TYPES IN CONTINUOUS LOGIC

for $\aleph_0$-categorical fmla $\Leftrightarrow$ continuous + Aut(M)-invariant

Def: $\mathrm{Def}(M) = \{ f(a,y) : M \to \mathbb{R} \mid f(x,y) \text{ formula}, a \in M^\omega \} \subseteq \mathcal{C}(M)$

definable predicates

If $a \in A^\omega$ for some $A \subseteq M$, y is a variable of length β, $f(x,y) : M^\omega \times M^\beta \to \mathbb{R}$ a fml. Then f_a is also called an A-definable predicate in the variable y.

Def: M metric structure, $A \subseteq M$ and y a variable of length β. A (complete) type over A in the variable y can be defined as a maximal ideal in the uniformly closed algebra of A-definable predicates in the variable y.
The type over A of $b \in M^\beta$ is defined by

$$tp(b/A) = \{ f_a \mid a \in A^\omega, f(x,y) \text{ fla. with } f(a,b) = 0 \}$$

Def: We say that a tuple $b \in M^\beta$ realizes a type $p \in S_y^M(A)$ if $tp(b/A) = p$.

Recall: M is $\emptyset$-saturated if M realizes all $f \in S_\omega^M(\emptyset)$

Fact: $\aleph_0$-categorical $\Rightarrow$ $\emptyset$-saturated

Def: $A \subseteq M$ is said to be definable if the predicate $P_A(x) = d(x, A)$ is definable.

Def: Given a G-space X,

$$RUC(X) = \{ f : X \to \mathbb{R} \mid \forall \varepsilon > 0\ \exists U_\varepsilon \in \mathcal{N}_{1_G} \mid \|gf - f\| < \varepsilon\ \forall g \in U_\varepsilon \}$$

If X is a metric space,

$$RUC_u(X) = RUC(X) \cap \{ f : X \to \mathbb{R} \mid f \text{ is unif. continuous} \}$$

Def: $\mathrm{Def}(M) = \{ f(a, y) : M \to \mathbb{R} \mid f(x,y) \text{ formula}, a \in M^\omega \} \subseteq \mathcal{C}(M)$

Lemma 1: M an $\aleph_0$-categorical metric struct, $G = \mathrm{Aut}(M)$.
Then $\mathrm{Def}(M) = RUC_u(M)$.

Proof: $\boxed{\subseteq}$ actually holds in all metric structures:
Let $f(x,y)$ be a formula and $a \in M^\omega$ a parameter. Let $\Delta_f^{\mathbb{R} \to \mathbb{R}}$ be the modulus of unif. continuity for f and take $U_\varepsilon \in \mathcal{N}_{1_G}$ s.t. $d(a, ga) < \Delta_f(\varepsilon)$ for $g \in U_\varepsilon$. So

$$\underbrace{\| g f_a - f_a \|}_{f_a(y) = f(a,y)} = \| f_{ga} - f_a \| < \varepsilon \quad \text{for all } g \in U_\varepsilon.$$

This shows that all $f \in \mathrm{Def}(M)$ are also in $RUC_u(M)$.

$\boxed{\supseteq}$ Let $h \in RUC_u(M)$ and let $a \in M^\omega$ enumerate a ctbl. dense subset of M. Define

$$f : Ga \times M \longrightarrow M \quad \text{by}$$
$$(ga, b) \longmapsto gh(b) = h(g^{-1}b).$$

$ga = g'a$

$\Downarrow$

$g = g'$ by density

This is well defined because a is dense in M

f is G-invariant (just compute it) and unif. cont.

Indeed

$$|f(ga,b) - f(g'a,b')| \leq \underbrace{|gh(b) - gh(b')|}_{} + \underbrace{|gh(b') - g'h(b')|}_{}$$

This is small if $b \approx b'$ since $d(g^{-1}b, g^{-1}b') = d(b,b')$ + unif. cont. of h.

Given $\varepsilon > 0$ we can find $U_\varepsilon \in \mathcal{N}_{1_G}$ s.t. $\|gh - g'h\| < \varepsilon$ whenever $g^{-1}g' \in U_\varepsilon$, since $h \in RUC(M)$. Now since a is dense we can find $\delta > 0$ s.t. $d(ga, g'a) < \delta \Rightarrow g^{-1}g' \in U_\varepsilon$.

Thus $d(ga, g'a) < \delta \Rightarrow |gh(b') - g'h(b')| < \varepsilon$

$\Longrightarrow$ We can control the second term as well and f is unif. continuous.

Now that we have unif. continuity we can extend f to $[a] \times M$ $(= \overline{Ga} \times M)$, and the extension is still G-invariant, so we can regard f as defined on $([a] \times M) /\!/ G \overset{\text{closed}}{\subseteq} (M^\omega \times M) /\!/ G$.

By Tietze we get a continuous extension $\tilde{f} : (M^\omega \times M) /\!/ G \to \mathbb{R}$ and now $h(x,y) : M^\omega \times M \to \mathbb{R}$ defined by $F = \tilde{f} \circ \tilde{\pi}$ is such that $F_a = h$. Since M is $\aleph_0$-categorical the G-invariant function f is in fact a formula, thus $h \in Def(M)$. $\blacksquare$

Prop / Def (Already in Martin's talk)

Let M be $\emptyset$-saturated, $f: M^\alpha \times M^\beta \to \mathbb{R}$ a formula, $A \subseteq M^\alpha$, $B \subseteq M^\beta$ be $\emptyset$-definable. If $M \preceq \tilde{M}$ we write $\tilde{A}, \tilde{B}, \tilde{f}$ for the interpretations in $\tilde{M}$. TFAE:

1) $\nexists\, r \neq s \in \mathbb{R}$, $(a_i)_{i<\omega} \subset A$, $(b_I)_{I \subseteq_{fin} \omega} \subset \tilde{B}$

such that
$$\tilde{f}(a_i, b_I) = \begin{cases} r & i \in I \\ s & i \notin I \end{cases}$$

2) For every indiscernible sequence $(a_i)_{i<\omega} \subset A$ and $b \in \tilde{B}$, $\lim_i \tilde{f}(a_i, b)$ exists.

3) Every sequence $(a_i)_{i<\omega} \subset A$ admits a subsequence $(a_{i_j})_{j<\omega}$ s.t.
$$\lim_j \tilde{f}(a_{i_j}, b) \text{ exists } \forall b \in \tilde{B}$$

We say that f is NIP on $A \times B$ if any of those conditions is satisfied.

Corollary: Let M be $\emptyset$-saturated and $A \subseteq M^\alpha$, $B \subseteq M^\beta$ be $\emptyset$-definable. Then f is NIP on $A \times B$ iff $\{\tilde{f}_a|_{B^*} \mid a \in A\}$ is seq. precompact in $\mathbb{R}^{B^*}$, where $\tilde{f}_a|_{B^*}$ is the extension of f_a to
$$B^* = \{p \in S_y(M) \mid P_B \in p\}$$

definable predicate that coincides with $d(x, B)$ on M^β

If $A' \subset A$ is dense, it is enough to check that $\{\tilde{f}_a|_{B^*} \mid a \in A'\}$ is seq. precompact in $\mathbb{R}^{B^*}$

Fact: Let Y be a compact G-space. Then $f \in \mathcal{C}(Y)$ is tame iff $Gf \subseteq \mathbb{R}^Y$ is seq. precompact.

If Y is not compact we say that f is tame if its extension to one (any) G-compactification of Y is tame

We finally obtain the connection between Tame and NIP:

Proposition: Let M be an $\aleph_0$-categorical structure. Then $h \in \mathrm{Tame}_u(M)$ iff $h = f_a$ for a formula $f(x,y)$ that is NIP on $[a] \times M$. More generally if $f(x,y)$ is a formula, $a \in M^\alpha$, $B \subset M^\beta$ is definable, we have $f_a|_B \in \mathrm{Tame}(B) \Leftrightarrow f(x,y)$ is NIP on $[a] \times B$

Proof: The first fact follows from the second by lemma 1: For fixed $f(x,y)$, a and B, the function $f_a \in \mathrm{RUC}_u(B)$ is tame iff its extension to $\overline{B}$ is tame, where $\overline{B}$ is the closure of B in $S_y(M)$ under the compactification $M^\beta \longrightarrow S_y(M)$.
Now take $A' = Ga$ and use the corollary, after showing that $\overline{B} = B^*$.

$\boxed{\overline{B} \subseteq B^*}$ is easy: if $b \in \overline{B}$, since B is definable, $P_B(b) = 0$.

$\boxed{B^* \subseteq \overline{B}}$: fix $p \in B^*$ and let b realize p in a separable elementary extension $M' \preceq M$.

Let $\varphi(z,y)$ be a formula, $c \in M^{|z|}$ and $\varepsilon > 0$.

By $\aleph_0$-categoricity, since M' is separable, there is an isomorphism $\sigma : M' \longrightarrow M$. Then $tp(c) = tp(\sigma c)$, which means by homogeneity, that there is $g \in Aut(M)$ with $d(c, g\sigma c) < \Delta_\varphi(\varepsilon)$. Hence $g\sigma b \in B$ and

Why?

modulus of unif. continuity

$$|\varphi(c,b) - \varphi(c, g\sigma b)| < \varepsilon \Rightarrow p \in \overline{B}$$

Since M is $\aleph_0$-categorical the projection $M^\beta \longrightarrow M^\beta /\!/ G$ is a compactification, and the functions coming from it are the continuous G-invariant ones, that is the $\emptyset$-definable predicates. We can thus identify $M^\beta \longrightarrow M^\beta /\!/ G$ with $tp : M^\beta \longrightarrow S_y^M(\emptyset)$.

This gives the following homogeneity property: if $tp(a) = tp(b)$, for $a, b \in M^\beta$ and $\varepsilon > 0$, there is $g \in Aut(M)$ with $d(a, gb) < \varepsilon$.