# NOTES ON HRUSHOVSKI'S "DEFINABILITY PATTERNS AND THEIR SYMMETRIES"

PIERRE SIMON

Those are notes written for the Berkeley model theory seminar, in which we read Hrushovski's paper. They are based on the talks I gave there.

I give a slightly different presentation of the paper and introduce my own terminology. The difference is that whereas Hrushovski considers a positive theory of patterns, we work here entirely in a space of types $S_x(M)$, which is a monster model (saturated and homogeneous) for that theory. So this is the same difference between working with the category of models of a complete theory $T$ and working only with elementary substructures of a fixed monster model. Here, it allows us to never mention positive logic. We just need to define a notion of morphism and work in the category of subsets of $S_x(M)$ equipped with those morphisms. This might make the constructions easier to understand, although I don't think it actually makes anything shorter. Hrushovski mentioned to me a few reasons why the point of view taken in the paper of defining the core as an e.c. model of a positive theory could be useful, one of them being that it makes it easier to talk about imaginary elements. For me, it is still easier to first understand the constructing inside $S_x(M)$, and then it is a relatively small step to see it as an e.c. model of a positive theory. I might include that at some point in those notes.

On the other hand, one advantage (for me) of this approach is that it makes it very similar to what is done it topological dynamics. In the $\omega$-categorical case, the construction is essentially that of a minimal ideal of the Ellis semi-group and seeing it in this way was for me a big help in making sense of it. I presented this in my first talk at the seminar, but it is not included in those notes. That might happen later.

I restrict here to the case of a complete first order theory and do not consider finite sets of formulas as Hrushovski does. Again, I might add some words about this in future versions.

Thanks to the participants of the seminar and to Krzysztof Krupinski for comments on those notes.

## 1. The pattern structure on type space and the core

Let $T$ be a complete first order theory in a language $L$, fixed throughout. Let $M$ be a model of $T$.

We define a relational structure on any type space $S_x(M)$, or more generally on products of type spaces $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$, where the $x_i$'s are variables ranging in possibly different sorts.

Let $\phi_1(x_1; t), \ldots, \phi_n(x_n; t), \theta(t)$ be $L$-formulas without parameters ($t$ a tuple of variables). Write $\bar{\phi} = (\phi_1, \ldots, \phi_n)$ and define $R_{(\bar{\phi},\theta)}$ to be the subset of $S_{x_1}(M) \times$

*Date*: April 22, 2020.

$\cdots \times S_{x_n}(M)$ consisting of tuples $(p_1, \ldots, p_n)$ for which there does not exist $b \in M^t$ with:

– $M \models \theta(b)$;
– $p_i \vdash \phi_i(x_i; b)$ for all $i \leq n$.

Some observations:

(1) We could do without $\theta$ and replace each $\phi_i$ by $\phi_i \wedge \theta$. However, it is more natural to have $\theta$ around and also we might later want to put restrictions on $\phi$ (such as being quantifier free) to define variations on this, while putting different, or no restrictions on $\theta$.
(2) If $\phi(x)$ is a formula over $\emptyset$, then the set of $p \in S_x(M)$ containing $\phi(x)$ is p-closed (even p-clopen).
(3) The union of two sets $R_\bullet$ is again of this form. More precisely $R_{(\bar{\phi},\theta)} \cup R_{(\bar{\phi}',\theta')} = R_{(\bar{\phi}'',\theta'')}$, with $\phi''_i(x_i; t_1\hat{\ }t_2) = \phi_i(x_i; t_1) \wedge \phi'_i(x_i; t_2)$ and $\theta''(t_1\hat{\ }t_2) = \theta(t_1) \wedge \theta(t_2)$.
(4) The set $R_{(\bar{\phi},\theta)}$ is a closed subset of $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$ (equipped with the product topology). It is invariant under all automorphisms of $M$.
(5) If $T$ is $\omega$-categorical and $M$ is the countable model, then any closed subsets of $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$ invariant under automorphisms of $M$ is a finite intersection of sets of the form $R_{(\bar{\phi},\theta)}$. I like to think of the sets $R_{(\bar{\phi},\theta)}$ as being in some sense the *0-definable* closed subset of $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$.

We will call a subset of the form $R_{(\bar{\phi},\theta)}$ an *atomic p-closed set*. A *p-closed set* is an intersection of atomic p-closed sets. By observation 3 above, p-closed sets form a topology on $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$, though that will not be very important. If we say closed or open, without the p-, then we always mean according to the usual topology.

*Remark* 1.1. Note that an open subset $C \subseteq S_x(M)$ is p-open if and only if the following holds: whenever the clopen set defined by $\phi(x; b)$ is included in $C$, there is $\theta(t) \in \text{tp}(b)$ such that $C$ contains all types which contain a formula $\phi(x; b')$ with $b' \models \theta(t)$. (So this expresses that $C$ is $\emptyset$-invariant in a strong way.) There is a similar condition for p-open subsets of product of type spaces.

*Remark* 1.2. Equality, as a subset of $S_x(M)^2$ is p-closed as it is the intersection of the sets $R_{(\phi(x;t), \neg\phi(x';t); t=t)}$ for $\phi \in L$.

**Proposition 1.3.** *The projection of a p-closed set is p-closed.*

*Proof.* Say that $R \subseteq S_x(M) \times S_y(M)$ is p-closed and let $\pi$ be the projection to the first coordinate. By compactness, the projection of an intersection of closed set is the intersection of the projections. We can therefore assume that $R$ is an atomic p-closed set, say $R = R_{(\phi(x,t), \psi(y;t); \theta(t))}$.

Let $p \in S_x(M) \setminus \pi(R)$. As $\pi(R)$ is closed in the usual topology, there is a formula $\phi(x; b) \in L(M)$ such that no type containing that formula lies in $\pi(R)$. By compactness, there is a formula $\psi(y; b') \in L(M)$ such that any $(p', q') \in S_x(M) \times S_y(M)$ such that $p(x) \vdash \phi(x; b)$ and $q(y) \vdash \psi(y; b')$ lies in the complement of $R$. Since $R$ is p-closed, there is $\theta(t\hat{\ }t') \in \text{tp}(b\hat{\ }b')$ such that $R \subseteq R_{(\phi(x;b), \psi(y;b'); \theta(t\hat{\ }t'))}$.

Let
$$\phi'(x; t) = \phi(x; t) \wedge (\exists y, t')(\theta(t\hat{\ }t') \wedge \psi(y; t')).$$
Then $R_{(\phi'(x;t); \theta(t))}$ contains $\pi(R)$ and does not contain $p$. This shows that $\pi(R)$ is p-closed. $\square$

In fact, more generally, we can project a p-closed set from a set of types in several variables.

**Proposition 1.4.** *The image of a p-closed set under the canonical map $S_{x_1,\dots,x_n}(M) \to S_{x_1}(M) \times \cdots \times S_{x_n}(M)$ is p-closed.*

*Proof.* The argument is the same as that for the previous proposition. □

Fix a variable $x$ and we define a new relational language $L_p(x)$. For each tuple $\bar{\phi} = (\phi_1(x;t), \dots, \phi_n(x;t))$ and formula $\theta(t)$ (all without parameters), let $R_{(\bar{\phi};\theta)}(x_1, \dots, x_n)$ be a relation symbol. Let $L_p(x)$ consist of all those symbols as the formulas $\phi_i$ and $\theta$ vary in $L$. The space $S_x(M)$ is made into an $L_p(x)$-structure with the natural interpretation. We can similarly define a language for a product of type spaces $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$. From now on, we will write $L_p(x)$ as just $L_p$, the variable being deduced from the context.

If $A, B$ are $L_p$-structures (in particular, subsets of $S(M)$), a *morphism* from $A$ to $B$ is a map $f : A \to B$ which is an $L_p$-morphism: if some relation $R_\bullet(\bar{a})$ holds for $\bar{a} \in A$, then also $R_\bullet(f(\bar{a}))$ holds. (Note that we do not say p-morphism, as those are the only kind of morphisms we will consider.) Morphisms need not be injective.

In the case of interest to us, where $A, B \subseteq S(M)$, we have that $f : A \to B$ is a morphism if and only if: for any finite $\bar{a} \in A^n$ and $R \subseteq S_x(M)^n$ p-closed, if $\bar{a} \in R$, then $f(\bar{a}) \in R$. Hence, one might want to think of morphisms as a form of *specializations*, since they preserve p-closed sets.

Given $\bar{a} \in S_x(M)^n$, defines the *locus* of $\bar{a}$, written $\operatorname{loc}(\bar{a})$, as the intersection of all the p-closed subsets of $S_x(M)^n$ containing $\bar{a}$ (equivalently, the smallest p-closed subset containing $\bar{a}$). A map $f : A \to B$ is a morphism

*Remark* 1.5. Let $A, B \subseteq S_x(M)$ and let $f : A \to B$ be a map. Fix an enumeration $\bar{a}$ of $A$. So $\bar{a}$ lies in some $S_x(M)^I$, for some index set $I$ (or we could use $A$ itself as index set). Then $f$ is a morphism if and only if $\operatorname{loc}(f(\bar{a})) \subseteq \operatorname{loc}(\bar{a})$, where $f(\bar{a})$ lies in the same product space. Equivalently, $f$ is a morphism if and only if we have $\operatorname{loc}(f(\bar{a}')) \subseteq \operatorname{loc}(f(\bar{a}))$ for every *finite* tuple $\bar{a}'$ from $A$.

*Remark* 1.6. Morphisms preserve types over $\emptyset$: more precisely, let $f : A \to S_x(M)$ be a morphism and $p \in A$, then the restriction of $p$ and $f(p)$ to $\emptyset$ are equal. This follows from observation 2 above.

**Proposition 1.7.** *The $L_p$-structure $S_x(M)$ is homogeneous for morphisms: if $A \subseteq M$ and $f : A \to S_x(M)$ is a morphism, then $f$ extends to an endomorphism of $S_x(M)$.*

*Proof.* Let $p_* \in S_x(M)$ and say we want to extend $f$ to $A \cup \{p_*\}$. Let $\bar{a}$ be an enumeration of $A$, indexed by some $\alpha$. Let $R$ be the locus of $\bar{a}\hat{\ }p_*$ in $S_x(M)^\alpha \times S_x(M)$. Let $R'$ be the projection of $R$ to $S_x(M)^\alpha$. By Proposition 1.3, $R'$ is p-closed. As $\bar{a} \in R'$, also $f(\bar{a}) \in R'$. By definition of $R'$, there is $q \in S_x(M)$ such that $f(\bar{a})\hat{\ }q \in R$. Then we can extend $f$ by sending $p_*$ to $q$. □

**Definition 1.8.** A subset $A \subseteq S_x(M)$ is *p-minimal* if any morphism $f : A \to S_x(M)$ is an $L_p$-isomorphism on its image.

Note in particular that any morphism defined on a p-minimal set is injective.

**Lemma 1.9.** *A subset $A \subseteq S_x(M)$ is p-minimal if and only if every finite $A_0 \subseteq A$ is p-minimal. In particular, an increasing union of p-minimal sets is p-minimal.*

*Proof.* It follows at once from the definitions that if every finite $A_0 \subseteq A$ is p-minimal, then $A$ is p-minimal, since if a map is not an isomorphism, there is a finite subset of the domain that witnesses it.

Conversely, assume that $A$ is p-minimal and let $A' \subseteq A$. If there is a morphism $f : A' \to S_x(M)$ that is not an $L_p$-isomorphism, then by Proposition 1.7 $f$ can be extended to an endomorphism of $S_x(M)$ which by restriction to $A$ contradicts p-minimality of $A$. □

*Remark* 1.10. Let $A \subseteq S_x(M)$ and let $\bar{a}$ be an enumeration of $A$ indexed by some $\alpha$. Then $A$ is p-minimal if and only if $\mathrm{loc}(\bar{a})$ is a minimal (under inclusion) non-empty p-closed subset of $S_x(M)^\alpha$. Indeed, if there is $R \subsetneq \mathrm{loc}(\bar{a})$ p-closed and $\bar{a}' \in R$, then the map sending $\bar{a}$ to $\bar{a}'$ defines a morphism on $A$ which is not an $L_p$-isomorphism.

We now state the main theorem, which defines the core of $T$. By a *retraction*, we mean a morphism which is the identity on its image.

**Theorem 1.11.** *There exists a p-minimal $J \subseteq S_x(M)$ and a retraction $f : S_x(M) \to J$. Furthermore, $J$ is unique up to $L_p$-isomorphism and its $L_p$-isomorphism type does not depend on the choice of the model $M$.*

We prove the theorem in several steps.

**Lemma 1.12.** *There exists a p-minimal $K \subseteq S_x(M)$ and a morphism $g : S_x(M) \to K$.*

*Proof.* Let $\bar{m}$ be an enumeration of $S_x(M)$, indexed by some ordinal $\lambda$. Let $R \subseteq S_x(M)^\lambda$ be a minimal non-empty p-closed set containing $\mathrm{loc}(\bar{m})$. This exists as p-closed sets are closed in the usual product topology and that topology is compact. Let $\bar{m}' \in R$. Write $\bar{m} = (m_i : i < \alpha)$ and $\bar{m}' = (m_i : i < \alpha)$. Then the map $f$ defined by sending $m_i$ to $m_i'$ is a morphism from $S_x(M)$ to $S_x(M)$. Its image $K$ is p-minimal. □

Let $J \subseteq S_x(M)$ be a p-minimal set, which is maximal under inclusion (amongst p-minimal subsets of $S_x(M)$). This exists by Lemma 1.9 and Zorn's lemma.

**Lemma 1.13.** *Let $f : J \to L$ be a morphism, with $L \subseteq S_x(M)$ p-minimal. Then $f$ is onto $L$ (and hence is an $L_p$-isomorphism).*

*Proof.* Assume not and let $J' \subseteq L$ be the image of $f$. Then $f : J \to J'$ is an $L_p$-isomorphism as $J$ is p-minimal. Let $g : J' \to J$ be the inverse of that map. By Proposition 1.7, $g$ extends to an endomorphism $\tilde{g}$ of $S_x(M)$. Let $L' = \tilde{g}(L)$. As $L$ is p-minimal, $\tilde{g}$ is injective on $L$ and $L'$ is isomorphic to $L$. Hence $L'$ strictly contains $J$ and is p-minimal. This contradicts maximality of $J$. □

Let now $g : S_x(M) \to K$ a morphism, with $K$ p-minimal. The restriction of $f$ to $J$ gives a map $\sigma := g|_J : J \to K$. By the previous lemma, it is an $L_p$-isomorphism. Hence $J$ is $L_p$-isomorphic to $K$ and this proves uniqueness of $J$. Let $f = \sigma^{-1} \circ g : S_x(M) \to J$. Then $f$ is a retraction from $S_x(M)$ to $J$.

It remains to prove independence from the choice of $M$.

**Lemma 1.14.** *Let $N \models T$, then there is a morphism $f : S_x(M) \to S_x(N)$.*

*Proof.* As $N$ and $M$ are elementarily equivalent, there is an index set $I$ and an utrafilter $\mathcal{U}$ on $I$ such that $N$ embeds elementarily in the utrapower $M^\mathcal{U}$ of $M$. For

$p \in S_x(M)$, we can define the ultrapower $p^{\mathcal{U}} \in S_x(M^{\mathcal{U}})$ as:

$$p^{\mathcal{U}} \vdash \phi(x;[a_i]) \iff \{i \in I : p \vdash \phi(x;a_i)\} \in \mathcal{U}.$$

Now the map $p \mapsto p^{\mathcal{U}}$ is easily seen to be a morphism from $S_x(M)$ to $S_x(M^{\mathcal{U}})$. Having identified $N$ with its image in $M^{\mathcal{U}}$, the restriction map $S_x(M^{\mathcal{U}}) \to S_x(N)$ is also a morphism (this follows at once from the definitions). Composing them, we get a morphism from $S_x(M)$ to $S_x(N)$. □

Now independence from the model is just a diagram-chasing argument.

Take morphisms $\sigma : S_x(M) \to S_x(N)$ and $\tau : S_x(N) \to S_x(M)$. Let $J_M$ and $J_N$ be the p-minimal sets constructed above for $M$ and $N$ respectively, and $f : S_x(M) \to J$, $g : S_x(N) \to J'$ be retractions. Let $J' = \sigma(J_M)$. Restricting $g$ to $J'$ gives a morphism $J' \to J_N$. Composing with $\tau$ and then with $f$ gives us a morphism $J' \to J$.

The composition $f \circ \sigma^{-1} \circ g \circ \sigma : S_x(M) \to S_x(M)$ maps $S_x(M)$ to $J$. Restricting to $J_M$, we see that it must be onto $J_M$ and an $L_p$-isomorphism on $J_M$. The image of $g \circ \sigma$ is a subset of $J_N$, hence $f(\sigma^{-1}(J_N)) = J_M$. Similarly, $g(\sigma(J_M)) = J_N$. Hence $f \circ \sigma^{-1}$ is an isomorphism on $J_N$ (otherwise $f \circ \sigma^{-1} \circ g \circ \sigma$ would not be an isomorphism on $J_M$). Therefore $J_N$ and $J_M$ are isomorphic.

This finishes the proof of the theorem.

**Definition 1.15.** The $L_p$-isomorphism type of the set $J$ given by the theorem is called the *core* of $T$, denoted core$(T)$.

We will write $\mathcal{J} = \text{core}(T)$, thinking of $\mathcal{J}$ as an $L_p$-structure and write $J$ for a subset of $S_x(M)$ isomorphic to $\mathcal{J}$ (equivalently, the image under a morphism $f : \mathcal{J} \to S_x(M)$).

Note that any $J' \subseteq S_x(M)$ isomorphic to the core satisfies the conclusion of the theorem: this follows from homogeneity (Proposition 1.7). We will refer to such a set as a core.

## 2. Properties of the core

2.1. **Types in the core.** Let $J \subseteq S_x(M)$ be a core.

**Lemma 2.1.** *Let $A \subseteq S_x(M)^n$. Then there exists $A' \subseteq J$, $L_p$-isomorphic to $A$ if and only if $A$ is p-minimal.*

*Proof.* If $A$ is p-minimal, then any retraction $f : S_x(M) \to J$ restricts to an $L_p$-isomorphism on $A$. This shows one direction; the converse follows from Lemma 1.9. □

**Lemma 2.2.** *The core is homogeneous: any morphism $f : A \to \mathcal{J}$, $A \subseteq \mathcal{J}$ extends to an ($L_p$-)automorphism of $\mathcal{J}$.*

*Proof.* Identify $\mathcal{J}$ with an image $J$ of it in $S_x(M)$. Then by homogeneity of $S_x(M)$, $f$ extends to an endomorphism $\tilde{f}$ of $S_x(M)$. Let $g : S_x(M) \to J$ be a retraction. Then $g \circ \tilde{f}$ sends $J$ to $J$ and extends $f$. Its restriction to $J$ is an automorphism. □

**Proposition 2.3.** *Let $p, q \in J$ with $p \neq q$. Then there is are formulas $\phi_1(x;t), \dots, \phi_n(x;t)$ and $\theta(t)$ such that there does not exist $b \in \theta(M)$ with*

$$p \vdash \phi_i(x;b) \leftrightarrow q \vdash \phi_i(x;b) \text{ for all } i.$$

*Proof.* As $\{p, q\}$ is p-minimal, $\mathrm{loc}(p, q) \subseteq S_x(M)^2$ is a minimal p-closed set and hence does not intersect the diagonal. Write $\mathrm{loc}(p, q) = R(x, x')$, thinking of it as a partial type in two variables. The following partial type over $M$ is inconsistent:

$$R(x, x') \wedge \mathrm{tp}(x/M) = \mathrm{tp}(x'/M).$$

By compactness, there is some finite $b \in M$ and formulas $\phi_1(x; t), \ldots, \phi_n(x; t)$ such that

$$R(x, x') \wedge \bigwedge_i \phi_i(x; b) \leftrightarrow \phi_i(x'; b)$$

is inconsistent. As $R$ is p-closed and using Remark 1.1 (on the complement of $R$), there is some $\theta(t) \in \mathrm{tp}(b)$ such that

$$R(x, x') \wedge \bigwedge_i \phi_i(x; b') \leftrightarrow \phi_i(x'; b')$$

is inconsistent for any $b' \in \theta(M)$. This gives what we want. □

2.1.1. *Definable types.*

**Proposition 2.4.** *Any type in $S_x(M)$ which is definable over $\mathrm{acl}^{eq}(\emptyset)$ is in $J$. Furthermore (at least if $M$ is sufficiently saturated), the $\mathrm{acl}^{eq}(\emptyset)$-definable types are precisely the types which are in all the cores of $S_x(M)$.*

*Proof.* If $p \in S_x(M)$ is definable over $\emptyset$, then $\{p\}$ is a p-closed set: it is defined as the intersection of the sets $R_{(\phi, \neg d_p \phi)}$, for all $\phi$, where

$$p \vdash \phi(x; b) \iff M \models d_p\phi(b).$$

Hence $\mathrm{loc}(p) = \{p\}$ is p-closed and minimal under inclusion (amongst non-empty p-closed sets). Therefore $J$ must intersect $\{p\}$ and hence contain $p$.

If $p$ is $\mathrm{acl}^{eq}(\emptyset)$-definable, then the set of conjugates of $p$ is minimal p-closed. (Some of this will be more clear later on.)

For the furthermore part, simply note that any type which is not $\mathrm{acl}^{eq}(\emptyset)$-definable has unboundedly many conjugates under the automorphism group of $M$ (if $M$ is homogeneous and saturated enough), hence cannot be in all the cores. Probably this also works for smaller $M$'s, but I didn't check. □

2.2. **The pp-topology.** Given a core $J \subseteq S_x(M)$, we have two induced topologies on $J$: the standard one and the p-topology. Only the p-topology survives to $\mathcal{J}$ (and its cartesian powers), since the standard topology might depend on the choice of embedding. The p-topology is not in general $T_1$ (recall that a topology is $T_1$ if singletons are closed). We now define a third topology which is $T_1$.

**Definition 2.5.** A subset $C(u) \subseteq \mathcal{J}$ is *pp-closed* if it is an intersection of sets of the form $R(u, q_1, \ldots, q_n)$, with $R \subseteq \mathcal{J}^{1+n}$ p-closed and $(q_1, \ldots, q_n) \in \mathcal{J}^n$.

We similarly define a pp-closed subset of $\mathcal{J}^k$ as an intersection of sets of the form $R(u_1, \ldots, u_k, q_1, \ldots, q_n)$ with $R \subseteq \mathcal{J}^{k+n}$ p-closed and $(q_1, \ldots, q_n) \in \mathcal{J}^n$.

In other words, a pp-closed set is a fiber of a p-closed set. One can check that the family of pp-closed sets is closed under finite unions (this follows from the fact that p-closed sets are closed under finite union and adding dummy variables), hence they are the closed sets of a topology. Note that similarly to the Zariski topology, the topology on $\mathcal{J}^2$ say is not defined as the product topology from $\mathcal{J}$. (This is also related to the fact that a 2-type is not a product of 1-types.)

Hrushovski really thinks of $\mathcal{J}$ as being a multi-sorted structure with each cartesian power being its own sort (and in fact, he has one sort for each finite set of formulas, but we do not do this here). This point of view that is useful to define the topology on $G$ below, but we will try to do things without explicitly taking that approach.

**Proposition 2.6.** *The pp-topology on $\mathcal{J}^n$ is compact and $T_1$.*

*Proof.* For simplicity of notations, assume $n = 1$. Let $p \in \mathcal{J}$. Since equality is p-closed in $\mathcal{J}^2$, the set $x = p$ is pp-closed by definition of the pp-topology. Hence $\mathcal{J}$ is $T_1$. (Note that equality in $\mathcal{J}^2$ is p-closed, but need not be closed in the product topology in $\mathcal{J}^2$, when $\mathcal{J}$ is equipped with the pp-topology. This is why we do not get that $\mathcal{J}$ is Hausdorff.)

Let $C_i(u)$ be a family of pp-closed subsets of $\mathcal{J}$ such that any finite subfamily has non-empty intersection. Without loss, $C_i(u) = R_i(u, \bar{q}_i)$ for some p-closed $R_i$ and tuple $\bar{q}_i \in \mathcal{J}$. Fix an image $J$ of $\mathcal{J}$ in $S_x(M)$ and identify them. Then we can see each $R_i(u, \bar{q}_i)$ as defining a closed subset of $S_x(M)$. By compactness of the standard topology, there is a type $p \in S_x(M)$ in the intersection $\bigcap_i R_i(u, \bar{q}_i)$. Let $f : S_x(M) \to J$ be a retraction. Then we also have $R_i(f(p), f(\bar{q}_i))$ for each $i$. As $f$ is a retraction, $f(\bar{q}_i) = \bar{q}_i$. Therefore $R_i(f(p), \bar{q}_i)$ holds for each $i$ and $p \in \bigcap_i C_i(u)$ as required. ☐

Let $G = Aut(\mathcal{J})$ be the group of $L_p$-automorphisms of the core. Since we know that any endomorphism of the core is an automorphism, $G$ is also the set of endomorphisms of $\mathcal{J}$.

We equip $G$ with the topology for which a basic open subset has the form $\{g \in G : \neg R(ga_1, \dots, ga_k, b_1, \dots, b_n)\}$ for some $a_1, \dots, a_k, b_1, \dots, b_n$ in $\mathcal{J}$ and $R$ a p-closed set in $\mathcal{J}^{k+n}$.

*Remark* 2.7. By Lemma 2.2, $G$ is non-trivial if and only if there are two types $p, q \in J$ such that $\{p\}$ and $\{q\}$ are $L_p$-isomorphic.

**Proposition 2.8.** *The group $G$ is compact $T_1$. For $g \in G$, left and right translation by $g$ are continuous on $G$; also inversion is continuous.*

*Proof.* Let $g \in G$. Then the singleton $\{g\}$ is equal to the intersection $\bigcap_{a \in \mathcal{J}} \{\sigma \in G : \sigma(a) = g(a)\}$. As equality is p-closed, all those sets are closed in $G$, hence $G$ is $T_1$.

Let us show compactness. Let $(g_i)_{i \in I}$ be a family of elements of $G$ and $\mathcal{U}$ an ultrafilter on $I$. We are looking for a limit to the family $(g_i)$ along $\mathcal{U}$. Fix an embedding $J \subseteq S_x(M)$ of $\mathcal{J}$. By homogeneity of $S_x(M)$, we can extend each $g_i$ to an endomorphism $\tilde{g}_i$ of $S_x(M)$. See each $\tilde{g}_i$ as living in the product

$$S_x(M)^{S_x(M)},$$

where $S_x(M)$ is equipped with its usual compact Hausdorff topology. Let $\tilde{g}_*$ be the limit $\lim_{\mathcal{U}} \tilde{g}_i$ in this product space (which is compact). Then $\tilde{g}_*$ is an endomorphism is $S_x(M)$ (this is a closed condition on $S_x(M)^{S_x(M)}$ as each p-closed subset of $S_x(M)^n$ is closed in the product topology). Let $f : S_x(M) \to J$ be a retraction and define $g_* = (f \circ \tilde{g}_*)|_J$. Then $g_*$ is an automorphism of $J$. It is also a limit of the family $g_i$ along $\mathcal{U}$: if say $\neg R(g_* a_1, \dots, g_* a_k, b_1, \dots, b_n)$ holds for $R$ p-closed in $S_x(M)$, then also $\neg R(\tilde{g}_* a_1, \dots, \tilde{g}_* a_k, b_1, \dots, b_n)$ holds as $f$ is a morphism, and then

$\neg R(g_i a_1, \dots, g_i a_k, b_1, \dots, b_n)$ is true for almost all $i$ since $R \subseteq S_x(M)^{k+1}$ is closed in the product topology.

Fix $g \in G$. For any $\bar{a} \in \mathcal{J}^k$, $h \in G$, and p-closed $R(\bar{x}, \bar{d})$, we have

$$R((g \cdot h)\bar{a}, \bar{d}) \iff R(h\bar{a}, g^{-1}(\bar{d})).$$

This shows that left-multiplication by $g$ is continuous: the preimage of the open set $\{\sigma \in G : \neg R(\sigma\bar{a}, \bar{d})\}$ is the open set $\{\tau \in G : \neg R(\tau\bar{a}, g^{-1}(\bar{d}))\}$. Similarly, for right multiplication write:

$$R((h \cdot g)\bar{a}, \bar{d}) \iff R(h \cdot (g\bar{a}), \bar{d}).$$

Finally, inversion is continuous as

$$R(g^{-1}\bar{a}, \bar{d}) \iff R(\bar{a}, g\bar{d}).$$

□

Let $\mathfrak{g} \trianglelefteq G$ be the set of *infinitesimal* elements of $G$ that is the set

$$\{g \in G : gU \cap U \neq \emptyset \text{ for all non-empty pp-open subsets } U \subseteq \mathcal{J}\}.$$

This is a closed normal subgroup of $G$.

**Proposition 2.9.** *The group $\mathcal{G} := G/\mathfrak{g}$ is compact Hausdorff.*

*Proof.* See Lemma C.1 in Appendix C of the paper. □

**Proposition 2.10.** *We have the following cardinality bounds:*

*(1)* $|\mathcal{J}| \leq 2^{|T|}$;
*(2)* $|G| \leq 2^{2^{|T|}}$;
*(3)* $|\mathcal{G}| \leq 2^{|T|}$.

*Proof.* The first point follows from the fact that $\mathcal{J}$ embeds in any given model. The second is a direct consequence of it. To see the third point, fix an embedding $J \subseteq S_x(M)$ inside some $M$ of size $|T|$. Then in the usual topology on $S_x(M)$, $J$ admits a dense subset $D \subseteq J$ of size $\leq |T|$. Then $D$ is also dense in the pp-topology on $J$ since a pp-open set is open in the usual topology. Let $g \in G$ and assume that $g$ fixes $D$ pointwise. Let $U \subseteq J$ be pp-open, non-empty. Then by density, there is $d \in U \cap D$ and then since $g(d) = d$, $d \in U \cap g(U)$ and $U \cap g(U)$ is not empty. Hence $g \in \mathfrak{g}$. This proves that $\mathcal{G}$ is at most as large as the set of bijections of $D$ which has size at most $2^{|T|}$. □

## 3. Lascar strong types

Let $\bar{M}$ be a monster model of $T$. We define the lascar neighbor relation ${L_1}^2$ on pairs of tuples of $\bar{M}$ as follows.

For $a, b \in \bar{M}^x$, say that $L_1^2(a, b)$ holds if there is a small model $M_0 \prec M$ such that $\mathrm{tp}(a/M_0) = \mathrm{tp}(b/M_0)$.

Fix some small model $M_0 \prec M$. Assume that $a, b \in \bar{M}^x$ are such that there is no $M_0' \prec M$ isomorphic to $M_0$ such that $\mathrm{tp}(a/M_0') = \mathrm{tp}(b/M_0')$. Then this translates to a certain partial type over $ab$ being inconsistent. By compactness, we get the following:

There is a formula $\theta(\bar{y})$ and a finite set $\Delta$ of formulas such that there is no $\bar{b} \in \theta(\bar{M})$ with $\mathrm{tp}_\Delta(a/\bar{b}) = \mathrm{tp}_\Delta(a'/\bar{b})$.

Conversely, if this holds, then there cannot be any submodel $M_1 \prec \bar{M}$ for which $\mathrm{tp}(a/M_1) = \mathrm{tp}(a'/M_1)$. From this it follows that in the definition of ${L_1}^2$, we could impose that $M_0$ has a fixed isomorphism type and obtain an equivalent definition. This also shows that $L_1^2(x,x')$ is type definable over $\emptyset$.

Given a model $M$ of $T$, define $L_1(x,x')$ as a binary relation on $S_x(M)$ by

$$L_1(p,q) \iff (\exists r)(L_1^2(r) \wedge r|_x = p \wedge r|_{x'} = q).$$

So $L_1$ is the projection of $L_1^2$ under the canonical projection $S_{x,x'}(M) \to S_x(M) \times S_{x'}(M)$. Since $L_1^2$ is type-definable over $\emptyset$, it is in particular p-closed in $S_{x,x'}(M)$ and by Proposition 1.4, $L_1$ is p-closed in $S_x(M)^2$. It follows in particular, that $L_1$ is also well-defined on $\mathcal{J}$.

Finally, note that $L_1(p,q)$ holds if and only if for each $\theta(\bar{y})$ and each finite $\Delta$, there are $a \models p$, $b \models q$ in $\bar{M}^x$ and $\bar{d} \in \theta(\bar{M})$ such that $\mathrm{tp}_\Delta(a/\bar{d}) = \mathrm{tp}_\Delta(b/\bar{d})$.

**Lemma 3.1.** *Let $f : S_x(M) \to S_x(M)$ be a morphism. Let $p,q \in S_x(M)$ such that $f(p) = f(q)$, then $L_1(p,q)$ holds.*

*Proof.* Assume $\neg L_1(p,q)$. Then there is a consistent $\theta(\bar{y})$ and a finite $\Delta$ such that we cannot find $a \models p$, $b \models q$ and $\bar{d} \in \theta(\bar{M})$ with $\mathrm{tp}_\Delta(a/\bar{d}) = \mathrm{tp}_\Delta(b/\bar{d})$. It follows in particular, that we cannot find such a $\bar{d}$ in $\theta(M)$. This is now saying that something is not represented in $(p,q)$, hence is a p-closed condition. As $f$ is a morphism, the same is true of $f(p)$ and $f(q)$. Pick $\bar{d} \in \theta(M)$. Then $f(p)$ and $f(q)$ have a different restriction to $\bar{d}$, hence $f(p) \neq f(q)$. □

**Corollary 3.2.** *Let $r : S_x(M) \to J$ be a retraction, then for any $p \in S_x(M)$, we have $L_1(p, r(p))$.*

*Proof.* This follows from the previous lemma taking $q = r(p)$ there, as $r(q) = q$. □

We define Lascar equivalence $L_\infty$, first as a relation on pairs of tuples of $\bar{M}$: say that $L_\infty^2(a,b)$ holds for $a,b \in \bar{M}^x$ if there exists a finite sequence $a = a_0, a_1, \ldots, a_n = b$ with $L_1^2(a_k, a_{k+1})$ for each $k$. Define similarly $L_\infty(x,x')$ as a binary relation on $S_x(M)$ by a similar condition: $L_\infty(p,q)$ holds if there exist $p = p_0, p_1, \ldots, p_n = q$ such that $L_1(p_k, p_{k+1})$ holds for each $k$.

The following expresses that the two notions coincide.

**Lemma 3.3.** *Given $a,b \in \bar{M}^x$ and $M_0 \prec \bar{M}$, $a$ and $b$ are Lascar-equivalent if and only if $L_\infty(\mathrm{tp}(a/M_0), \mathrm{tp}(b/M_0))$ holds in $S_x(M)$.*

*Proof.* We may assume that $L_1(\mathrm{tp}(a/M_0), \mathrm{tp}(b/M_0))$ holds. Then there are $a' \equiv_{M_0} a$ and $b' \equiv_{M_0} b$ such that $L_1^2(a',b')$ holds. But then we have $L_1^2(a,a') \wedge L_1^2(a',b') \wedge L_1^2(b',b)$ and therefore $L_\infty^2(a,b)$ holds. The converse is clear by definition of $L_1$. □

Let $\mathrm{Las}_{x,M}$ be the set of $L_\infty$-equivalence classes in $S_x(M)$. Define similarly $\mathrm{Las}_{x,\mathcal{J}}$ as the set of $L_\infty$-classes in $\mathcal{J}$.

Note that for $M \prec N$, the restriction map gives an identification of $\mathrm{Las}_{x,N}$ and $\mathrm{Las}_{x,M}$. Hence more generally, if we have a fixed monster model $\bar{M}$, then $\mathrm{Las}_{x,M}$ can be canonically identified with $\mathrm{Las}_{x,N}$ for any two $M, N \prec \bar{M}$, and is also canonically identified with the quotient of $\bar{M}^x$ by $L_\infty^2$.

**Lemma 3.4.** *An embedding $j : \mathcal{J} \to J \subseteq S_x(M)$ defines a bijection between $\mathrm{Las}_{x,\mathcal{J}}$ and $\mathrm{Las}_{x,M}$.*

*Proof.* We have an injection $\mathrm{Las}_{x,\mathcal{J}} \to \mathrm{Las}_{x,M}$, since the relation $L_1$ on $\mathcal{J}$ is the same as the restriction of $L_1$ on $S_x(M)$ to $J$. The fact that this is a bijection follows from the existence of a retraction $r : S_x(M) \to J$ and Corollary 3.2. □

**Lemma 3.5.** *There exists an embedding $J \subseteq M^*$ of $\mathcal{J}$ in a saturated model $M^*$ and a small model $M \prec M^*$ such that for any $g \in G$, seen as a set of bijections of $J$, there is $\sigma \in Aut(M^*)$ such that $\sigma(p)|_M = g(p)|_M$ for all $p \in J$.*

*Proof.* Fix a small model $M$ and a core $J_0 \subseteq S_x(M)$. Let $M' \succ M$ containing a realization of each type in $J_0$: for each $p \in J_0$, say that $a_p \in M'$ realizes $p$. Let $(M', M)$ be the structure obtained by expanding $M'$ with a new unary predicate naming the subset $M$. Let $(M', M) \prec (M'', M^*)$ be a sufficiently saturated elementary extension. For each $p \in J_0$, let $p_* = \operatorname{tp}(a_p/M^*)$. Let $J = \{p_* : p \in J_0\}$. By elementarity of the extension, one can see that $J$ is $L_p$-isomorphic to $J_0$ (note also that the restriction map $S_x(M^*) \to S_x(M)$ sends $J$ to $J_0$). Let $g \in G$ and see $g$ as a bijection of $J$. Fix some $p \in J$ and we construct $\sigma \in Aut(M^*)$ such that $\sigma(p)|_M = g(p)|_M$. We can do the same with all $p$'s in $J$ at once to obtain what we want.

Given any $\bar{b} \in M$ and $\phi(x; \bar{b}) \in g(p)$, we can find $\bar{b}' \in M^*$ such that $p \models \phi(x; \bar{b}')$. This follows from the fact that $p \mapsto g(p)$ is an $L_p$-isomorphism, so the same formulas are represented in $p$ and $g(p)$. By saturation of the pair $(M'', M^*)$, this also holds for $\phi(x; \bar{b})$ replaced by an arbitrary conjunction of formulas and for $\bar{b}$ an infinite tuple. In particular, taking $\bar{b}$ to be an enumeration of $M$ we obtain that there is some $M_0 \prec M^*$ isomorphic to $M$ such that $\sigma(p|_{M_0}) = g(p)|_M$, where $\sigma \in Aut(M^*)$ is any automorphism extending the given isomorphism from $M_0$ to $M$. This gives $\sigma(p)|_M = \sigma(p)|_{\sigma(M_0)} = \sigma(p|_{M_0}) = g(p)|_M$ as required. □

Let $\operatorname{Gal}_{\mathcal{J}}$ be the group $G$ quotiented by the group of elements that preserve all Lascar strong types in $\mathcal{J}$.

Define also $\operatorname{Gal}_x$ to be the group of automorphisms of $\bar{M}$ quontiented by the subgroup of automorphisms inducing the identity on $\operatorname{Las}_{x,\bar{M}}$.

**Proposition 3.6.** *Fix a retraction $r : S_x(\bar{M}) \to J$ to a core and identify $\mathcal{J}$ with $J$. The map $\alpha : \operatorname{Aut}(\bar{M}) \to G$, $\sigma \mapsto r\sigma$ induces an isomorphism of groups $\operatorname{Gal}_x \to \operatorname{Gal}_{\mathcal{J}}$.*

*Proof.* We identify Lascar strong types in $\mathcal{J}$ and in $\bar{M}$ using the identification of $\mathcal{J}$ with $J$. Let us first assume that $J$ is chosen so that the conclusion of Lemma 3.5 holds. Let $g \in G$, seen as a bijection of $J$. Then there is a small model $M \prec \bar{M}$ and $\sigma \in \operatorname{Aut}(\bar{M})$ such that $\sigma(p)|_M = g(p)|_M$ for each $p \in J$. It follows that $L_1(\sigma(p), g(p))$ holds, hence $L_1(r\sigma(p), rg(p))$ also since $L_1$ is p-closed and $r$ preserves p-closed sets. But $rg(p) = g(p)$ since $g(p) \in J$, so we have $L_1(\alpha(\sigma)(p), g(p))$. Since this is true for all $p \in J$ and all Lascar strong types have a representative in $J$, this shows that $\alpha(\sigma)$ and $g$ induce the same permutation on Lascar strong types. Hence the quotient map $\bar{\alpha} : \operatorname{Aut}(\bar{M}) \to \operatorname{Gal}_{\mathcal{J}}$ is surjective.

Next, we show that $\bar{\alpha}$ is a group homomorphism. Given $p \in S_x(M)$ and $\tau \in \operatorname{Aut}(\bar{M})$, we have $L_1(r\tau(p), \tau(p))$ by Corollary 3.2. Thus $L_1(\sigma r\tau(p), \sigma\tau(p))$ as $\sigma$ is an automorphism and $L_1(r\sigma r\tau(p), r\sigma\tau(p))$ as $L_1$ is p-closed and $r$ preserves p-closed sets. This implies that $\bar{\alpha}$ is a homomorphism of groups.

Now by the first paragraph of this proof, the kernel of $\bar{\alpha}$ consists precisely of the elements which act as the identity on the set of Lascar strong types. Hence $\bar{\alpha}$ induces an isomorphism $\operatorname{Gal}_x \to \operatorname{Gal}_{\mathcal{J}}$.

Finally, take another core $J' \subseteq S_x(\bar{M})$ and a retraction $r' : S_x(\bar{M}) \to J'$. We have that $r' = r'fr$ for some automorphism $f$ of $J$. Since $r$ and $r'$ preserve Lascar strong types, so does $f$. Define $\alpha' : \sigma \mapsto r'\sigma$. Then $\alpha'(\sigma) = r'f\alpha(\sigma)$

since $r' = r'fr$. Now $r'f$ induces an isomorphism between $\mathrm{Aut}(J)$ and $\mathrm{Aut}(J')$ (as $L_p$-automorphisms) and preserves Lascar strong types. It follows that $r'f$ induces an isomorphism between the quotients of $\mathrm{Aut}(J)$ and $\mathrm{Aut}(J')$ respectively by the subgroup of elements preserving Lascar strong types. Therefore $\alpha'$ also induces an isomorphism $\mathrm{Gal}_x \to \mathrm{Gal}_{\mathcal{J}}$. □

**Proposition 3.7.** *If $g \in \mathfrak{g}$ then $g$ maps to the identity in* $\mathrm{Gal}_{\mathcal{J}}$.

*Proof.* Identify $\mathcal{J}$ with some $J \subseteq S_x(M)$. Let $g \in \mathfrak{g}$ and take $p \in S_x(M)$. Fix some consistent formula $\theta(\bar{y})$ and finite set $\Delta$ of formulas in variables $(x; \bar{y})$. Let $R(u, v)$ be the relation on $S_x(M)$ that holds of a pair $(p, q)$ if there does not exist $\bar{b} \in \theta(M)$ over which $p$ and $q$ have the same $\Delta$-part. Then $R$ is p-closed. Thus $\neg R(p, v)$ is a pp-neighborhood of $p$ consisting of types $q \in S_x(M)$ for which there is $\bar{b} \in \theta(M)$ such that $p$ and $q$ have the same $\Delta$-part over $\bar{b}$. As $g \in \mathfrak{g}$, there is $r \in S_x(M)$ such that $\neg R(p, r) \wedge \neg R(p, g(r))$ holds. We also have $\neg R(g(p), g(r))$. Let $a \models p$ and $b \models g(p)$. We can find $c \in \bar{M}^x$ such that $L_1^2(a, c) \wedge L_1^2(b, c)$ hold: indeed, for a finite $\theta(\bar{y})$ and $\Delta$, we can just take $c$ realizing $g(r)$ as above and then use compactness to find $c$ as required. This shows that $L_\infty^2(a, b)$ holds, hence so does $L_\infty(p, g(p))$. Therefore $g$ acts as the identity on Lascar types. □

Putting the last two results together, we get an epimorphism $\mathcal{G} \to \mathrm{Gal}_x$.

Note that usually, the Lascar group is defined as the quotient of $\mathrm{Aut}(\bar{M})$ by the subgroup generated by the automorphisms fixing a model. It is well known that this is the same as $\mathrm{Gal}_x$ for $x$ large enough, but we include a proof for completeness.

**Proposition 3.8.** *Assume that $x$ is large enough to be able to enumerate a model. Let $\mathrm{Aut}_f(\bar{M}) \trianglelefteq \mathrm{Aut}(\bar{M})$ be the subgroup generated by automorphisms fixing a model pointwise. Then $\mathrm{Aut}(\bar{M}) / \mathrm{Aut}_f(\bar{M})$ is equal to* $\mathrm{Gal}_x$.

*Proof.* It is clear that the elements of $\mathrm{Aut}_f(\bar{M})$ map to the identity in $\mathrm{Gal}_x$. It remains to show the converse. Let $\sigma \in \mathrm{Aut}(\bar{M})$ act as the identity on Lascar strong types for the variable $x$. Let $a \in \bar{M}^x$ contain the enumeration of a submodel. Then $L_\infty^2(a, \sigma(a))$ holds, hence there is a sequence $a = a_0, a_1, \ldots, a_n = \sigma(a)$ such that $L_1^2(a_k, a_{k+1})$ holds for all $k$. For each such $k$, by definition of $L_1^2$, there is a model $M_k \prec \bar{M}$ such that $a_k \equiv_{M_k} a_{k+1}$. Let $\sigma_k$ be an automorphism fixing $M_k$ pointwise and sending $a_k$ to $a_{k+1}$. Then $\sigma_k \in \mathrm{Aut}_f(\bar{M})$. Define $\sigma' = \sigma_{n-1} \circ \cdots \circ \sigma_0$. So $\sigma' \in \mathrm{Aut}_f(\bar{M})$ and $\sigma'(a) = \sigma(a)$. Now $\sigma'^{-1}\sigma$ fixes $a$ and since $a$ contains a model, this implies $\sigma^{-1}\sigma \in \mathrm{Aut}_f(\bar{M})$. So finally $\sigma \in \mathrm{Aut}_f(\bar{M})$. □

## 4. Examples

- If $T$ is stable, then there is a unique core $J \subseteq S_x(M)$ consisting of all $\mathrm{acl}^{eq}(\emptyset)$-definable types (equivalently types that do not fork over $\emptyset$). Indeed, we know from Proposition 2.4 that those types are in the core. Furthermore a retraction to this set is given by sending any type to the unique non-forking extension of its strong type over $\emptyset$. (One way to see that is that given $p \in S_x(M)$, the non-forking extension $q$ of $p|_\emptyset$ is an average type of an indiscernible sequence of realizations of $p$ and hence must be larger in the fundamental order, or in other words $p \mapsto q$ is a an $L_p$-morphism. The same argument works for several types at once, taking realizations of all of them and a non-forking extension of the type of the whole tuple of $\emptyset$.)

- If $T$ is DLO and $|x| = 1$, then $J$ consists of the two 0-definable types. The same thing holds for the random graph.

  If we take a circular order, then there are no 0-definable type. In this case, $J$ is a singleton, and can be any non-realized type.

University of California, Berkeley, Mathematics Department, Evans Hall, Berkeley, CA, 94720-3840, USA

*E-mail address*: pierre.simon@berkeley.edu