AN ARGUMENT OF SHKREDOV IN THE FINITE FIELD SETTING

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Abstract. An argument of Skhredov can be adapted to show that if $A \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n$ does not contain a triple $((x,y),(x+d,y),(x,y+d))$ with $d \neq 0$ then $2^{-2n}|A| \ll (\log n)^{-1/25}$. The argument was outlined in the author's survey article [1]. The purpose of this note is to collect together the proofs that were omitted from that survey.

This is designed to be read as a supplement to [1, §5], and does not make a lot of sense independently.

1. A preliminary lemma

Lemma 1.1. *Let $\tau, \kappa \in (0,1)$ be parameters, and let $S_1, \dots, S_k \subseteq V_n$ be sets. Write $\varepsilon(k) = \kappa 2^k \tau^{-k/2}$. Suppose that $|S_i| = \sigma_i N$ and that each S_i is $\varepsilon(k-1)$-uniform. Then for at least $(1 - 2^k\tau)N^k$ of the k-tuples $(x_1, \dots, x_k) \in V_n$ we have*

$$|\sum_d S_1(x_1+d)S_2(x_2+d)\dots S_k(x_k+d) - \sigma_1\dots\sigma_k N| \leqslant \varepsilon(k)N. \qquad (1.1)$$

Proof. We proceed by induction on k, the case $k=1$ being trivial. For a fixed choice of $u = (x_1, \dots, x_{k-1})$ write $F_u(d) = S_1(x_1+d)S_2(x_2+d)\dots S_{k-1}(x_{k-1}+d)$. Then

$$\sum_d S_1(x_1+d)S_2(x_2+d)\dots S_k(x_k+d) = (F_u * S_k)(x_k).$$

Now observe that

$$\sum_{x_k}\left((F_u * S_k)(x_k) - \sigma_k \sum_d F_u(d)\right)^2 = N^{-1}\sum_{\xi \neq 0}|\widehat{F_u}(\xi)|^2|\widehat{S_k}(\xi)|^2 \leqslant \varepsilon(k-1)^2N^3. \qquad (1.2)$$

Using the induction hypothesis, for at least $(1-2^{k-1}\tau)N^{k-1}$ values of u we have

$$|\sum_d F_u(d) - \sigma_1\dots\sigma_{k-1}N| \leqslant \varepsilon(k-1)\tau N.$$

For these values of u, we have from (1.2) that

$$\sum_{x_k}((F_u * S_k)(x_k) - \sigma_1\dots\sigma_k N)^2 \leqslant 4\varepsilon(k-1)^2N^3.$$

For such a value of u, then, the number of x_k for which $|(F_u * S_k)(x_k) - \sigma_1\dots\sigma_k N| > \varepsilon(k)N$ is no more than

$$\frac{4\varepsilon(k-1)^2}{\varepsilon(k)^2}N \leqslant 2^{k-1}\tau N.$$

The total number of $(x_1, \dots, x_k)$ which are exceptions to (1.1) is thus at most

$$2^{k-1}\tau N^k + 2^{k-1}\tau N^k = 2^k\tau N^k,$$

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which is what we wanted to prove. □

2. Proof of Proposition 5.6

For any three functions $f_1, f_2, f_3 : \mathcal{S} \to [-1,1]$ write $T(f_1,f_2,f_3) = \mathbb{E}(f_1(x,y)f_2(y+z,y)f_3(x,x+z)|x,y,z \in H)$. This operator T is clearly trilinear, and furthermore $|H|^3 T(A,A,A)$ is the number of corners in A. Write $f = A - \alpha$, and observe that $T(A,A,A) = T(f,A,A) + \alpha T(1,A,A)$. We are working under the assumption that $\|f\|_\square^4 \leqslant 2^{-8}\alpha^{12}$; for brevity, write $\eta = 2^{-8}\alpha^{12}$.

Now if we write $g(z) = \mathbb{E}(\mathbf{1}_A(s,t)|s,t \in H, s+t = z)$ we clearly have $T(1,A,A) = \mathbb{E}g(z)^2$, and this is at least $\alpha^2\beta_1^2\beta_2^2$ by the Cauchy-Schwarz inequality. Thus we have

$$T(A,A,A) \geqslant T(f,A,A) + \alpha^3\beta_1^2\beta_2^2. \tag{2.1}$$

To conclude the proof we place an upper bound on $T(f,A,A)$ by invoking the as yet unused hypothesis that $\|f\|_\square$ is small.

To do this, we apply the Cauchy-Schwarz inequality twice. First of all we have

$$\begin{aligned} T(f,A,A) &= \mathbb{E}\bigg(A(y+z,y)\mathbb{E}(E_1(y+z)f(x,y)A(x,x+z)\,|\,x)\,|\,y,z\bigg) \\ &\leqslant \mathbb{E}(E_1(y+z)E_2(y)|y,z)^{1/2} \times \qquad\qquad (2.2) \\ &\quad \times \mathbb{E}(E_1(y+z)A(x,x+z)A(x',x'+z)f(x,y)f(x',y)|x,x',y,z)^{1/2}. \end{aligned}$$

Next, observe that

$$\begin{aligned} &\mathbb{E}(E_1(y+z)A(x,x+z)A(x',x'+z)f(x,y)f(x',y)|x,x',y,z) \\ =\ &\mathbb{E}\bigg(A(x,x+z)A(x',x'+z)\mathbb{E}(E_1(y+z)E_2(x+z)E_2(x'+z)f(x,y)f(x',y)|y)|x,x',z\bigg) \\ \leqslant\ &\mathbb{E}\bigg(E_1(x)E_1(x')E_2(x+z)E_2(x'+z)|x,x',z\bigg)^{1/2} \times \\ &\times\ \mathbb{E}\bigg(\omega(x,x',y,y')f(x,y)f(x',y)f(x,y')f(x',y')|x,x',y,y'\bigg)^{1/2}, \qquad (2.3) \end{aligned}$$

where

$$\omega(x,x',y,y') = \mathbb{E}(E_1(x+z)E_1(x'+z)E_2(y+z)E_2(y'+z)|z).$$

Since $\mathbb{E}(E_1(y+z)E_2(y)|y,z) = \beta_1\beta_2$ and (by a simple adaptation of [1, Lemma 3.2]

$$\mathbb{E}(E_1(x)E_1(x')E_2(x+z)E_2(x'+z)|x,x',z) \leqslant 2\beta_1^2\beta_2^2,$$

we get from (2.2),(2.3) that

$$T(f,A,A)^4 \leqslant 2\beta_1^4\beta_2^4\mathbb{E}(\omega(x,x',y,y')f(x,y)f(x',y)f(x,y')f(x',y')|x,x',y,y'). \tag{2.4}$$

A short check confirms that our uniformity assumptions on E_1 and E_2 are sufficient to apply the case $k = 4$ of Lemma 1.1 with $\tau = 2^{-4}\beta_1^4\beta_2^4\eta$ and $\kappa = 2^{-12}\beta_1^{12}\beta_2^{12}\eta^3$. This allows us to conclude that the weight function $\omega(x,x',y,y')$ is roughly constant, in the sense that $|\omega(x,x',y,y') - \beta_1^2\beta_2^2| \leqslant \beta_1^4\beta_2^4\eta$ with probability at least $1 - \beta_1^4\beta_2^4\eta$. This allows us to estimate the right-hand-side of (2.4) using the fact that $\|f\|_\square$ is small. Indeed it is easy to see that

$$\left|\mathbb{E}(\omega(x,x',y,y') - \beta_1^2\beta_2^2)f(x,y)f(x',y)f(x,y')f(x',y')|x,x',y,y'\right| \leqslant 3\beta_1^4\beta_2^4\eta,$$

and so (2.4) and the smallness of $\|f\|_\square$ imply that

$$|T(f,A,A)| \leqslant 2\beta_1^2\beta_2^2\eta^{1/4}.$$

Comparing with (2.1), we get

$$T(A,A,A) \geqslant (\alpha^3\beta_1^2\beta_2^2 - 2\beta_1^2\beta_2^2\eta^{1/4}) \geqslant \alpha^3\beta_1^2\beta_2^2/2,$$

which is what we wanted to prove. □

3. PROOF OF PROPOSITION 5.7

Shkredov employs a spectral argument to prove this. We take a more elementary approach, which is perhaps best phrased in the language of graph theory.

Suppose that $|E_1| = M_1$, $|E_2| = M_2$. We define a certain bipartite graph G associated with the set A. This will be a graph on disjoint vertex sets X, Y, which we consider to be copies of E_1, E_2. We will abuse notation somewhat by identifying X with E_1 and Y with E_2. We say that $xy \in E(G)$ precisely if $(x,y) \in A$, and observe that $|E(G)| = \alpha M_1 M_2$. We may associate to any sub-product set $\mathcal{S}' = F_1 \times F_2$ the pair of sets $F_1 = X' \subseteq X$ and $F_2 = Y' \subseteq Y$. The density of $\delta_{\mathcal{S}'(A)}$ is then simply the edge density

$$\delta(X',Y') := \mathbb{E}(\mathbf{1}_{xy\in E(G)} | x \in X', y \in Y'),$$

and so our objective is to make this large for sets X', Y' which are not too small. Note also that the number of rectangles in A, $\|A\|_\square^4$, is precisely $C_4(G)$, the number of 4-cycles in G.

We will use some standard notation of graph theory. If v is a vertex then $\mathcal{N}(v)$ denotes the neighbourhood of v, whilst $d(v) = |\mathcal{N}(v)|$ is the degree of v. It will be important in later arguments to assume that $d(v)$ is roughly constant on both X and Y. Fortunately, if $d(v)$ is not roughly constant in this sense then it is relatively easy to find sets X', Y' such that $\delta(X',Y')$ is large.

Lemma 3.1. *Let $\epsilon_1, \epsilon_2 \in (0,1)$, and suppose that there are either at least $\epsilon_1 M_1$ vertices $x \in X$ such that $|d(x) - \alpha M_2| > \epsilon_2 M_2$, or else at least $\epsilon_2 M_2$ vertices $y \in Y$ such that $|d(y) - \alpha M_1| > \epsilon_1 M_1$. Then we may find $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geqslant \min(\epsilon_1/2, \epsilon_2/2)M_1$, $|Y'| \geqslant \min(\epsilon_1/2, \epsilon_2/2)M_2$ such that $\delta(X',Y') \geqslant \alpha + \epsilon_1\epsilon_2/2$.*

Proof. By symmetry we may assume that we are in the first situation, that is at least $\epsilon_1 M_1$ vertices $x \in X$ have $|d(x) - \alpha M_2| > \epsilon_2 M_2$. Suppose first that $d(x) > (\alpha + \epsilon_2)M_2$ for at least $\epsilon_2 M_1/2$ values of $x \in X$. Let X' be the set of such x, and set $Y' = Y$. Then it is clear that $\delta(X',Y') \geqslant \alpha + \epsilon_2$.

Alternatively, suppose that $d(x) < (\alpha - \epsilon_2)M_2$ for at least $\epsilon_1 M_1/2$ values of $x \in X$. Let X_0 be the set of such x, and set $X' = X \setminus X_0$, $Y' = Y$. Set $|X'| = \kappa M$. Counting edges in G, we have the bound

$$(\alpha - \epsilon_2)(1-\kappa) + \kappa \geqslant \alpha,$$

which quickly leads to $\kappa \geqslant \epsilon_2$. Furthermore we know that $\kappa \leqslant 1 - \epsilon_1/2$, and so

$$\delta(X',Y') \geqslant \frac{\alpha M - (\alpha - \epsilon_2)|X_0|}{|X'|} = \alpha + \epsilon_2(\frac{1}{\kappa} - 1) \geqslant \alpha + \epsilon_1\epsilon_2/2.$$ □

We are now more-or-less free to work under the assumption that $d(v)$ is roughly constant on both classes X and Y. We may use this to deduce, from the fact that $\|A-\alpha\|_\square$ is large, that A has significantly more than $\alpha^4 M_1^2 M_2^2$ rectangles.

Lemma 3.2. *Suppose that $\|A-\alpha\|_\square^4 \geqslant \eta$, and that the degrees $d(v)$ are roughly constant in the sense that $|d(x)-\alpha M_2| \leqslant \eta M_2/56$ for all but at most $\eta M_1/56$ values of $x \in X$, and $|d(y)-\alpha M_1| \leqslant \eta M_1/56$ for all but at most $\eta M_2/56$ values of $y \in Y$. Then $\|A\|_\square^4 \geqslant \alpha^4 + \eta/2$.*

Proof. Write $f = A - \alpha$, and expand $\|A\|_\square^4$ as a sum of 16 terms by writing $A = \alpha + f$. There is a main term α^4, the term $\|f\|_\square^4$ which we know to be at least η, plus 14 other terms which after a change of variables if necessary may each be written in the form

$$\alpha\mathbb{E}(f(x',y)g(x,y')h(x',y')|x,x' \in X, y,y' \in Y), \tag{3.1}$$

where $\|g\|_\infty, \|h\|_\infty \leqslant 1$. We estimate each of these terms by

$$|\mathbb{E}(f(x',y)g(x,y')h(x',y')|x,x' \in X, y,y' \in Y)| \leqslant \mathbb{E}(|\mathbb{E}(f(x',y)|y \in Y)| \mid x' \in X). \tag{3.2}$$

But $\mathbb{E}(f(x',y)|y \in Y)$ is exactly $(d(x')-\alpha M_2)/M_2$, which bounded in absolute value by $\eta/56$ with the possible exception of $\eta M_1/56$ values of x'. Thus

$$\mathbb{E}(|\mathbb{E}(f(x',y)|y \in Y)| \mid x' \in X) \leqslant \eta/56 + \eta/56 \leqslant \eta/28,$$

which leads in view of (3.2) to the upper bound of $\eta/28$ for the expression (3.1). The sum of fourteen such expressions appearing in the expansion of $\|A\|_\square^4$ does not, therefore, contribute more than $\eta/2$. □

Lemma 3.3. *Suppose that $\|A\|_\square^4 \geqslant \alpha^4 + \eta/2$. Suppose also that the degrees of the graph G are roughly constant in the sense that $|d(x)-\alpha M_2| \leqslant \eta M_2/32$ with the possible exception of $\eta M_1/8$ values of $x \in X$, and $|d(y)-\alpha M_1| \leqslant \eta M_1/32$ with the possible exception of $\eta M_2/8$ values of $y \in Y$. Then there are sets $X' \subseteq X$. $Y' \subseteq Y$ such that $|X'| \geqslant \eta M_1/32$, $|Y'| \geqslant \eta M_2/32$ and such that $\delta(X',Y') \geqslant \alpha + \eta/8$.*

Proof. For each pair $(x,y) \in X \times Y$ write $e(x,y)$ for the number of edges between $\mathcal{N}(x)$ and $\mathcal{N}(y)$. It is clear that

$$\sum_{xy \in E(G)} e(x,y) = C_4(G) \geqslant (\alpha^4 + \eta/2)M_1^2 M_2^2. \tag{3.3}$$

Let X_0 be the set of all $x \in X$ for which $|d(x)-\alpha M_2| \leqslant \eta M_2/32$, and define $Y_0 \subseteq Y$ similarly. By assumption we have $|X_0^c| \leqslant \eta M_1/8$, $|Y_0^c| \leqslant \eta M_2/8$ and so the total number of edges incident to $X_0^c \cup Y_0^c$ is at most $\eta M_1 M_2/8$. Thus at the cost of replacing η by $\eta/2$ we may ignore such edges and replace (3.3) by

$$\sum_{xy \in E(G): x \in X_0, y \in Y_0} e(x,y) \geqslant (\alpha^4 + \eta/4)M_1^2 M_2^2.$$

In particular, there are choices of $x \in X_0, y \in Y_0$ for which

$$e(x,y) \geqslant (\alpha^3 + \eta/4\alpha)M_1 M_2.$$

Setting $Y' = \mathcal{N}(x)$, $X' = \mathcal{N}(y)$ and observing that $|X'| \leqslant (\alpha + \eta/32)M_1$, $|Y'| \leqslant (\alpha + \eta/32)M_2$, we establish

$$\delta(X',Y') \geqslant \frac{\alpha^3 + \eta/4\alpha}{(\alpha + \eta/32)^2} \geqslant \alpha + \eta/8.$$

Applying the fact that $x \in X_0, y \in Y_0$ once more, we have the crude bounds $|X'| \geqslant \alpha M_1/2 > \eta M_1/32$ and $|Y'| > \eta M_2/32$. $\Box$

Proof of Proposition 5.7. Suppose first that there are at least $\eta M_1/56$ vertices $x \in X$ for which $|d(x) - \alpha M_2| > \eta M_2/56$. Then we may apply Lemma 3.1 to get sets F_1, F_2 with $|F_i| \geqslant 2^{-8}\eta M_i$ such that the density of A on $\mathcal{S}' = F_1 \times F_2$ is at least $\alpha + 2^{-14}\eta^2$. The same applies under the assumption that there are at least $\eta M_2/56$ vertices $y \in Y$ for which $|d(y) - \alpha M_1| > \eta M_1/56$.

If neither of these two alternatives holds then Lemma 3.2 is applicable, and we deduce that $\|A\|_{\Box}^4 \geqslant \alpha^4 + \eta/2$. This is one of the hypotheses of Lemma 3.3. The other assumption, that $d(v)$ is roughly constant, is also satisfied and so we may find F_1, F_2 with $|F_i| \geqslant \eta M_i/32$ such that the density of A on $\mathcal{S}' = F_1 \times F_2$ is at least $\alpha + \eta/8$. $\Box$

4. PROOF OF PROPOSITION 5.8

Set $\delta = \delta_1\delta_2$. We will describe an algorithm, leading to a decomposition of $W \times W$ into *cells*

$$W \times W = \bigcup_{i \in \mathcal{I}} C^{(i)}, \qquad C^{(i)} = (W^{(i)} + t_1^{(i)}) \times (W^{(i)} + t_2^{(i)}). \tag{4.1}$$

After the jth stage of the algorithm the index set will be $\mathcal{I}_j$, and each $W^{(i)}$ will be a subspace of dimension at least $n - j$. The $(j+1)$st stage of the algorithm involves partitioning some of the cells $C^{(i)}$ into four subcells, in which the subspaces $W^{(i)}$ are reduced in dimension by one. To describe our algorithm we need some notation. For each i write $D_1^{(i)} = F_1 \cap (W^{(i)} + t_1^{(i)})$, $D_2^{(i)} = F_2 \cap (W^{(i)} + t_2^{(i)})$, and define the two densities

$$\delta_1^{(i)} = \mathbb{P}(x \in D_1^{(i)} | x \in W^{(i)} + t_1^{(i)}),$$
$$\delta_2^{(i)} = \mathbb{P}(x \in D_2^{(i)} | x \in W^{(i)} + t_2^{(i)}).$$

Write $\delta^{(i)} = \delta_1^{(i)}\delta_2^{(i)}$ for the density of $F_1 \times F_2$ on $C^{(i)}$. Note that

$$\sum_i \operatorname{meas}(C^{(i)})\delta^{(i)} = \delta.$$

We say that $C^{(i)}$ has *expired* if $\delta^{(i)} < \delta\tau/2$. We divide the unexpired cells into two classes. $C^{(i)}$ is a *uniform cell* if both $D_1^{(i)} - t_1^{(i)}$ and $D_2^{(i)} - t_2^{(i)}$ are σ-uniform as subsets of $W^{(i)}$. Otherwise, it is *non-uniform*. Note that by definition all non-uniform cells are unexpired. Finally, if $\mathcal{J} \subseteq \mathcal{I}$ corresponds to any subcollection of cells, we define the measure of $\mathcal{J}$, $\operatorname{meas}(\mathcal{J})$, by

$$\operatorname{meas}(\mathcal{J}) = \mathbb{P}(x \in \bigcup_{i \in \mathcal{J}} C^{(i)} | x \in W \times W).$$

The algorithm may now be described. At step j we will have an indexing set $\mathcal{I}_j$ and a partition $\mathbb{F}_2^n \times \mathbb{F}_2^n = \bigcup_{i \in \mathcal{I}_j} C^{(i)}$. The cells $C^{(i)}$ can be expired, uniform or non-uniform, and we denote the corresponding subsets of $\mathcal{I}_j$ by $\mathcal{E}_j$, $\mathcal{U}_j$ and $\mathcal{N}_j$ respectively. If it so happens that

$$\operatorname{meas}(\mathcal{N}_j) < \tau\delta/4 \tag{4.2}$$

then we `STOP` the algorithm at step j. Otherwise, we subdivide the cells in $\mathcal{N}_j$, whilst leaving the cells in $\mathcal{E}_j$ and $\mathcal{U}_j$ unmodified. Note that in order for a cell $C^{(i)}$ to lie in $\mathcal{N}_j$ it must have undergone a subdivision at every step of the algorithm so far, which means that $\dim W^{(i)} = n - j$ for such cells. Without loss of generality there are at least $\frac{\tau}{8}2^{2j}$ values of i for which $D_1^{(i)} - t_1^{(i)}$ is not σ-uniform as a subset of $W^{(i)}$. Recall that $\delta_1^{(i)}$ denotes the density of $D_1^{(i)} - t_1^{(i)}$ on $W^{(i)}$. Using [1, Lemma 3.4 (1)], we may choose a codimension one subspace $H^{(i)} \subseteq W^{(i)}$ such that the mean square of the densities of $D_1^{(i)} - t_1^{(i)}$ on $H^{(i)}$ and $W^{(i)} \setminus H^{(i)}$ is at least $\delta_1^{(i)2} + \sigma^2$. The cell $C^{(i)} = (W^{(i)} + t_1^{(i)}) \times (W^{(i)} + t_2^{(i)})$ gets divided into four subcells which we temporarily denote by $C^{(i_1)}, C^{(i_2)}, C^{(i_3)}$ and $C^{(i_4)}$. Each is a coset of $H^{(i)} \times H^{(i)}$. Observe that

$$\frac{1}{4}(\delta_1^{(i_1)2} + \delta_1^{(i_2)2} + \delta_1^{(i_3)2} + \delta_1^{(i_4)2}) \geqslant \delta^2 + \sigma^2. \tag{4.3}$$

We claim that the algorithm just described terminates after at most $16\sigma^{-2}\delta^{-1}\tau^{-1}$ steps. To see this, define the *index* $\mathcal{I}$ of the partition (4.1) to be the quantity

$$\operatorname{ind}(\mathcal{I}) = \frac{1}{2}\sum_{i \in \mathcal{I}} \operatorname{meas}(\{i\})(\delta_1^{(i)2} + \delta_2^{(i)2}),$$

a kind of mean square density. With this definition and (4.3) one sees that $\operatorname{ind}(\mathcal{I}_{j+1}) \geqslant \operatorname{ind}(\mathcal{I}_j) + \frac{1}{16}\tau\delta\sigma^2$. It being clear that necessarily $\operatorname{ind}(\mathcal{I}) \leqslant 1$, one indeed sees that after at most $16\sigma^{-2}\delta^{-1}\tau^{-1}$ steps the algorithm must `STOP`.

Suppose that the algorithm stops at step K, $K \leqslant 16\sigma^{-2}\delta^{-1}\tau^{-1}$. This means that $\operatorname{meas}(\mathcal{N}_K) < \tau\delta/4$. We will show that one of the cells of the partition (4.1) gives us a subspace W with the properties claimed in the proposition. Observe that so far the set $A \subseteq F_1 \times F_2$ has played no role. That is about to change, and in anticipation let us define

$$\alpha^{(i)} = \mathbb{P}((x_1, x_2) \in A | (x_1, x_2) \in C^{(i)} \cap (F_1 \times F_2)).$$

Note that the density of $A \cap C^{(i)}$ relative to $C^{(i)}$ is $\alpha^{(i)}\delta^{(i)}$.

To start with, we observe that

$$\sum_{i \in \mathcal{E}_K} \operatorname{meas}(C^{(i)})\alpha^{(i)}\delta^{(i)} < \delta\tau/2,$$

which means that

$$\sum_{i \in \mathcal{U}_K \cup \mathcal{N}_K} \operatorname{meas}(C^{(i)})\alpha^{(i)}\delta^{(i)} > \delta(\alpha + \tau) - \delta\tau/2 \geqslant \delta(\alpha + \tau/2).$$

Suppose, as a hypothesis for contradiction, that $\alpha^{(i)} < \alpha + \tau/4$ for all $i \in \mathcal{U}_K$. Then we would have

$$\begin{aligned} \delta(\alpha + \tau/2) &\leqslant \sum_{i \in \mathcal{U}_K} \operatorname{meas}(C^{(i)})\alpha^{(i)}\delta^{(i)} + \sum_{i \in \mathcal{N}_K} \operatorname{meas}(C^{(i)})\alpha^{(i)}\delta^{(i)} \\ &< (\alpha + \tau/4)\sum_i \operatorname{meas}(C^{(i)})\delta^{(i)} + \tau\delta/4 \\ &= \delta(\alpha + \tau/2), \end{aligned}$$

a contradiction. Thus there is at least one value of $i \in \mathcal{U}_K$ for which $\alpha^{(i)} \geqslant \alpha + \tau/4$. Taking $W = W^{(i)}$, $t_1 = t_1^{(i)}$, $t_2 = t_2^{(i)}$ we have now simultaneously satisfied (1),(2) and

(3) (Observe that (1) is a consequence of the fact that $i \in \mathcal{U}_K$, and so in particular $C^{(i)}$ is not expired). □

References

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