$, say $\mathcal{L}(\mathcal{M})$-formulas ϕ and ϕ' are **α-equivalent**, $\phi \approx_\alpha \phi'$, if $\mathrm{MR}(\phi\Delta\phi') < \alpha$.

Lemma 16.7. *$\approx_\alpha$ is an equivalence relation.*

Proof. Reflexivity and symmetry are clear. Transitivity follows from Lemma 16.4(ii) and the logical tautology

$$(\phi\Delta\phi'') \rightarrow ((\phi\Delta\phi') \vee (\phi'\Delta\phi'')).$$

□

Definition 16.8. An $\mathcal{L}(\mathcal{M})$-formula ϕ is **α-strongly-minimal** if $\mathrm{MR}(\phi) = \alpha \in \mathrm{On}$ and for every $\mathcal{L}(\mathcal{M})$-formula ψ, either $\mathrm{MR}(\phi \wedge \psi) < \alpha$ or $\mathrm{MR}(\phi \wedge \neg\psi) < \alpha$.

Lemma 16.9. *Let ϕ be an $\mathcal{L}(\mathcal{M})$-formula. If $MR(\phi) = \alpha \in \mathrm{On}$, there are $d \in \omega$ and α-strongly-minimal $\mathcal{L}(\mathcal{M})$-formulas $\psi_1, \ldots, \psi_d$ such that $\phi \leftrightarrow_T \bigvee_i \psi_i$ and $\psi_i \vdash_T \neg\psi_j$ for $i \neq j$.*

This number d is uniquely determined. The ψ_i are unique up to $\approx_\alpha$.

Proof. Suppose ϕ admits no such decomposition. In particular ϕ is not α-strongly-minimal, so say $\mathrm{MR}(\phi \wedge \psi) = \alpha = \mathrm{MR}(\phi \wedge \neg\psi)$. If both $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ admitted such a decomposition, then so would ϕ; so at least one does not. Continuing in this way, we obtain an infinite family of disjoint $MR = \alpha$ subsets of $\phi(\mathcal{M})$, contradicting $\mathrm{MR}(\phi) = \alpha$.

For the uniqueness: if ψ' is α-strongly-minimal and $\psi' \vdash_T \phi$, then by Lemma 16.4(ii) $MR(\psi' \wedge \psi_i) = \alpha$ for some i, and then $\psi' \approx_\alpha (\psi' \wedge \psi_i) \approx_\alpha \psi_i$. So up to $\approx_\alpha$, the ψ_i are precisely the α-strongly-minimal formulas implying ϕ. □

Definition 16.10. Let ϕ be an $\mathcal{L}(\mathcal{M})$-formula.

If $\mathrm{MR}(\phi) \in \mathrm{On}$, the **Morley degree** $\mathrm{MD}(\phi)$ is the number d in Lemma 16.9. If $\mathrm{MR}(\phi) \in \{-\infty, +\infty\}$, set $\mathrm{MD}(\phi) := 0$.

By Lemma 16.3(i), $\mathrm{MD}(\phi(\overline{x},\overline{a}))$ depends only on $\mathrm{tp}(\overline{a})$, and in particular not on the choice of $\aleph_0$-saturated model $\mathcal{M}$.

Set $\mathrm{MRD}(\phi) := (\mathrm{MR}(\phi), \mathrm{MD}(\phi)) \in \mathrm{On}^{\pm\infty} \times \omega$. We consider $\mathrm{On}^{\pm\infty} \times \omega$ as a well-ordered set with the lexicographic order .

Remark 16.11. ϕ is strongly minimal iff $\mathrm{MRD}(\phi) = (1, 1)$.

Lemma 16.12. *Let ϕ, ϕ' be $\mathcal{L}(\mathcal{M})$-formulas with $\phi(\mathcal{M}) \cap \phi'(\mathcal{M}) = \emptyset$. Then*

$$\mathrm{MRD}(\phi \vee \phi') = \begin{cases} \max(\mathrm{MRD}(\phi), \mathrm{MRD}(\phi')) & (\mathrm{MR}(\phi) \neq \mathrm{MR}(\phi')) \\ (\mathrm{MR}(\phi), \mathrm{MD}(\phi) + \mathrm{MD}(\phi')) & (\mathrm{MR}(\phi) = \mathrm{MR}(\phi')) \end{cases}.$$

Proof. Exercise. □

Proposition 16.13. *T is totally transcendental iff $\mathrm{MR}(\phi) < +\infty$ for all $\mathcal{L}(\mathcal{M})$-formulas ϕ.*

Proof. $\Rightarrow$: Suppose $\mathrm{MR}(\phi) = +\infty$. Then $\mathrm{MR}(\phi) > (2^{|T|})^+$, so ϕ splits into $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ each of rank $\geq (2^{|T|})^+$, and hence by Lemma 16.5 of rank $+\infty$. So we obtain a binary tree contradicting total transcendence.

$\Leftarrow$: Suppose $(\phi_s)_{s\in 2^{<\omega}}$ is a binary tree of $\mathcal{L}(\mathcal{M})$-formulas. Say $\mathrm{MRD}(\phi_s) = \inf_s \mathrm{MRD}(\phi_s)$. By assumption, $\mathrm{MR}(\phi_s) \in \mathrm{On}$. But then $\mathrm{MRD}(\phi_s) \geq \mathrm{MRD}(\phi_{s0} \vee \phi_{s1})$, and Lemma 16.12 yields a contradiction.

□

Definition 16.14. If X is a $\mathcal{M}$-definable set, i.e. $X = \phi(\mathcal{M})$ for some $\mathcal{L}(\mathcal{M})$-formula ϕ, we set $\mathrm{MRD}(X) := \mathrm{MRD}(\phi)$.

Lemma 16.15. *Let $f : X \to Y$ be an $\mathcal{M}$-definable bijection of $\mathcal{M}$-definable sets. Then $MRD(X) = \mathrm{MRD}(Y)$.*

Proof. By induction on MRD. Exercise. □

Proposition 16.16 (Baldwin-Saxl, tt case)**.** *Let $(G;\cdot)$ be a totally transcendental group. Then there is no infinite descending chain of G-definable subgroups $G = G_0 > G_1 > \ldots$.*

Proof. Suppose $(G_i)_i$ is such. Each G_i contains G_{i+1} and a distinct (hence disjoint) coset g_iG_{i+1}, and $\mathrm{MRD}(g_iG_{i+1}) = \mathrm{MRD}(G_{i+1})$ since $x \mapsto g_ix$ is a definable bijection. Now $\mathrm{MR}(G_i) < \infty$ by Proposition 16.13, so by Lemma 16.12, $\mathrm{MRD}(G_i) > \mathrm{MRD}(G_{i+1})$. So we contradict well-orderedness of $\mathrm{On}^{\pm\infty} \times \omega$. □

Fact 16.17 (Macintyre)**.** *Any totally transcendental field $(K;+,\cdot)$ is algebraically closed.*

Conjecture 16.18 (Cherlin-Zilber Algebraicity Conjecture)**.** *If $(G;\cdot)$ is an infinite simple group with $\mathrm{MR}^{(G;\cdot)}(G) < \omega$, then G is an algebraic group over an algebraically closed field.*

Definition 16.19. Let $A \subseteq \mathcal{M}$.

- For $p \in S(A)$, $\mathrm{MRD}(p) := \inf_{\phi\in p} \mathrm{MRD}(\phi)$.
- For $a \in \mathcal{M}$, $\mathrm{MRD}(a/A) := \mathrm{MRD}(\mathrm{tp}(a/A))$.

Lemma 16.20. *Let ϕ be a consistent $\mathcal{L}(\mathcal{M})$-formula.*

(i) $\mathrm{MR}(\phi) = \max\{\mathrm{MR}(p) : \phi \in p \in S(\mathcal{M})\}$.

(ii) If $\mathrm{MR}(\phi) \in \mathrm{On}$*, then there are precisely* $\mathrm{MD}(\phi)$ *types* $\phi \in p_i \in S(\mathcal{M})$ *with* $\mathrm{MR}(p_i) = \mathrm{MR}(\phi)$.

Proof. (i) It suffices to find $p \in S(\mathcal{M})$ with $\mathrm{MR}(p) = \mathrm{MR}(\phi)$. If $\mathrm{MR}(\phi) = +\infty$, any $p \ni \phi$ has $\mathrm{MR}(p) = +\infty$. Otherwise, let $\phi' \vdash \phi$ with $\mathrm{MRD}(\phi') = (\mathrm{MR}(\phi), 1)$. Then $p_{\phi'} := \{\psi : \mathrm{MR}(\phi' \wedge \psi) = \mathrm{MR}(\phi')\}$ is complete and $\mathrm{MR}(p) = \mathrm{MR}(\phi') = \mathrm{MR}(\phi)$.

(ii) By Lemma 16.9, any such type p_i contains some ϕ' as in (i), and then $p_i = p_{\phi'}$.

□

Lemma 16.21. *Let $A \subseteq \mathcal{M} \vDash T$. If $\bar{b} \in \mathrm{acl}(A \cup \bar{a})$, then* $\mathrm{MR}(\bar{b}/A) \leq \mathrm{MR}(\bar{a}/A)$.

Proof. By induction on $\alpha := \mathrm{MR}(\overline{a}/A)$. The result is immediate if $\alpha = +\infty$, so suppose $\alpha \in \mathrm{On}$.

Say $\psi(\overline{x},\overline{y}) \in \mathrm{tp}(\overline{a},\overline{b}/A)$ with $\vDash \forall \overline{x}.\ \exists^{\leq d}\overline{y}.\ \psi(\overline{x},\overline{y})$ and $\mathrm{MR}(\exists \overline{y}.\ \psi(\overline{x},\overline{y})) = \alpha$.

Suppose $(\delta_i(\overline{y}))_{i\in\omega}$ are $\mathcal{L}(\mathcal{M})$-formulas defining disjoint subsets of $\exists\overline{x}.\psi(\overline{x},\overline{y})$.

Let $\epsilon_i(\overline{x}) := \exists\overline{y}.\ (\psi(\overline{x},\overline{y}) \wedge \delta_i(\overline{y}))$. The conjunction of any $d+1$ of the ϵ_i is inconsistent, and it follows that $\mathrm{MR}(\epsilon_{i_0}) < \alpha$ for some i_0; indeed, otherwise each ϵ_i is in at least one of the finitely many $\mathrm{MR} = \alpha$ types on $\exists\overline{y}.\ \psi(\overline{x},\overline{y})$ given by Lemma 16.20(ii), so one contains infinitely many ϵ_i, contradicting the inconsistency of any $d+1$ of them.

Now say $A' \subseteq_{\mathrm{fin}} \mathcal{M}$ is such that ψ and δ_{i_0} are both $\mathcal{L}(A')$-formulas. By Lemma 16.20(i) and $\aleph_0$-saturation, let $\overline{b}' \in \delta_{i_0}(\mathcal{M})$ be such that $\mathrm{MR}(\overline{b}'/A') = \mathrm{MR}(\delta_{i_0})$. Then there is $\overline{a}' \in \epsilon_{i_0}(\mathcal{M})$ such that $\mathcal{M} \vDash \psi(\overline{a}',\overline{b}')$. So by the inductive assumption, $\mathrm{MR}(\delta_{i_0}) = \mathrm{MR}(\overline{b}'/A') \leq \mathrm{MR}(\overline{a}'/A') < \alpha$. Hence $\mathrm{MR}(\overline{b}/A) \leq \mathrm{MR}(\exists\overline{x}.\ \psi(\overline{x},\overline{y})) \leq \alpha$, as required. □

16.2 Morley rank in a strongly minimal theory

Let T be a strongly minimal theory. Let $\mathcal{M} \vDash T$ be $\aleph_0$-saturated.

Definition 16.22. Let $A \subseteq \mathcal{M}$ and $\overline{a} \in \mathcal{M}^{<\omega}$. Then $\dim(\overline{a}/A)$ is the maximal n such that $\overline{a}' \vDash \mathfrak{p}_A^{(n)}$ for some subtuple $\overline{a}'$ of $\overline{a}$.

Theorem 16.23. *Let $A \subseteq \mathcal{M}$ and $\overline{a} \in \mathcal{M}^{<\omega}$. Then $\mathrm{MR}(\overline{a}/A) = \dim(\overline{a}/A)$.*

Proof. Let $n := \dim(\overline{a}/A)$, and let $\overline{a}'$ be a subtuple with $\overline{a}' \vDash \mathfrak{p}_A^{(n)}$. Then by the maximality, $a_i \in \mathrm{acl}(A \cup \overline{a}')$ for all i. So $\mathrm{acl}(A \cup \overline{a}') = \mathrm{acl}(A \cup \overline{a})$, and so by Lemma 16.21 $\mathrm{MR}(\overline{a}/A) = \mathrm{MR}(\overline{a}'/A)$. So we may assume $\overline{a} = \overline{a}' \vDash \mathfrak{p}_A^{(n)}$ and $n > 0$. We may inductively assume that $\mathrm{MR}(\overline{a}/Aa_1) = \dim(\overline{a}/Aa_1) = n-1$.

Let $\psi(\overline{x}) \in \mathfrak{p}_A^{(n)}$. Set $\psi'(\overline{x},y) := (\psi(\overline{x}) \wedge x_1 \doteq y)$. Then $\overline{a} \vDash \psi'(\overline{x},a_1)$, so $\mathrm{MR}(\psi'(\overline{x},a_1)) \geq n-1$. By Lemma 16.3(i), $\mathrm{MR}(\psi'(\overline{x},a_1')) \geq n-1$ for any $a_1' \vDash \mathfrak{p}_A$; these formulas are pairwise inconsistent, and so $\mathrm{MR}(\psi) \geq n$.

By the result for $m < n$, we have inductively that for every $B \subseteq \mathcal{M}' \vDash T$, every type $p \in S_n(B)$ with $p \neq \mathfrak{p}_B^{(n)}$ has rank $\mathrm{MR}(p) < n$.

Let $\theta(\overline{x})$ be an $\mathcal{L}(\mathcal{M})$-formula with $\theta \vdash \psi$ and $\mathrm{MR}(\theta) = n$. By Lemma 16.20(i) applied to θ and the previous paragraph, we have $\mathrm{MR}(\mathfrak{p}_{\mathcal{M}}^{(n)}) = n$. Then again by Lemma 16.20(i) and the previous paragraph, $\mathrm{MR}(\overline{x} \doteq \overline{x}) = n$. So $\mathrm{MR}(\psi) = n$. Hence $\mathrm{MR}(\mathfrak{p}_A^{(n)}) = n$. □

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