On Runge's Theorem

We spent several lectures studying approximation of functions on the real line by polynomials. We proved, for example, that on a closed interval *every* continuous function can be uniformly approximated by polymomials. In the complex plane, the situation is different. Polynomials are holomorphic, and hence any sequence of polynomials which converges uniformly on an open set converges to a holomorphic function on that set. This follows from Morera's theorem: you should take the opportunity to recall the statement and proof of this theorem, as well as why it supplies a justification of the preceding sentence. Thus, broadly speaking, a function f which is the uniform limit of polynomials had better be holomorphic. However, this is not enough: on the interior of the domain $K = \{z \in \mathbb{C} : \frac{1}{2} \leqslant z \leqslant 2\}$, the function $f(z) = \frac{1}{z}$ is holomorphic, but it is not the uniform limit of any sequence of polynomials. This is because the integral of f around the contour $\gamma(t) = e^{2\pi i t}$, $0 \leqslant t \leqslant 1$, is equal to $2\pi i$, but the integral of any polynomial around γ is zero by Cauchy's theorem. The problem is that f does not extend to a holomorphic function *inside* γ, and this is possible because K as a "hole".

Runge's theorem states that in a sense these two ways in which a function can fail to be uniformly approximable by polynomials are the only ones.

Theorem 1. *Suppose that K is a compact subset of $\mathbb{C}$, and that f is a function taking complex values which is holomorphic on some domain Ω containing K. Suppose that $\mathbb{C} \setminus K$ is path-connected. Then f is uniformly approximable by polynomials.*

The proof of this theorem splits naturally into two parts.

Part 1: Prove that f is uniformly approximated by rational functions, all of whose poles lie outside of K. In fact, we can assume that each of these is a sum of functions of the form $\frac{c}{\alpha - z}$, where $\alpha \notin K$.

Part 2: Prove that every function of the form $f(z) = \frac{c}{\alpha - z}$ can be uniformly approximated by polynomials.

Why does Runge's theorem follow from these observations? What is needed is the following simple exercise. Here, and in the rest of what follows, we say that a function is u.a.p. if it can be uniformly approximated by polynomials on K.

Lemma 1. *Suppose that f and g are u.a.p. The so are $f + g$, fg and λf for any $\lambda \in \mathbb{C}$. Suppose furthermore than (f_n) is a sequence of u.a.p. functions, converging uniformly on K to a function f. Then f is also u.a.p.*

Let us turn to the proof of parts 1 and 2.

To motivate the proof of part 1, let us recall how the proof of Taylor's theorem goes. Taylor's theorem implies certain cases of Runge's theorem, specifically those in which Ω can be taken to be an open disc in the complex plane. (For the general case, however, it is useless – try using it to show that $f(z) = \frac{1}{z}$ can be uniformly approximated by polynomials on a "keyhole" set K of the form $\{z \in \mathbb{C} : \varepsilon \leqslant |z| \leqslant 10, \varepsilon \leqslant \arg z \leqslant 2\pi - \varepsilon\}$).

The proof of Taylor's theorem goes as follows. Suppose, for simplicity of notation, that f is holomorphic on an open set containing the unit circle contour γ mentioned above. Then for $|z| < 1$ we have *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw.$$

This may be rewritten as

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)dw}{w} (1 + \frac{z}{w} + \frac{z^2}{w^2} + \dots)$$

by the geometric series formula. The sum of the first N terms is a polynomial

$$p_N(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)dw}{w} (1 + \frac{z}{w} + \frac{z^2}{w^2} + \dots + \frac{z^N}{w^N}).$$

Since $|w| = 1$ on γ, the error $|f(z) - p_N(z)|$ is bounded above by $C \sum_{n>N} |z|^n$, which tends to zero uniformly on any compact set K contained inside γ.

Consider now a general compact set K contained in an open set Ω. The set Ω may not be disc-shaped, and so Cauchy's integral formula cannot be used in such an obvious way. Drawing some pictures, however, it is "obvious" that there ought to be a nice contour γ which circumscribes K, and which lies entirely in Ω and goes around each point of K precisely once. We would then have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw.$$

By the definition of path integral, this gives

$$f(z) = \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt.$$

We could then approximate the integral by a finite sum sampled at the points n/N, for some large N, to get

$$f(z) \approx \frac{1}{2\pi i} \frac{1}{N} \sum_{n=0}^{N-1} \frac{f(\gamma(n/N))}{\gamma(n/N) - z} \gamma'(n/N). \tag{0.1}$$

The right-hand side is now a rational function, and in fact it is a sum of functions of the form $\frac{c}{\alpha - z}$. The poles α are at the points $\gamma(n/N)$, which, because they lie on the path γ, are outside K.

What must we do to make this rigorous? We must justify two things. Firstly, drawing pictures is insufficient evidence for the existence of a nice contour γ. Secondly, we must explain the meaning of the $\approx$ symbol in (0.1). We want, of course, for it to mean that the right-hand side converges uniformly to f on K, as $N \to \infty$.

In lectures, I constructed an appropriate path γ as follows (I recommend you draw or remind yourself of pictures as you follow this). Since K is compact and $\mathbb{C} \setminus \Omega$ is closed, there is some $\delta > 0$ such that $|x - y| \geqslant \delta$ whenever $x \in K$ and $y \notin \Omega$. Take a rectilinear grid with squares of sidelength $\delta/10$ (say). Each such square has a boundary, which we consider to be a contour (traversed anticlockwise). Let $C_1, \ldots, C_J$ be the complete collection of these square contours having some intersection with K. Then we have

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{J} \int_{C_j} \frac{f(w)}{w - z} dw \tag{0.2}$$

whenever $z \in K$, unless z actually lies on one of the contours C_j. Why? Well, z must lie inside precisely one of the contours C_j, and then

$$f(z) = \frac{1}{2\pi i} \int_{C_j} \frac{f(w)}{w - z} dw$$

by Cauchy's integral formula. For other contours $C_{j'}$ we have simply

$$\frac{1}{2\pi i} \int_{C_{j'}} \frac{f(w)}{w - z} dw = 0,$$

by Cauchy's formula (since $f(w)/(w - z)$ is holomorphic inside $C_{j'}$).

Consider again the expression (0.2). Each contour C_j is comprised of four edges. However, in forming the sum over all $j = 1, \ldots, J$, the contributions from many of these edges cancel out. Indeed, if an edge of some contour meets K then it will be traversed twice, once in each direction. Therefore the sum (0.2) collapses to a sum

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{J'} \int_{E_j} \frac{f(w)}{w - z} dw, \tag{0.3}$$

where each E_j is now an "edge", that is to say a straight line segment of length $\delta/10$, which lies entirely in Ω but does not intersect K.

We will operate with this union of edges E_j as a substitute for the contour γ, whose existence was somewhat speculative and not rigorously justified. Note (draw yourself an appropriate picture!) that the union of the edges E_j may not be connected, but this will be of no consequence. Although we verified the formula (0.3) only when $z \in K$ does not lie on some contour C_j, it is actually true for all values of $z \in K$. This is because both sides are continuous functions of $z \in K$ (we leave the detailed proof of this as an exercise).

We turn now to the issue of approximating these integrals along E_j with discrete sums. Let us begin by reminding ourselves of a slightly simpler setting, in which we want to approximate

$$\int_0^1 F(t)dt \approx \frac{1}{N}\sum_{n=0}^{N-1} F(n/N),$$

where $F : [0,1] \to \mathbb{C}$ is some continuous function. The error in making this approximation is easily seen (draw a graph if this is helpful) to be bounded by $\sup_{|x-y|\leqslant 1/N} |F(x) - F(y)|$. By uniform continuity, this tends to 0 as $N \to \infty$.

The situation we care about is the one in which the integral to be approximated is

$$\frac{1}{2\pi i}\int_0^1 \frac{f(\phi(t))}{\phi(t)-z}dt,$$

where $\phi(t)$ is a parametrisation of one of the straight-line segments E_j. This is covered by the preceding remarks, because the function $t \mapsto f(\phi(t))/(\phi(t)-z)$ is continuous. However, we also want to know that the rate of convergence as $N \to \infty$ is *uniform in* $z \in K$. To get this additional fact, note that the function

$$F(z,t) = \frac{f(\phi(t))}{\phi(t)-z}$$

is continuous on $K\times[0,1]$, since $\phi(t)$ lies on E_j, which does not intersect K. However $K\times[0,1]$ is compact, and so F is uniformly continuous. This means that $\sup_{|x-y|\leqslant 1/N} |F(z,x) - F(z,y)| \to 0$, uniformly in $z \in K$. This is the statement we need in order to approximate the integrals in (0.3) by sums of rational functions.

This completes our discussion of part 1 of the proof of Runge's theorem.

Remember that the task in part 2 is to show that if $\alpha \notin K$ then the function $\frac{1}{\alpha - z}$ is u.a.p. The strategy for proving this is rather beautiful. Writing $\mathcal{S}$ for the set of all $\alpha \in \mathbb{C} \setminus K$ for which this is so, we shall establish that

(i) $\mathcal{S}$ is not empty;
(ii) if $\alpha \in \mathcal{S}$ and β is "near" α then $\beta \in \mathcal{S}$;
(iii) hence $\mathcal{S}$ is all of $\mathbb{C} \setminus K$.

The proof of (i) is not difficult. If $|\alpha|$ is very large then we have

$$\frac{1}{\alpha - z} = \frac{1}{\alpha}(1 + \frac{z}{\alpha} + \frac{z^2}{\alpha^2} + \dots)$$

by the geometric series formula. This sum is uniformly convergent on the domain $|z| \leqslant \frac{1}{2}|\alpha|$ (say), and if $|\alpha|$ is large enough this domain will contain K.

The proof of (ii) (which of course we have not stated precisely yet) is in many ways quite similar. Note that

$$\frac{1}{\beta - z} = \frac{1}{(\alpha - z)(1 - \frac{\alpha-\beta}{\alpha - z})}.$$

However if $|\alpha - \beta| < |\alpha - z|$ then we may expand

$$\frac{1}{1 - \frac{\alpha-\beta}{\alpha-z}} = 1 + \frac{\alpha - \beta}{\alpha - z} + \big(\frac{\alpha - \beta}{\alpha - z}\big)^2 + \dots.$$

Furthermore convergence is uniform in $z \in K$ if $|\alpha - \beta| < d(\alpha, K) = \inf_{z \in K} |\alpha - z|$. However, each partial sum

$$1 + \frac{\alpha - \beta}{\alpha - z} + \big(\frac{\alpha - \beta}{\alpha - z}\big)^2 + \cdots + \big(\frac{\alpha - \beta}{\alpha - z}\big)^N$$

is u.a.p., by repeated applications of Lemma 1 and the assumption that $1/(\alpha - z)$ is u.a.p..

We have established the followiung precise version of (ii): if $\alpha \in \mathcal{S}$, and if $|\beta - \alpha| < d(\alpha, K)$, then $\beta \in \mathcal{S}$.

Finally, we address (iii). Let $\beta \in \mathbb{C} \setminus K$ be arbitrary; our aim is to show that $\beta \in \mathcal{S}$. Let us start with some arbitrary $\alpha \in \mathcal{S}$, which we know exists by (i). Since $\mathbb{C} \setminus K$ is path-connected, there is a continuous path $\phi : [0, 1] \to \mathbb{C} \setminus K$ with $\phi(0) = \alpha$ and $\phi(1) = \beta$. Since ϕ is continuous and $[0, 1]$ is compact, the image $\phi([0, 1])$ is also compact and hence closed. Therefore, since $\phi([0, 1])$ is disjoint from K, there is some $\delta > 0$ such that $d(\phi(t), K) \geqslant \delta$ for all $t \in [0, 1]$. Furthermore, since ϕ is uniformly continuous, there is some finite set of points $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $|\phi(t_{i+1}) - \phi(t_i)| < \delta$ for all i.

We know that $\phi(t_0) = \alpha \in \mathcal{S}$. By repeated application of (the precise form of) (ii), we conclude in turn that $\phi(t_1), \phi(t_2), \dots$ all lie in $\mathcal{S}$. Finally, we see that $\phi(t_m) = \phi(1) = \beta$ does indeed lie in $\mathcal{S}$.

This concludes the proof of part 2 of Runge's theorem, and hence the whole theorem.