Free and linear representations of outer automorphism groups of free groups

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2012

This thesis is dedicated to Magda

Acknowledgements

First and foremost the author wishes to thank his supervisor, Martin R. Bridson.

The author also wishes to thank the following people: his family, for their constant support; David Craven, Cornelia Drutu, Marc Lackenby, for many a helpful conversation; his office mates.

Abstract

For various values of n and m we investigate homomorphisms

$$\mathrm{Out}(F_n) \to \mathrm{Out}(F_m) \text{ and } \mathrm{Out}(F_n) \to \mathrm{GL}_m(\mathbb{K}),$$

i.e. the *free* and *linear representations* of $\mathrm{Out}(F_n)$ respectively.

By means of a series of arguments revolving around the representation theory of finite symmetric subgroups of $\mathrm{Out}(F_n)$ we prove that each homomorphism $\mathrm{Out}(F_n) \to \mathrm{GL}_m(\mathbb{K})$ factors through the natural map

$$\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}(H_1(F_n, \mathbb{Z})) \cong \mathrm{GL}_n(\mathbb{Z})$$

whenever $n = 3, m < 7$ and $\mathrm{char}(\mathbb{K}) \notin \{2, 3\}$, and whenever

$$n > 5, m < \binom{n+1}{2}$$

and

$$\mathrm{char}(\mathbb{K}) \notin \{2, 3, \ldots, n+1\}.$$

We also construct a new infinite family of linear representations of $\mathrm{Out}(F_n)$ (where $n > 2$), which do not factor through π_n. When n is odd these have the smallest dimension among all known representations of $\mathrm{Out}(F_n)$ with this property.

Using the above results we establish that the image of every homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ is finite whenever $n = 3$ and $n < m < 6$, and of cardinality at most 2 whenever $n > 5$ and $n < m < \binom{n}{2}$. We further show that the image is finite when $\binom{n}{2} \leqslant m < \binom{n+1}{2}$.

We also consider the structure of normal finite index subgroups of $\mathrm{Out}(F_n)$. If N is such then we prove that if the derived subgroup of the intersection of N with the Torelli subgroup $\overline{\mathrm{IA}}_n < \mathrm{Out}(F_n)$ contains some term of the lower central series of $\overline{\mathrm{IA}}_n$ then the abelianisation of N is finite.

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Chapter 1

Introduction

1.1 Aims and results

Finitely generated free and free abelian groups form two cornerstones of the theory of infinite groups. Both classes have been studied extensively (and successfully) over the years. One particular aspect of great interest for a group theorist is the symmetries of such groups, i.e. the structure of their automorphism groups.

In the abelian case, the automorphisms form the groups $\mathrm{GL}_n(\mathbb{Z})$ for varying n. These groups are of fundamental importance in numerous areas of mathematics; from the perspective of a geometric group theorist the interplay between $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{R})$ (most prominently Margulis' superrigidity) is particularly interesting.

Automorphisms of free groups share many properties with those of free abelian groups. A curious aspect of this is that often one can obtain a result for $\mathrm{GL}_n(\mathbb{Z})$ using the ambient Lie group, and then prove the analogous result for $\mathrm{Aut}(F_n)$, even though there is no underlying Lie group in this case!

In this work we have focused on a property shared by $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Aut}(F_n)$, namely on the existence of embeddings

$$\mathrm{GL}_n(\mathbb{Z}) \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{Z}) \text{ and } \mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_{n+1}).$$

Every injection $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1}$ induces one such, and so does each inclusion $F_n \hookrightarrow F_{n+1}$ as a free factor of F_{n+1}. In the case of free groups, we can also find a characteristic subgroup $F_m < F_n$ for some $m > n$, which by restriction gives an embedding $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_m)$.

Every group acts on itself by conjugation, and this way maps to its own automorphism group. If the group is abelian (like $\mathbb{Z}^n$), this action is trivial. In the case of centre-free groups (like F_n for $n \geqslant 2$), this map is injective. It is also easy to see that the image is a normal subgroup of the automorphism group; it is generally referred to as the group of inner automorphisms. Quotienting by the group of inner automorphisms leaves us with the outer automorphism group. In the case of the free groups this is denoted by $\mathrm{Out}(F_n)$.

One easily checks that the embeddings $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_m)$ described above do not descend to embeddings $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$. A natural question arises: do such embeddings exist? Or, more generally, what can we say about homomorphisms $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$, that is about the *free representation theory* of $\mathrm{Out}(F_n)$? These questions have seen some partial answers in the work of Khramtsov [16], Bogopol'skii–Puga [5], Aramayona–Leininger–Souto [1], and Bridson–Vogtmann [8]. Our approach is based on [8], but the much more extensive use of representation theory allows us to obtain stronger results.

Our investigations into this problem yield three results:

Theorem 4.2.7. *Suppose $\phi : \mathrm{Out}(F_3) \to \mathrm{Out}(F_5)$ is a homomorphism. Then the image of ϕ is finite.*

Theorem 4.4.7. *Let $n, m \in \mathbb{N}$ be distinct, $n \geqslant 6$, $m < \binom{n}{2}$, and let*

$$\phi : \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

be a homomorphism. Then the image of ϕ is contained in a copy of $\mathbb{Z}_2$, the finite group of order two.

Theorem 4.4.9. *Let $n, m \in \mathbb{N}$ be distinct, with n even and at least 6. Let*

$$\phi : \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

be a homomorphism. Then the image of ϕ is finite, provided that

$$\binom{n}{2} \leqslant m < \binom{n+1}{2}.$$

Previously, the best known bound (for $n > 8$) was $m \leqslant 2n$ (when n is even), and $m \leqslant 2n - 2$ (when n is odd) – see [8, Theorem C].

Let us remark here that Theorem 4.4.9 suggests that there is no 'non-abelian analogue of taking external squares': one can obtain an interesting map

$$\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_{\binom{n}{2}}(\mathbb{Z})$$

by taking the external square of $\mathbb{Z}^n$, but one cannot find an analogous map in the case of $\mathrm{Out}(F_n)$ (at least for n even).

The key ingredient in all of these theorems is the following commutative diagram

$$\begin{array}{ccccccc} G & \longrightarrow & \mathrm{Isom}(X) & \longrightarrow & \mathrm{GL}(H_1(X,\mathbb{Z})) & \longrightarrow & \mathrm{GL}(H_1(X,\mathbb{K})) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{Out}(F_n) & \xrightarrow{\phi} & \mathrm{Out}(F_m) & \xrightarrow{\pi_m} & \mathrm{GL}_m(\mathbb{Z}) & \xrightarrow{i_m^{\mathbb{K}}} & \mathrm{GL}_m(\mathbb{K}) \end{array}$$

where $G < \mathrm{Out}(F_n)$ is a finite subgroup, and X is a finite graph with fundamental group F_m on which G acts.

As the above diagram indicates, our method relies heavily on the structure of torsion subgroups of $\mathrm{Out}(F_n)$. This is in fact unavoidable, as none of the above theorems stay true if we replace $\mathrm{Out}(F_n)$ by a torsion-free finite-index subgroup! We discuss this issue in more detail in Chapter 4.

Note that the curved arrow at the very bottom of the diagram gives us a $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$. Thus, knowing the linear representation theory of $\mathrm{Out}(F_n)$ gives us obstructions to its free representations.

Linear representations of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ have been studied previously by Potapchik–Rapinchuk [22] and Grunewald–Lubotzky [13]. Our considerations yield three results:

Theorem 3.2.11. *Suppose V is a $\mathbb{K}$-linear representation of $\mathrm{Out}(F_3)$ of dimension at most 6, where the characteristic of $\mathbb{K}$ is not 2 or 3. Then the representation factors through the natural projection $\pi_3 : \mathrm{Out}(F_3) \to \mathrm{GL}_3(\mathbb{Z})$.*

Theorem 3.3.3. *Let $\mathbb{K}$ be a field of characteristic equal to zero or greater than $n+1$. Suppose $\phi : \mathrm{Out}(F_n) \to \mathrm{GL}(V)$ is an m-dimensional $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$, where $n \geqslant 6$ and $m < \binom{n+1}{2}$. Then ϕ factors through the natural projection $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

Theorem 3.3.4. *Let $\mathbb{K}$ be a field of characteristic equal to zero or greater than 5. Suppose $\phi\colon \mathrm{Out}(F_n) \to \mathrm{GL}(V)$ is an m-dimensional $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$, where $n \in \{4,5\}$ and $m < 2n+1$. Then ϕ factors through the natural projection $\pi_n\colon \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

Note that, as before, none of these theorems remain true if we allow ourselves to replace $\mathrm{Out}(F_n)$ by a subgroup of finite index. For details see [13].

The immediate question that comes to mind after seeing the above is: do there exist linear representations of $\mathrm{Out}(F_n)$ which do not factor through

$$\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \ ?$$

The answer turns out to be affirmative. An easy construction follows from the fact that $\mathrm{Out}(F_n)$ is residually finite: take a finite quotient H of $\mathrm{Out}(F_n)$ in which an element of $\overline{\mathrm{IA}}_n$, i.e. the kernel of $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$, is not the identity. Then take any representation of H.

Because of the above it is natural to restrict our attention to representations of $\mathrm{Out}(F_n)$ with infinite image. The first examples of such a representations not factoring through π_n follows from the work of Bridson–Vogtmann [8]. We give a different construction, which creates a new infinite family of such representations of $\mathrm{Out}(F_n)$. When n is odd our construction gives representations of smaller dimension than those previously known.

In Chapter 5 we investigate a different aspect of $\mathrm{Out}(F_n)$, namely the structure of its finite-index subgroups. There is an interesting (and open) problem connected with these subgroups, namely if all of them have finite abelianisations. We offer the following partial result, where by the Torelli subgroup of $\mathrm{Out}(F_n)$ is the kernel of the natural map $\pi_n \colon \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.

Theorem 5.3.2. *Let $\{X_i\}$ be the lower central series of $\overline{\mathrm{IA}}_n$, the Torelli subgroup of $\mathrm{Out}(F_n)$, with $\overline{\mathrm{IA}}_n = X_0$. Let $N \trianglelefteq \mathrm{Out}(F_n)$ be a normal subgroup of finite index. If there exists j such that $X_j \leqslant (N \cap X_0)' = [N \cap X_0, N \cap X_0]$, then the abelianisation of N is finite.*

1.1.1 Asymptotics

The results contained in Chapters 3 and 4 of this paper can be viewed as a a first step in the search for three functions $\alpha_{\mathbb{K}}, \beta, \gamma : \mathbb{N} \to \mathbb{N}$, where $\alpha_{\mathbb{K}}(n)$ is the lowest number such that $\mathrm{Out}(F_n)$ has an $\alpha(n)$-dimensional $\mathbb{K}$-linear representation with infinite image which does not factor through $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$, $\beta(n)$ is the lowest number not equal to n such that there exists a homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_{\beta(n)})$ with infinite image (i.e. a free representation with infinite image), and $\gamma(n)$ is the lowest number not equal to n such that there exists an embedding $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_{\gamma(n)})$ (i.e. a faithful free representation).

Our results (together with the work previously done by Bogopol'skii–Puga and Bridson–Vogtmann) show that, asymptotically, each of these functions (at least for suitable field $\mathbb{K}$) is at least quadratic and at most exponential. The exact nature of the functions remains, however, a deep mystery.

1.2 Structure of the thesis

We start by introducing necessary notation in Chapter 1, Section 1.3.

Chapter 2 is devoted to stating the classical results from the representation theory of symmetric and general linear groups that we shall need. Section 2.3 of this chapter gives us a couple of important results about the presentation and representations of $\mathrm{GL}_n(\mathbb{Z}_2)$.

In Chapter 3 we start investigating linear representations of $\mathrm{Out}(F_n)$. First we consider the case $n = 3$ (Section 3.2), then the more general case of $n \geqslant 6$ (Section 3.3). Finally, we look into representations of $\mathrm{Out}(F_n)$ which do not factor through $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ (Section 3.4).

Chapter 4, in which we consider free representations of $\mathrm{Out}(F_n)$, has a similar structure to its predecessor: Section 4.2 deals with the case $n = 3$, and Sections 4.3 and 4.4 deal with the general case. In Section 4.5 we discuss the two known ways of constructing embeddings $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$ for $m > n$.

The final chapter, Chapter 5, contains some results on the abelianisation of finite index subgroups of $\mathrm{Out}(F_n)$.

The author's results concerning $\mathrm{Out}(F_3)$ have appeared in [17], and the results concerning $\mathrm{Out}(F_n)$ for $n \geqslant 4$ have appeared in [18]. All results presented are the author's original work unless their source (and authorship) is specified. In particular most of Chapter 2, the first half of Section 3.4, and Section 4.5 were not proven by the author, and are included for the sake of completeness.

1.3 Notation and conventions

Definition 1.3.1 (Graphs)**.** We say that X is a *graph* if and only if it is a 1-dimensional CW complex. The 1-cells of X will be called *edges*, the 0-cells will be called *vertices*. The sets of vertices and edges of a graph will be denoted by $V(X)$ and $E(X)$ respectively. The points of intersection of an edge with the vertex set are referred to as *endpoints* of the edge.

We will equip X with the standard path metric in which the length of each edge is 1.

Given two graphs X and Y, a function $f : X \to Y$ is a *morphism of graphs* if and only if f is a continuous map sending $V(X)$ to $V(Y)$, and sending each open edge in X either to a vertex in Y or isometrically onto an open edge in Y.

When we say that a group G *acts on a graph* X, we mean that it acts by graph morphisms.

We say that a graph X is *directed* if and only if it comes equipped with a map $o : E(X) \to X$ such that $o(e)$ is a point on the interior of e of distance $\frac{1}{3}$ from one of its endpoints. We also define $\iota, \tau : E(X) \to V(X)$ by setting $\tau(e)$ to be the endpoint of e closest to $o(e)$, and $\iota(e)$ to be the endpoint of e farthest from $o(e)$. Note that we allow $\iota(e) = \tau(e)$.

The *rank* of a connected graph is defined to be the size of a minimal generating set of its fundamental group (which is a free group).

Remark 1.3.2. Let G be a group. We will adopt the following notation:

- for two elements $g, h \in G$, we define $g^h = h^{-1}gh$;
- for two elements $g, h \in G$, we define $[g, h] = ghg^{-1}h^{-1}$;

We will also use $\mathbb{Z}_k$ to denote the cyclic group of order k.

Definition 1.3.3. Let us introduce the following notation for some elements of $\mathrm{Aut}(F_n)$, the automorphism group of F_n, where F_n is the free group on $\{a_1, \dots, a_n\}$:

$$\epsilon_i : \begin{cases} a_i & \mapsto & a_i^{-1}, \\ a_j & \mapsto & a_j, \quad j \neq i \end{cases}, \qquad \sigma_{ij} : \begin{cases} a_i & \mapsto & a_j, \\ a_j & \mapsto & a_i, \\ a_k & \mapsto & a_k, \quad k \notin \{i, j\} \end{cases},$$

$$\rho_{ij} : \begin{cases} a_i & \mapsto & a_i a_j, \\ a_k & \mapsto & a_k, \quad k \neq i \end{cases}, \qquad \lambda_{ij} : \begin{cases} a_i & \mapsto & a_j a_i, \\ a_k & \mapsto & a_k, \quad k \neq i \end{cases}.$$

Let us also define $\Delta = \prod_{i=1}^n \epsilon_i$ and

$$\sigma_{(n+1)i} = \sigma_{i(n+1)} : \begin{cases} a_i & \mapsto & a_i^{-1}, \\ a_j & \mapsto & a_j a_i^{-1}, \quad j \neq i \end{cases}.$$

We are going to use the same symbols to denote the images of those elements under the natural projection

$$p_n : \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n).$$

We will feel free to abuse the notation even more by using the same symbols to denote the images under the natural map $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.

Below we give an explicit presentation of $\mathrm{Out}(F_n)$, the outer automorphism group of F_n:

Theorem 1.3.4 (Gersten's presentation [12]). *Suppose* $n \geqslant 3$. *The group* $\mathrm{Out}(F_n)$ *is generated by* $\{\epsilon_1, \rho_{ij}, \lambda_{ij} \mid i, j = 1, \dots, n,\ i \neq j\}$, *with relations*

- $[\rho_{ij}, \rho_{kl}] = [\lambda_{ij}, \lambda_{kl}] = 1$ *for* $k \notin \{i, j\}, l \neq i$;
- $[\lambda_{ij}, \rho_{kl}] = 1$ *for* $k \neq j, l \neq i$;
- $[\rho_{ij}^{-1}, \rho_{jk}^{-1}] = [\rho_{ij}, \lambda_{jk}] = [\rho_{ij}^{-1}, \rho_{jk}]^{-1} = [\rho_{ij}, \lambda_{jk}^{-1}]^{-1} = \rho_{ik}^{-1}$ *for* $k \notin \{i, j\}$;
- $[\lambda_{ij}^{-1}, \lambda_{jk}^{-1}] = [\lambda_{ij}, \rho_{jk}] = [\lambda_{ij}^{-1}, \lambda_{jk}]^{-1} = [\lambda_{ij}, \rho_{jk}^{-1}]^{-1} = \lambda_{ik}^{-1}$ *for* $k \notin \{i, j\}$;
- $\rho_{ij}\rho_{ji}^{-1}\lambda_{ij} = \lambda_{ij}\lambda_{ji}^{-1}\rho_{ij}, \ (\rho_{ij}\rho_{ji}^{-1}\lambda_{ij})^4 = 1$;
- $[\epsilon_1, \rho_{ij}] = [\epsilon_1, \lambda_{ij}] = 1$ *for* $i, j \neq 1$;
- $\rho_{12}^{\epsilon_1} = \lambda_{12}^{-1}, \ \rho_{21}^{\epsilon_1} = \rho_{21}^{-1}$;
- $\epsilon_1^2 = 1$;
- $\prod_{i \neq j} \rho_{ij}\lambda_{ij}^{-1} = 1$ *for each fixed* j.

Note the actions of $\mathrm{Aut}(F_n)$ on F_n and of $\mathrm{Out}(F_n)$ on the conjugacy classes of F_n are **on the left**.

Definition 1.3.5. Let us define some standard homomorphisms:

- $p_n : \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n) \cong \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ is the quotient map, where $\mathrm{Inn}(F_n)$ is the group of inner automorphisms of F_n;
- $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Out}(F_n)/\langle\langle \rho_{ij}\lambda_{ij}^{-1} \mid i, j = 1, \ldots, n, \ i \neq j \rangle\rangle$ is the quotient map. Its kernel will be denoted by $\overline{\mathrm{IA}}_n$, and referred to as the Torelli subgroup of $\mathrm{Out}(F_n)$. We will use IA_n to denote the kernel of the composition $\pi_n \circ p_n$, and refer to it as the Torelli subgroup of $\mathrm{Aut}(F_n)$;
- $i_n^{\mathbb{K}} : \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{K})$ is the natural embedding induced by the unique unital ring homomorphism $\mathbb{Z} \to \mathbb{K}$;
- we adopt the convention $i_n = i_n^{\mathbb{C}}$;
- $\det : \mathrm{GL}_n(\mathbb{Z}) \to \mathbb{Z}_2$ denotes the determinant map, where $\mathbb{Z}_2$ is identified with the multiplicative group $\{1, -1\}$;
- we will also slightly abuse the notation and use $\det : \mathrm{Out}(F_n) \to \mathbb{Z}_2$ to denote the composition $\det \circ \pi_n$. The kernel of this homomorphism will be denoted by $\mathrm{SOut}(F_n)$.

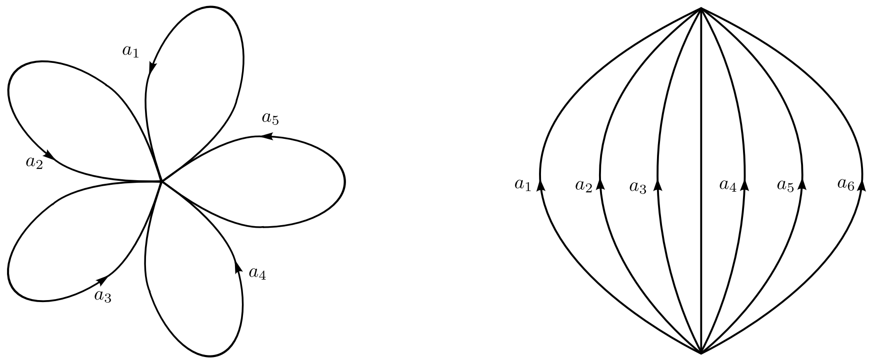


Figure 1.3.7: The 5-rose and 7-cage graphs

Definition 1.3.6. Let us define some finite subgroups of $\mathrm{Out}(F_n)$:

$$\begin{aligned} S_n &\cong \langle\{\sigma_{ij} \mid i,j=1,\dots,n,\ i\neq j\}\rangle \\ S_{n+1} &\cong \langle\{\sigma_{ij} \mid i,j=1,\dots,n+1,\ i\neq j\}\rangle \\ \mathbb{Z}_2^n \rtimes S_n \cong W_n &= \langle\{\epsilon_1,\sigma_{ij} \mid i,j=1\dots,n,\ i\neq j\}\rangle \\ \mathbb{Z}_2 \times S_{n+1} \cong G_n &= \langle\{\Delta,\sigma_{ij} \mid i,j=1\dots,n+1,\ i\neq j\}\rangle. \end{aligned}$$

We do not give distinctive names to the groups on the right in the first two lines; instead, we will usually refer to them as respectively $S_n < W_n$ and $S_{n+1} < G_n$. More generally, whenever we mention S_n or S_{n+1} as subgroups of $\mathrm{Out}(F_n)$, we mean these two groups.

Note that we abuse notation by using S_n to denote the abstract symmetric group of degree n as well. We will denote its maximal alternating subgroup by A_n.

We will often talk about the natural action of S_n and A_n on $\{1,2,\dots,n\}$. When doing so in the case of S_n, we will always mean the action in which

$$\sigma_{ij}(k) = \begin{cases} j & \text{if} \quad k=i \\ i & \text{if} \quad k=j \\ k & \text{if} \quad k\neq i,j \end{cases}.$$

In the case of A_n, we will mean the restriction of the described action to $A_n < S_n$.

Observe that the subgroup W_n is the automorphism group of the n-rose, that is the graph with one vertex and n edges, whereas the subgroup G_n is the automorphism group of the $(n+1)$-cage, that is the graph with two vertices and $n+1$ edges, such that each edge has both vertices as its endpoints (see Figure 1.3.7). Choosing the right isomorphism between the fundamental groups of these graphs and F_n induces the embeddings $W_n, G_n < \mathrm{Out}(F_n)$.

In the case of $\mathrm{Out}(F_3)$, let us also define V_4 and A_4 to be the Klein 4-group and the alternating group of degree 4 satisfying

$$V_4 < A_4 < S_4 < G_3 < \mathrm{Out}(F_3).$$

Note that, if $i, j \leqslant n$, we have

$$\epsilon_i \sigma_{ij} = \lambda_{ij} \lambda_{ji}^{-1} \rho_{ij} = \rho_{ij} \rho_{ji}^{-1} \lambda_{ij},$$

and the subgroup $S_n < \mathrm{Out}(F_n)$ defined above acts on the sets

$$\begin{aligned} &\{\epsilon_i \mid i = 1 \ldots, n\}, \\ &\{\rho_{ij} \mid i, j = 1 \ldots, n,\ i \neq j\}, \text{ and} \\ &\{\lambda_{ij} \mid i, j = 1 \ldots, n,\ i \neq j\} \end{aligned}$$

by permuting the indices in the natural way.

Chapter 2

Some classical representation theory

In this chapter we recall some basic facts from the (classical) representation theory of symmetric groups and general linear groups over $\mathbb{C}$. Most of the results can be found in [11].

Let us first prepare the ground by giving necessary definitions.

Definition 2.0.1 (Representations)**.** Given a field $\mathbb{K}$ and a $\mathbb{K}$-vector space V, a group homomorphism $\phi : G \to \mathrm{GL}(V)$ is called a $(\dim V)$-*dimensional* $\mathbb{K}$-*linear representation* of G. We say that ϕ is *reducible* if and only if there exists a G-invariant non-trivial proper subspace of V. If no such subspace exists, we say that ϕ is *irreducible*.

A map $f : U \to V$, where $\phi : G \to \mathrm{GL}(U)$ and $\psi : G \to \mathrm{GL}(V)$ are representations, is a *morphism* of representations if and only if it is G-equivariant, that is

$$\forall g \in G \, : \, f \circ \phi(g) = \psi(g) \circ f.$$

We say that f is an *isomorphism* if and only if there exists a morphism $f' : V \to U$ such that $ff' = \mathrm{id}_U$ and $f'f = \mathrm{id}_V$.

Note that we will often refer to V as the representation. We will equally often call V a G-module.

Let us define one important representation.

Definition 2.0.2. We say that a representation $\phi : G \to \mathrm{GL}(V)$ is *trivial* if and only if $\dim V = 1$ and $\ker \phi = G$.

We are now ready to state a useful lemma.

Lemma 2.0.3 (Schur's Lemma)**.** *Suppose*

$$f : U \to V$$

is a morphism of G-representations. Suppose further that U is irreducible. Then either $f(U)$ is trivial or f is an isomorphism onto its image.

2.1 Linear representations of symmetric groups

Definition 2.1.1 (Partitions)**.** Given a natural number n, we say that

$$\mu = (\mu_1, \mu_2, \ldots, \mu_k)$$

is a *partition* of n if and only if $\mu_i \in \mathbb{N}$ for each i,

$$\sum_{i=1}^{k} \mu_i = n$$

and $\mu_i \geqslant \mu_{i+1} > 0$ for all $i \geqslant 1$.

For brevity we will write μ_i^α to mean a subsequence $\mu_i, \mu_{i+1}, \ldots, \mu_{i+\alpha-1}$ whenever $\mu_i = \mu_{i+j}$ for all $0 \leqslant j \leqslant \alpha - 1$.

Now let us state the main result of the theory.

Theorem 2.1.2. *Let $n \in \mathbb{N}$ and let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > n$. Then the isomorphism types of irreducible $\mathbb{K}$-linear representations of S_n are in a one-to-one correspondence with partitions of n.*

We shall give names to a number of particularly useful representations of S_n.

Definition 2.1.3. When V is an irreducible representation of S_n corresponding to a partition μ, we say that

- V is the *trivial* representation if and only if $\mu = (n)$;
- V is the *determinant* representation if and only if $\mu = (1^n)$;
- V is the *standard* representation if and only if $\mu = (n-1, 1)$;
- V is the *signed standard* representation if and only if $\mu = (2, 1^{n-1})$.

When V is isomorphic to direct sum of the trivial and standard representation, we say it is the *permutation* representation. Similarly, when it is isomorphic to a direct sum of the determinant and signed standard representation, we say it is the *signed permutation* representation.

We shall now give a topological way of thinking about some of the above representations.

Example 2.1.4. Let R be the n-rose, with petals labeled by letters $\{a_1, a_2, \ldots, a_n\}$, and let S_n act on R, so that the action is by orientation preserving graph morphisms, and is natural on the indices of the labels. Then S_n acts on $H_1(R, \mathbb{K}) \cong \mathbb{K}^n$, and the representation is isomorphic to the permutation representation.

Now let C be the n-cage, with all ribs labeled by letters $\{a_1, a_2, \ldots, a_n\}$, and let S_n act on R, so that the action is again by orientation preserving graph morphisms, and is natural on the indices of the labels. Then S_n acts on $H_1(C, \mathbb{K}) \cong \mathbb{K}^{n-1}$, and the representation is isomorphic to the standard representation.

Definition 2.1.5 (Natural bases)**.** We say the basis for $H_1(R, \mathbb{K})$ coming from the cycles labeled

$$a_1, a_2, \ldots, a_n$$

is the *natural basis* for the permutation module. Similarly, we say that the basis for $H_1(C, \mathbb{K})$ coming from the cycles labeled

$$a_1 a_n^{-1}, a_2 a_n^{-1}, \ldots, a_{n-1} a_n^{-1}$$

is the *natural basis* for the standard module.

Proposition 2.1.6 (Branching rule)**.** *Suppose we have an irreducible S_{n+1}-module V corresponding to a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of $n+1$. Then, as an S_n-module, V is isomorphic to the direct sum of all S_n representations corresponding to partitions $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_{k'})$ of n, such that*

$$\sum_{i=1}^{k} |\mu'_i - \mu_i| = 1$$

with the convention that $\mu'_j = 0$ whenever $j > k'$.

The above proposition is extremely useful, as the following example is supposed to exemplify.

Example 2.1.7. Suppose V is the standard representation of S_{n+1}, so it corresponds to the partition $(n, 1)$. There are two partitions of n that satisfy the above condition, namely (n) and $(n-1, 1)$. Hence, as an S_n-module,

$$V = V_{\mathrm{tr}} \oplus V_{\mathrm{std}},$$

where V_{tr} is isomorphic to the trivial, and V_{std} to the standard S_n-module.

2.2 Schur functors and general linear groups

We now introduce a useful tool of representation theory.

Definition 2.2.1 (Schur functors)**.** Let $\mathrm{Rep}_G^{\mathbb{K}}$ be the category of $\mathbb{K}$-linear representations of a group G, and let μ be a partition of a natural number n. Then we define the *Schur functor* $\mathbb{S}_\mu\colon \mathrm{Rep}_G^{\mathbb{K}} \to \mathrm{Rep}_G^{\mathbb{K}}$ corresponding to μ by its action on the objects:

$$\mathbb{S}_\mu U = U^{\otimes n} \otimes_{S_n} V_\mu,$$

where V_μ is the irreducible S_n-module corresponding to μ, and where S_n acts on $U^{\otimes n}$ by permuting the factors in the natural way, and G acts on V_μ trivially and on $U^{\otimes n}$ diagonally.

We have seen above how we can classify all irreducible representations of symmetric groups. We can do a very similar thing with general linear groups over $\mathbb{C}$.

Theorem 2.2.2. *Suppose U is a $\mathbb{C}$-linear irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ for some n. Then U is isomorphic to $\mathbb{S}_\mu V$ for some partition μ (not necessarily of n), where $V = \mathbb{C}^n$ is the standard $\mathrm{GL}_n(\mathbb{C})$-module, that is $\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$.*

Let us list a number of useful facts about $\mathrm{GL}_n(\mathbb{C})$-modules.

Proposition 2.2.3. *Suppose $U = \mathbb{S}_\mu V$ is an irreducible representation of*

$$\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V),$$

with $\mu = (\mu_1, \mu_2, \dots, \mu_k)$. Then

1. *U is the trivial representation whenever $\mu_{n+1} \neq 0$;*
2. *$\mathbb{S}_{(1^n)}V$ is the determinant representation, that is the one given by*

$$\mathrm{GL}_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C});$$

3. *$U \cong \mathbb{S}_{\mu'}V \otimes \mathbb{S}_{(1^n)}V$, where $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_k)$ satisfies $\mu_i + 1 = \mu'_i$ for all $i \leqslant n$ (again with the convention that $\mu_i = 0$ whenever $i > k$);*
4. *if U is non-trivial, its dimension is given by the formula*

$$\dim U = \prod_{1 \leqslant i < j \leqslant n} \frac{\mu_i - \mu_j + i - j}{i - j} \ .$$

2.3 Representations of $\mathrm{GL}_n(\mathbb{Z}_q)$

In this section we will be concerned with the groups $\mathrm{GL}_n(\mathbb{Z}_q)$, where q is a prime power.

Let us first mention an extremely useful theorem of Mennicke:

Theorem 2.3.1 (Mennicke [21])**.** *The group* $\mathrm{SL}_n(\mathbb{Z}_q)$ *satisfies*

$$\mathrm{SL}_n(\mathbb{Z}_q) = \mathrm{SL}_n(\mathbb{Z})/\langle\langle \rho_{ij}^q \mid i,j = 1,\, i \neq j\, \ldots, n\rangle\rangle.$$

Note that here we have abused the notation be using ρ_{ij} to denote $\pi_n(\rho_{ij})$, which is of course an elementary matrix.

Let us note an immediate corollary.

Corollary 2.3.2. *The group* $\mathrm{GL}_n(\mathbb{Z}_2)$ *satisfies*

$$\mathrm{GL}_n(\mathbb{Z}_2) = \mathrm{Out}(F_n)/\langle\langle \epsilon_i \mid i = 1, \ldots, n\rangle\rangle.$$

Proof. This follows directly from three observations, namely that $\mathrm{GL}_n(\mathbb{Z}_2) = \mathrm{SL}_n(\mathbb{Z}_2)$, that

$$\rho_{ij}^2 = (\rho_{ij}^{\epsilon_j}\rho_{ij}^{-1})^{-1},$$

and that the image of ϵ_i is trivial in $\mathrm{GL}_n(\mathbb{Z}_2)$. ∎

We will also need a result about representations of $\mathrm{GL}_n(\mathbb{Z}_p)$, for prime p, due to Landazuri and Seitz:

Theorem 2.3.3 (Landazuri–Seitz [19])**.** *Suppose we have a non-trivial, irreducible projective representation* $\mathrm{PSL}_n(\mathbb{Z}_p) \to \mathrm{PGL}(V)$*, where* $n \geqslant 3$*,* p *is prime, and* V *is a vector space over a field* $\mathbb{K}$ *of characteristic other than* p*. Then*

$$\dim V \geqslant \begin{cases} 2 & \text{if} \quad (n,p) = (3,2) \\ p^{n-1} - 1 & \text{otherwise} \end{cases}.$$

We offer an extension of their theorem for algebraically closed fields of characteristic 0.

Theorem 2.3.4. *Let* V *be a non-trivial, irreducible* $\mathbb{K}$*-linear representation of* $\mathrm{SL}_n(\mathbb{Z}_q)$*, where* $n \geqslant 3$*,* q *is a power of a prime* p*, and where* $\mathbb{K}$ *is an algebraically closed field of characteristic 0. Then*

$$\dim V \geqslant \begin{cases} 2 & \text{if} \quad (n,p) = (3,2) \\ p^{n-1} - 1 & \text{otherwise} \end{cases}.$$

Proof. Let $\phi : \mathrm{SL}_n(\mathbb{Z}_q) \to \mathrm{GL}(V)$ denote our representation. Consider Z, the subgroup of $\mathrm{SL}_n(\mathbb{Z}_q)$ generated by diagonal matrices with all non-zero entries equal. Note that Z is the centre of $\mathrm{SL}_n(\mathbb{Z}_q)$. Hence V splits as an $\mathrm{SL}_n(\mathbb{Z}_q)$-module into intersections of eigenspaces of all elements of Z. Since V is irreducible, we conclude that $\phi(Z)$ lies in the centre of $\mathrm{GL}(V)$.

First suppose that $q = p$. Consider the composition

$$\mathrm{SL}_n(\mathbb{Z}_q) \to \mathrm{GL}(V) \to \mathrm{PGL}(V).$$

We have just showed that Z lies in the kernel of this composition, and so our representation descends to a representation of $\mathrm{PSL}_n(\mathbb{Z}_p) \cong \mathrm{SL}_n(\mathbb{Z}_p)/Z$. This new, projective representation is still irreducible. It is also non-trivial, as otherwise V would have to be a 1-dimensional non-trivial $\mathrm{SL}_n(\mathbb{Z}_q)$-representation. There are no such representations since $\mathrm{SL}_n(\mathbb{Z}_q)$ is perfect when $p = q$. Now Theorem 2.3.3 yields the result.

Suppose now that $q = p^m$, where $m > 1$. Let $N \trianglelefteq \mathrm{SL}_n(\mathbb{Z}_q)$ be the kernel of the natural map $\mathrm{SL}_n(\mathbb{Z}_q) \to \mathrm{SL}_n(\mathbb{Z}_p)$. As an N-module, by Maschke's Theorem, V splits as

$$V = \bigoplus_{i=1}^{k} U_i$$

where each $U_i \neq \{0\}$ is a direct sum of irreducible N-modules, and irreducible submodules $W \leqslant U_i, W' \leqslant U_j$ are isomorphic if and only if $i = j$.

Observe that we get an induced action of $\mathrm{SL}_n(\mathbb{Z}_q)/N \cong \mathrm{SL}_n(\mathbb{Z}_p)$ on the set $\{U_i, U_2, \ldots, U_k\}$. As V is an irreducible $\mathrm{SL}_n(\mathbb{Z}_q)$-module, the action is transitive.

Since our result holds for $q = p$, and since an action of a group on a finite set S induces a representation on the vector space with basis S, we conclude that either $k = 1$, or

$$k \geqslant \begin{cases} 2 & \text{if} \quad (n,p) = (3,2) \\ p^{n-1} - 1 & \text{otherwise} \end{cases} .$$

In the latter case, as $\dim U_i \geqslant 1$ for all i, we get $\dim V \geqslant k$ and our result follows.

Let us henceforth assume that $k = 1$. We have

$$V = U_1 = \bigoplus_{j=1}^{l} W,$$

where W is an irreducible N-module.

Note that we have the alternating group $A_n < \mathrm{SL}_n(\mathbb{Z}_q)$ satisfying $A_n \cap N = \{1\}$, and so $A_n < \mathrm{SL}_n(\mathbb{Z}_p)$ in a natural way. Let $\sigma \in A_n$ be an element of order

$$o(\sigma) \in \{2, 3\}.$$

Consider the group $M = \langle N, \sigma \rangle < \mathrm{SL}_n(\mathbb{Z}_q)$. Note that $M \cong N \rtimes \mathbb{Z}_{o(\sigma)}$. The module V splits as a direct sum of M-modules. Frobenius Reciprocity (see e.g. [25, Corollary 4.1.17]) tells us that the multiplicity (let us call it m) of W (as an N-module) in each of the irreducible M-modules is equal to the multiplicity of that M-module in the M-module induced from the N-module W. Hence m^2 is the multiplicity of W in the M-module induced from the N-module W. But the latter is bounded above by $o(\sigma)$ and $o(\sigma) \in \{2, 3\}$, and so $m = 1$.

This shows in particular that W is an irreducible M-module. It also shows that the M-module induced from W contains a submodule isomorphic to W. Since

$$M \cong N \rtimes \mathbb{Z}_{o(\sigma)},$$

an easy calculation shows that σ acts on this copy of W as a scalar multiple of the identity matrix, i.e. via a central matrix. Hence σ commutes with N when acting on V. Since the above statement is true for each $\sigma \in A_n$ of order 2 or 3, we conclude that ϕ factors through $\mathrm{SL}_n(\mathbb{Z}_q)/[N, A_n]$. Note that we need to consider elements σ of order 3 when we are dealing with the case $n = 4$.

Theorem 2.3.1 tells us that N is normally generated (as a subgroup of $\mathrm{SL}_n(\mathbb{Z}_q)$) by all elements of the form ρ_{ij}^p. Now $\mathrm{SL}_n(\mathbb{Z}_q)$ itself is generated by all elements ρ_{ij}. Observe that

$$\forall \sigma \in A_n \, : \, \phi(\rho_{\alpha\beta}^{-1} \rho_{ij}^p \rho_{\alpha\beta}) = \phi(\sigma^{-1} \rho_{\alpha\beta}^{-1} \rho_{ij}^p \rho_{\alpha\beta} \sigma) = \phi(\rho_{\sigma(\alpha)\sigma(\beta)}^{-1} \rho_{ij}^p \rho_{\sigma(\alpha)\sigma(\beta)}).$$

Choose $\sigma \in A_n$ such that $\sigma(\alpha) = i$ and $\sigma(\beta) = j$. We conclude that $\phi(N)$ lies in the centre of $\phi\big(\mathrm{SL}_n(\mathbb{Z}_q)\big)$. In particular, $\phi(N)$ is abelian, and hence (as $\mathbb{K}$ is algebraically closed) $\dim W = 1$, as W is an irreducible N-module. Since V is a direct sum of N-modules isomorphic to W, the group N acts via matrices in the centre of $\mathrm{GL}(V)$. Hence N lies in the kernel of the composition

$$\mathrm{SL}_n(\mathbb{Z}_q) \xrightarrow{\phi} \mathrm{GL}(V) \longrightarrow \mathrm{PGL}(V) \, .$$

We have already shown that Z lies in this kernel, and so our representation descends to a projective representation of $\mathrm{PSL}_n(\mathbb{Z}_p)$. If we can show that this representation is non-trivial, we can then apply Theorem 2.3.3 and our proof will be finished.

Suppose that this projective representation is trivial. This means that V is a 1-dimensional, non-trivial $\mathrm{SL}_n(\mathbb{Z}_q)$-representation. This is however impossible, since the abelianisation of $\mathrm{SL}_n(\mathbb{Z}_q)$ is trivial when $n \geqslant 3$. □

In our considerations we will need to use the following elementary calculation.

Proposition 2.3.5. *Let A be the kernel of the map $\mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z}_2)$ induced by the surjection $\mathbb{Z} \to \mathbb{Z}_2$. Let V be the standard, 3-dimensional $\mathbb{K}$-linear representation of $\mathrm{GL}_3(\mathbb{Z})$. Suppose further that $\mathbb{K}$ is a field of characteristic 0 or at least 3. Then $W = \mathbb{S}_{(3,3,1)}V$ is irreducible as an A-module.*

Proof. Note that $W = \mathbb{S}_{(1^3)}V \otimes \mathbb{S}_{(2)}V^*$, that is it is the tensor of the determinant representation and the second symmetric power of the dual representation.

Let $U \leqslant W$ be an irreducible A-submodule of W, and let $\{v_1, v_2, v_3\}$ be the standard dual basis of V. Suppose $v \in U \smallsetminus \{0\}$. Then

$$v = \sum_{i \leqslant j} \mu_{ij}\, v_i \otimes v_j$$

for some collection of scalars μ_{ij}.

We are going to abuse notation by using the symbols ϵ_i and ρ_{ij} to denote the images of respective elements under $\pi : \mathrm{Out}(F_3) \to \mathrm{GL}_3(\mathbb{Z})$. Note that $\epsilon_i \epsilon_j \in A$ and $\rho_{ij}^2 \in A$ for each appropriate $i \neq j$. Now

$$\epsilon_1\epsilon_2(v) - v = -2\,\mu_{23}\, v_2 \otimes v_3 - 2\,\mu_{13}\, v_1 \otimes v_3$$

and hence

$$\epsilon_1\epsilon_3\big(\epsilon_1\epsilon_2(v) - v\big) - v = 4\,\mu_{13}\, v_1 \otimes v_3.$$

Hence, if $\mu_{ij} \neq 0$ for some $i \neq j$, then $v_i \otimes v_j \in U$.

Furthermore

$$\rho_{13}^2(v_1 \otimes v_3) - v_1 \otimes v_3 = -2\, v_1 \otimes v_1$$

and

$$\rho_{23}^2(v_1 \otimes v_3) - v_1 \otimes v_3 = -2\, v_1 \otimes v_2,$$

and therefore if $\mu_{ij} \neq 0$ for some $i \neq j$, then $U = V$.

Suppose that

$$v = \sum_i \mu_{ii}\, v_i \otimes v_i.$$

Without loss of generality let us assume that $\mu_{11} \neq 0$. Then

$$\rho_{21}^2(v) - v = -\mu_{11}\big(4\, v_1 \otimes v_2 - 4\, v_2 \otimes v_2\big) = v' \in U.$$

We can now apply our argument to v' and conclude that $U = V$. □

Chapter 3

Linear representations of $\mathrm{Out}(F_n)$

In this chapter we will investigate the structure of linear representations of $\mathrm{Out}(F_n)$, over a variety of fields. Our approach is based on using the linear representation theory of the finite subgroup $W_n \cong \mathbb{Z}_2^n \rtimes S_n$, and then on developing a convenient way of dealing with the combinatorics of a linear representation of $\mathrm{Out}(F_n)$. Both of these techniques will be discussed in the following section.

3.1 Representations of W_n and diagrams

Definition 3.1.1. Let V be a representation of W_n. Let $N = \{1, \ldots, n\}$. Define

- for each $I \subseteq N$, $E_I = \{v \in V \mid \epsilon_i v = (-1)^{\chi_I(i)} v\}$, where χ_I is the characteristic function of I;
- $V_i = \bigoplus_{|I|=i} E_I$.

We will slightly abuse notation, and often omit parentheses writing E_1 for $E_{\{1\}}$, etc.

Lemma 3.1.2. *Let V be a representation of W_n. Then $\dim V_i = \binom{n}{i} \dim E_I$, where $|I| = i$.*

Proof. The symmetric group $S_n < W_n$ acts on $\{\epsilon_1, \ldots, \epsilon_n\}$ by permuting the indices in the natural way. Hence its action on V_i will permute subspaces E_I by permuting subsets of N of cardinality i. Thus each E_I, for a fixed size of I, has the same dimension. □

Lemma 3.1.3. *Let V be a $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$, where $\mathbb{K}$ is a field of characteristic other than 2. Then, with the notation above, we have*

$$V = \bigoplus_{i=0}^{n} V_i$$

and for each $i \neq j$, $J \subseteq N \smallsetminus \{i,j\}$ we have

$$\rho_{ij}(E_J \oplus E_{J\cup\{i\}} \oplus E_{J\cup\{j\}} \oplus E_{J\cup\{i,j\}}) = E_J \oplus E_{J\cup\{i\}} \oplus E_{J\cup\{j\}} \oplus E_{J\cup\{i,j\}}.$$

An identical statement holds for λ_{ij}.

Proof. The first statement follows directly from the fact that we can simultaneously diagonalise commuting involutions ϵ_i, since we are working over a field $\mathbb{K}$ whose characteristic is not 2.

For the second statement, let us note that $[\rho_{ij}, \epsilon_k] = 1$ for each $k \notin \{i,j\}$. Hence for each $I \subseteq N$:

$$\rho_{ij}(E_I) \leqslant \bigoplus_{J \triangle I \subseteq \{i,j\}} E_J,$$

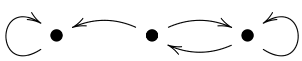
where $A \triangle B$ denotes the symmetric difference of two sets A and B. An identical argument works for λ_{ij}. □

To help us visualise the combinatorics of representations of $\mathrm{Out}(F_n)$ we introduce a calculus of diagrams.

Definition 3.1.4. Suppose V is a finite dimensional, $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$ over any field $\mathbb{K}$, and let $x \in \mathrm{Out}(F_n)$. Let us use the notation of Definition 3.1.1. We define *the minimal diagram for x over V* (often abbreviated to the minimal diagram for x) to be a directed graph D with the vertex set equal to a subset S of the power set of $N = \{1, 2, \ldots, n\}$, where $I \in S$ if and only if $E_I \neq \{0\}$, and the edge set given by the following rule: there is a directed edge from I to J if and only if $r_J\big(x(E_I)\big) \neq \{0\}$, where $r_J : \bigoplus_{K \subseteq N} E_K \to E_J$ is the natural projection.

We also say that a graph D' is *a diagram for x over V* if and only if the minimal diagram D for x is a subgraph of D'.

In practice, when realising these diagrams in terms of actual pictures, we are going to align vertices corresponding to subsets of N of the same cardinality in horizontal lines; each such line will correspond to some V_i. We are also going to represent edges as follows: if two vertices are joined by two directed edges, we are going to draw one edge without any arrowheads between them; we are not going to draw edges from a vertex to itself – instead, if a vertex does not have such a loop, then all edges emanating from it will be drawn with a tail (see the example below);

 becomes .

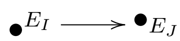
To get a firmer grip on these diagrams, let us have a look at a number of facts one can easily deduce from (not necessarily minimal) diagrams.

Remark 3.1.5. Let D_0 be a connected component of D, a diagram for x. Let $v \in \bigoplus_{I\in V(D_0)} E_I$ be a vector. Then $v = \sum v_I$ where $v_I \in E_I$. Let $J \notin V(D_0)$. Note that there are no edges between J and $V(D_0)$, and so $r_J\big(x(E_I)\big) = \{0\}$ for all $I \in V(D_0)$. Hence $x(v_I) \in \bigoplus_{I\in V(D_0)} E_I$ and therefore

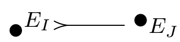
$$x(\bigoplus_{I\in V(D_0)} E_I) = \bigoplus_{I\in V(D_0)} E_I.$$

The following illustrates the relationship between our diagrams and matrices.

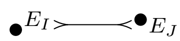
Example 3.1.6. Suppose we have a diagram for x with at least two vertices, I and J say, such that the union of the connected components containing these two vertices does not contain any other vertex. Fix a basis for E_I and E_J. The following illustrates the way the x action on $E_I \oplus E_J$ (seen as a matrix) depends on the diagram:



corresponds to $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$,

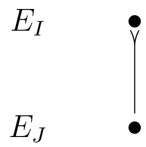


to $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$, and



to $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.

Example 3.1.7. Suppose we have a diagram D for x such that D has a connected component with only two vertices, I and J say, as depicted below.



Remark 3.1.5 tells us that $x|_{E_I\oplus E_J}$ is an isomorphism. Let $\{v_1, \ldots, v_k\}$ be a basis for E_I. Our diagram tells us that $r_I\big(x(v_i)\big) = 0$ for each i, and so $x(v_i) \in E_J$. Since x is an isomorphism, we immediately see that

$$\{x(v_1), \ldots, x(v_k)\}$$

is a linearly independent set, and hence $\dim E_J \geqslant \dim E_I$.

Example 3.1.8. Let D be a diagram for x. Note that $r_J\big(x^{\epsilon_i}(v)\big) = \pm r_J\big(x(v)\big)$ whenever $v \in E_I$ for some I. Therefore $r_J\big(x^{\epsilon_i}(E_I)\big) = \{0\}$ if and only if

$$r_J\big(x(E_I)\big) = \{0\},$$

and so D is also a diagram for x^{ϵ_i}.

Now consider $\sigma \in S_n$. We have

$$r_J\big(x^\sigma(E_I)\big) = \sigma^{-1}\Big(r_{\sigma(J)}\big(x(E_{\sigma(I)})\big)\Big),$$

and so the image of D under the graph morphism induced by I $\to \sigma(I)$ is a diagram for x^σ.

We will use the last example very often, for instance to relate diagrams for ρ_{21} with ones for $\rho_{21}^{-1} = \rho_{21}^{\epsilon_1}$ or $\rho_{31} = \rho_{21}^{\sigma_{23}}$.

3.2 The case of $\mathrm{Out}(F_3)$

First let us investigate the linear representations of $\mathrm{Out}(F_3)$ in the lowest dimensions, namely 1 and 2.

Proposition 3.2.1. *Suppose $\phi : \mathrm{Out}(F_3) \to G$ is a group homomorphism such that its kernel contains V_4. Then ϕ factors as*

$$\begin{array}{ccc} \mathrm{Out}(F_3) & \xrightarrow{\ \phi\ } & G \\ {\scriptstyle \det}\downarrow & \nearrow & \\ \mathbb{Z}_2 & & \end{array}$$

and so is determined by the image of ϵ_1.

Proof. Since V_4 lies in the kernel, we have $\phi(\sigma_{14}) = \phi(\sigma_{23})$. Hence, using $[\rho_{21}, \rho_{31}] = 1$ and $\epsilon_1\sigma_{14} = \rho_{21}\rho_{31}$, we get

$$\phi(\rho_{21}) = \phi(\rho_{21}^{\rho_{21}\rho_{31}}) = \phi(\rho_{21}^{\epsilon_1\sigma_{14}}) = \phi(\rho_{21})^{\phi(\epsilon_1)\phi(\sigma_{14})} = \phi(\rho_{21})^{\phi(\epsilon_1)\phi(\sigma_{23})} = \phi(\rho_{31}^{-1})$$

and thus, using $[\rho_{21}, \epsilon_3] = 1$,

$$\phi(\rho_{21}) = \phi(\rho_{21}^{\epsilon_3}) = \phi(\rho_{31}^{-1})^{\phi(\epsilon_3)} = \phi(\lambda_{31}).$$

Now

$$\phi(\rho_{31}^{-1}) = \phi([\rho_{32}^{-1}, \rho_{21}^{-1}]) = [\phi(\rho_{32}^{-1}), \phi(\lambda_{31}^{-1})] = 1.$$

Thus ρ_{31} lies in the kernel of ϕ. We can however conjugate ρ_{31} to each ρ_{ij} using S_3, and so all elements ρ_{ij} lie in the kernel. The result follows. □

An immediate consequence of the above is the following.

Lemma 3.2.2. *Let V be a 2-dimensional $\mathbb{K}$-linear representation of* $\mathrm{Out}(F_3)$*, where* $\mathrm{char}(\mathbb{K}) \neq 2$*. Then the representation factors as*

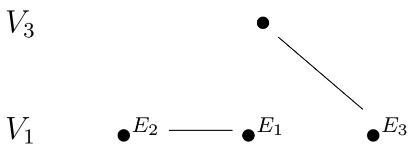
$$\begin{array}{ccc} \mathrm{Out}(F_3) & \xrightarrow{\phi} & \mathrm{GL}_2(\mathbb{K}) \\ {\scriptstyle \det}\downarrow & \nearrow & \\ \mathbb{Z}_2 & & \end{array}$$

and so is determined by the image of ϵ_1.

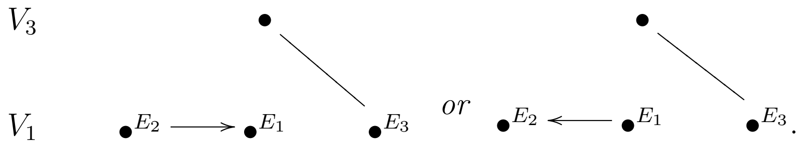
Proof. There are at most three irreducible $\mathbb{K}$-linear representations of S_4 of dimension at most 2: the trivial representation (corresponding to partition (4)), the determinant representation (corresponding to partition (1^4)), and the one given by a partition $(2, 2)$ (note that the latter might not be irreducible when $\mathrm{char}(\mathbb{K}) = 3$). In all three cases, the action of $V_4 < S_4$ is trivial. This implies that we have satisfied all the requirements of Proposition 3.2.1, and the result follows. □

Now we shall establish a number of lemmata which will come to our aid in the attempt to classify linear representations of $\mathrm{Out}(F_3)$ of dimension at most 6.

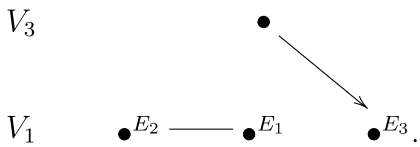
Lemma 3.2.3. *Let $\phi : \mathrm{Out}(F_3) \to \mathrm{GL}(V)$ be a representation such that we have a diagram for ρ_{21} of the form*



where $\dim E_i = 1$ *for all i. Then ρ_{21} has a diagram*



Proof. Suppose for a contradiction that $r_1\big(\rho_{21}(E_2)\big) \neq \{0\}$ and $r_2\big(\rho_{21}(E_1)\big) \neq \{0\}$. We claim that then ρ_{21} has a diagram



Once we have proven the above claim, take $x \in E_2$. Then, by our assumptions, $r_1\big(\rho_{21}(x)\big) \neq 0$ and so (again by assumption)

$$r_3\big(\rho_{31}\rho_{21}(x)\big) \neq 0.$$

But $\rho_{31}(x) \in E_2$ and thus $\rho_{21}\rho_{31}(x) \in E_2 \oplus E_1$. This contradicts the relation $[\rho_{21}, \rho_{31}] = 1$, and our proof is complete.

Now, to prove the claim, let $v_1 \in E_1 \setminus \{0\}$. Then $v_3 = \sigma_{23} r_2\big(\rho_{21}(v_1)\big)$ generates E_3. Now if

$$r_{1,2,3}\big(\rho_{21}(v_3)\big) = 0$$

then we have proven our claim. If not, let $U = \langle u \rangle$ be a subspace of V_3 of dimension 1, where $u = r_{1,2,3}\big(\rho_{21}(v_3)\big)$. Note that $\rho_{21}v_3 \in u + E_3$, and so

$$\rho_{21}^{-1}(-v_3) = \epsilon_1 \rho_{21} \epsilon_1(-v_3) \in u + E_3.$$

In particular, $\rho_{21}^{-1}(-v_3) - v_3' = u$ for some $v_3' \in E_3$. Now

$$\rho_{21}u = \rho_{21}\big(\rho_{21}^{-1}(-v_3) - v_3'\big) \in E_3 \oplus U,$$

and hence $U \oplus E_3$ is ρ_{21}-invariant.

Let us rewrite $\rho_{21}\rho_{31} = \rho_{31}\rho_{21}$ as

$$\rho_{21}\sigma_{23}\rho_{21}\sigma_{23} = \sigma_{23}\rho_{21}\sigma_{23}\rho_{21}$$

which yields $[\rho_{21}\sigma_{23}\rho_{21}, \sigma_{23}] = 1$. But now v_1 lies in the μ-eigenspace of σ_{23} where $\mu = \pm 1$, and hence so does $\rho_{21}\sigma_{23}\rho_{21}(v_1)$. Note that a diagram chase gives us the following

$$\begin{aligned} r_{1,2,3}\big(\rho_{21}\sigma_{23}\rho_{21}(v_1)\big) &= r_{1,2,3}\Big(\rho_{21} r_3\big(\sigma_{23}\rho_{21}(v_1)\big)\Big) \\ &= r_{1,2,3}\Big(\rho_{21}\sigma_{23} r_2\big(\rho_{21}(v_1)\big)\Big) \\ &= r_{1,2,3}\Big(\rho_{21}(v_3)\Big) \\ &= u. \end{aligned}$$

Therefore, as V_3 is S_3-invariant, U lies in the μ-eigenspace of σ_{23}.

Note that the eigenspaces of Δ are $V_0 \oplus V_2$ and $V_1 \oplus V_3$, and on each Δ acts as ± 1. Hence, since $V_0 \oplus V_2 = \{0\}$, we see that $[\phi(\rho_{21}), \phi(\Delta)] = 1$. Therefore, since $\rho_{ij}^{\Delta} = \lambda_{ij}$, the elements ρ_{ij} and λ_{ij} act identically for each i and j. This in turn implies that $[\phi(\rho_{21}), \phi(\rho_{23})] = 1$ since $[\rho_{21}, \lambda_{23}] = 1$. Rewriting the first relation as before we get $[\phi(\rho_{21}\sigma_{13}\rho_{21}), \phi(\sigma_{13})] = 1$.

Let $v_2 \in E_2 \smallsetminus \{0\}$. Then $\langle r_{1,2,3}(\rho_{21}\sigma_{23}\rho_{21}(v_2))\rangle = U$ as before. The group S_3 can act on V_1 in two ways: via the permutation or the signed permutation representation. In each case however, if E_1 is in the μ-eigenspace of σ_{23}, then E_2 is in the μ-eigenspace of σ_{13}. Thus $\sigma_{13}(v_2) = \mu v_2$ and so U lies in the μ-eigenspace of σ_{13}. Therefore it also lies in the μ-eigenspace of $\sigma_{12} = \sigma_{23}^{\sigma_{13}}$, just like E_3. This shows that

$$\phi(\rho_{12})|_{E_3 \oplus V_3} = \phi(\rho_{21}^{\sigma_{12}})|_{E_3 \oplus V_3} = \phi(\rho_{21})|_{E_3 \oplus V_3},$$

and therefore that

$$\phi(\rho_{21})|_{E_3 \oplus V_3} = \phi(\rho_{21}\rho_{12}^{-1}\lambda_{21})|_{E_3 \oplus V_3} = \phi(\epsilon_2\sigma_{12})|_{E_3 \oplus V_3}.$$

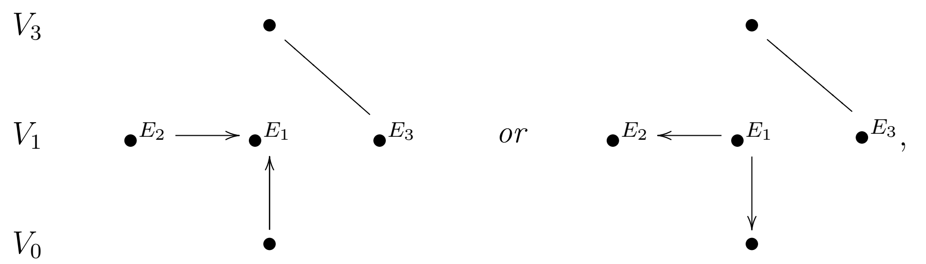
But $\epsilon_2\sigma_{12}(E_3) = E_3$ and so ρ_{21} has a diagram of the form claimed. □

We shall now focus on five- and six-dimensional representations of $\mathrm{Out}(F_3)$.

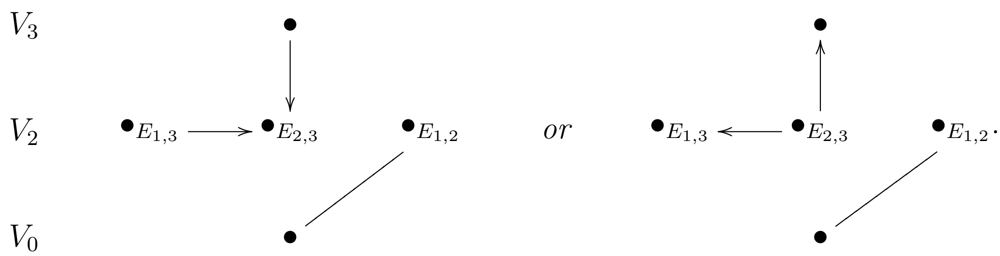
Lemma 3.2.4. *Let V be a $\mathbb{K}$-linear, six-dimensional representation of $\mathrm{Out}(F_3)$, where $\mathrm{char}(\mathbb{K}) \neq 2$. Suppose that, with notation of Definition 3.1.1,*

$$\dim V_1 \oplus V_2 \leqslant 3.$$

Note that $V_1 = \{0\}$ or $V_2 = \{0\}$. Then, if $V_2 = \{0\}$, we have a (not necessarily minimal) diagram for ρ_{21} of the form



and if $V_1 = \{0\}$ of the form



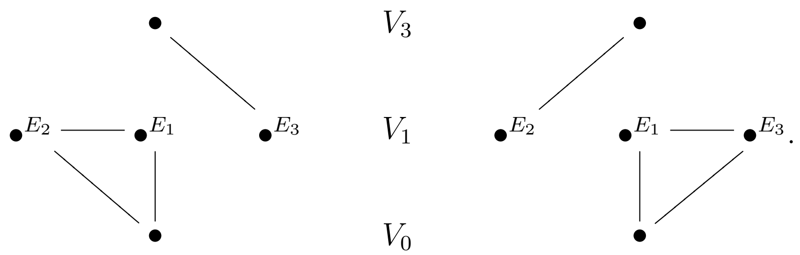
In both cases at least one of $V_0 \oplus V_2$ and $V_1 \oplus V_3$ is $\mathrm{Out}(F_3)$-invariant.

Proof. Lemma 3.1.2 tells us that the dimensions of V_1 and V_2 are divisible by 3. Hence, by assumption, at least one of V_1 and V_2 is trivial. If both of them are trivial then Lemma 3.1.3 immediately tells us that the decomposition $V = V_0 \oplus V_3$ is preserved by each ρ_{ij} and λ_{ij}. Thus the minimal diagram for ρ_{21} is a subdiagram of all the above.

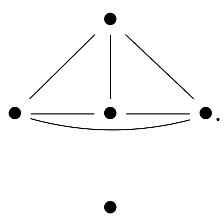
Suppose one of V_1, V_2 is non-trivial. Without loss of generality let us assume $\dim V_1 \neq 0$. Again by assumption we see that $\dim V_1 = 3$, and hence $\dim E_i = 1$ for all i.

Our strategy here is to start with the most general possible diagram for ρ_{21}, and then gradually add restriction until we arrive at one of the diagrams described above.

Lemma 3.1.3 allows us to conclude that we have the following diagrams for ρ_{21} and ρ_{31} respectively:



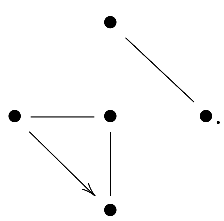
The element Δ lies in the centre of G_3, and so in particular $[\Delta, \epsilon_1\sigma_{14}] = 1$. This implies that $\epsilon_1\sigma_{14}$ preserves the eigenspaces of Δ, which happen to be the direct sums of all subspaces V_i with the index i of a given parity (even for the $(+1)$- and odd for the (-1)-eigenspace). Hence the following is a diagram for $\epsilon_1\sigma_{14}$:



But, in $\mathrm{Out}(F_3)$, we have $\epsilon_1\sigma_{14} = \rho_{31}\rho_{21}$, and Remark 3.1.5 tells us that

$$\rho_{31}\Big(r_2\big(\rho_{21}(V_0)\big)\Big) \leqslant E_2 \oplus V_3.$$

We can therefore conclude that $r_2\big(\rho_{21}(V_0)\big) = \{0\}$, and so that we have a diagram for ρ_{21} as follows:



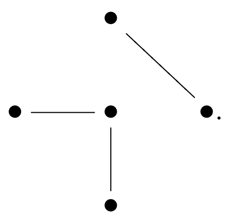
Again, by Remark 3.1.5, $\rho_{31}|_{E_2 \oplus V_3}$ is an isomorphism. Hence there exists

$$v \in E_2 \oplus V_3$$

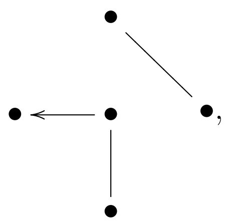
such that $\langle \rho_{31}(v) \rangle = E_2$. Since $v \in E_2 \oplus V_3 \leqslant V_1 \oplus V_3$, also $\epsilon_1\sigma_{14}(v) \in V_1 \oplus V_3$. Now

$$\epsilon_1\sigma_{14}(v) = \rho_{21}\rho_{31}(v)$$

and so we conclude that ρ_{21} has a diagram

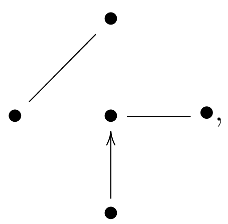


Note that ρ_{21} either has a diagram

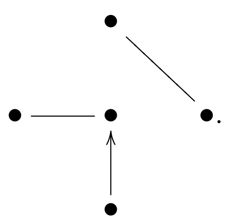


or $r_1\big(\rho_{21}(E_2)\big) = E_1$, since $\dim E_1 = 1$.

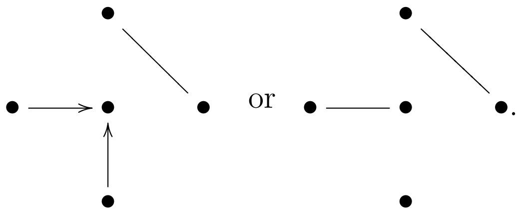
If $\rho_{21}(E_2)$ projects surjectively onto E_1, applying $\rho_{31}\rho_{21} = \epsilon_1\sigma_{14}$ to E_2 yields a diagram for ρ_{31} of the form



and, after conjugating by σ_{23} (see Example 3.1.8), ρ_{21} has a diagram

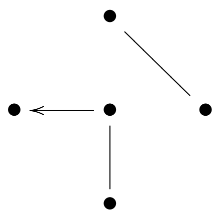


Requiring $\rho_{21}^{\epsilon_1} = \rho_{21}^{-1}$ yields two possibilities for a diagram for ρ_{21}:

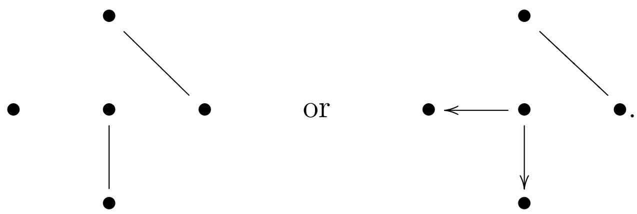


The first diagram is as required. The second diagram gives a required diagram after applying Lemma 3.2.3.

We still have to consider the case of a diagram

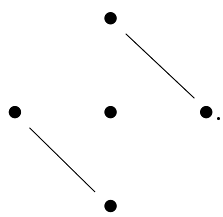


for ρ_{21}. Applying $\epsilon_1\sigma_{14} = \rho_{21}\rho_{31}$ to V_0 yields a diagram for ρ_{21} of the form



The second of these diagrams is as required.

Let us now focus on the first of the above diagrams. Note that, by Example 3.1.8, this is also a diagram for λ_{21}, and that a diagram for ρ_{12}^{-1} is as follows:



Let v_1 be a generator of E_1. Apply $\epsilon_2\sigma_{12} = \rho_{21}\rho_{12}^{-1}\lambda_{21}$ to v_1 and observe that $\epsilon_2\sigma_{12}(v_1) = v_2$, a generator of E_2. Now let x be the E_1 component of $\lambda_{21}(v_1)$. Note that $\rho_{12}^{-1}\lambda_{21}(v_1)$ has a non-trivial E_1 component if and only if x is not zero. But such a non-trivial component yields a non-zero component in $E_1 \oplus V_0$ of $\rho_{21}\rho_{12}^{-1}\lambda_{21}(v_1)$.

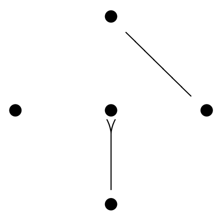
This is impossible, since $\epsilon_2\sigma_{12}(v_1) = v_2$ has no such components. Thus $x = 0$, $\lambda_{21}(v_1)$ lies in V_0, and

$$\rho_{12}^{-1}|_U : U \to E_2$$

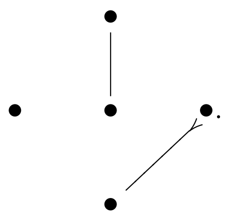
is an isomorphism, where $U = \langle \lambda_{21}(v_1) \rangle$. Hence $\rho_{12}|_{E_2} : E_2 \to U$ is an isomorphism as well.

We claim that $\rho_{ij}^{\pm 1}|_{E_j}, \lambda_{ij}^{\pm 1}|_{E_j} : E_j \to U$ are all isomorphisms. We have established this for λ_{21} and ρ_{12}. Conjugating by ϵ_1 and ϵ_2 establishes the claim also for $\rho_{21}^{-1}, \rho_{21}, \lambda_{21}^{-1}, \rho_{12}^{-1}, \lambda_{12}$ and λ_{12}^{-1}. Using the fact that $\epsilon_1\sigma_{14} = \rho_{31}\rho_{21}$ preserves $V_1 \oplus V_3$ we immediately conclude that the claim also holds for ρ_{31}^{-1}, and hence in particular also for ρ_{13} (repeating the argument above). Now the relation $\epsilon_3\sigma_{34} = \rho_{13}\rho_{23}$ establishes the claim for ρ_{23}, and the claim follows.

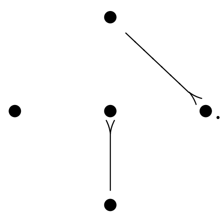
Our calculations enable us to deduce that diagrams for ρ_{21} and λ_{23} respectively are as follows



and



But ρ_{21} and λ_{23} commute, and this together with the fact that $\rho_{21}(E_1) = U = \lambda_{23}^{-1}(E_3)$ yields a diagram for ρ_{21} of the form



In particular Example 3.1.7 implies that $\dim V_0 \neq 0$.

Now let us define $U_2 = \rho_{21}(E_3) \leqslant V_3$. Note that $\dim U_2 = 1$. Since ρ_{21} commutes with λ_{23}, examining the respective diagrams yields $\lambda_{23}(U_2) = E_1$. Now, observing that each ϵ_i preserves each subspace of E_I, we see that in fact for all i

$$U_2 = \rho_{2i}^{\pm 1}(E_j) = \lambda_{2i}^{\pm 1}(E_j)$$

where j satisfies $\{i, j\} = \{1, 3\}$. We can define U_1 and U_3 similarly.

The relations $[\rho_{21}, \rho_{31}] = [\rho_{23}, \rho_{13}] = 1$, together with the structure of our diagrams, tell us that $U_2 \cap (U_1 + U_3) = \{0\}$. The relation $[\rho_{32}, \rho_{12}] = 1$ informs us that

$U_1 \cap U_3 = \{0\}$ and so that $\dim(U_1 \oplus U_2 \oplus U_3) = 3$. This is a contradiction, since $V_0 \neq \{0\}$ and so $\dim V_3 \leqslant 2$.

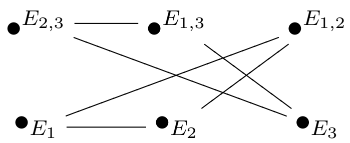
We have thus shown that ρ_{21} has a diagram as claimed. Observe that, since the subgroup $W_3 < \mathrm{Out}(F_3)$ preserves each V_i by construction, having a diagram for ρ_{21} of the form described in the statement of this lemma immediately implies that at least one of $V_0 \oplus V_2$ and $V_1 \oplus V_3$ is preserved by $\mathrm{Out}(F_3) = \langle W_3, \rho_{21} \rangle$. □

Lemma 3.2.5. *Let V be a $\mathbb{K}$-linear, six-dimensional representation of $\mathrm{Out}(F_3)$, where $\mathbb{K}$ is a field of characteristic other than 2 or 3. Suppose that $\dim(V_1 \oplus V_2) = 6$. Then V splits into $V = V_1 \oplus V_2$ as an $\mathrm{Out}(F_3)$-module.*

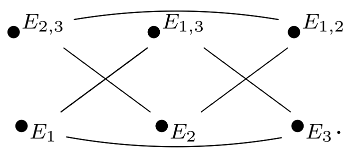
Proof. If $\dim V_1 = 6$ or $\dim V_2 = 6$ then the result is trivial.

Suppose that $\dim V_1 = \dim V_2 = 3$ and so $V = V_1 \oplus V_2$ as a vector space. We know (using Maschke's Theorem and our assumption on $\mathrm{char}(\mathbb{K})$) that each V_i (for $i = 1, 2$) is either a sum of standard and trivial or a sum of signed standard and determinant representations of S_3; we can therefore pick vectors $v_i \in E_i, w_i \in E_{\{1,2,3\} \smallsetminus \{i\}}$ so that each $v_i - v_j$ and $w_i - w_j$ is an eigenvector of an element of $S_3 \smallsetminus \{1\}$.

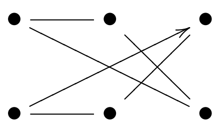
We have a diagram for ρ_{21} of the form



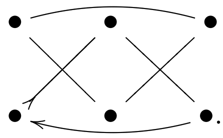
and analogously one for ρ_{31} of the form



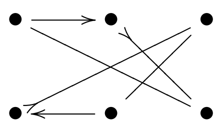
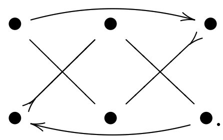
Since S_4 commutes with Δ, its action has to preserve the $(+1)$-eigenspace of Δ (which is equal to V_1 in our case) as well as the (-1)-eigenspace (which equals V_2 in this case). We also have $[\epsilon_1, \Delta] = 1$, and so $\epsilon_1 \sigma_{14} = \rho_{31}\rho_{21}$ preserves V_2. Hence, evaluating $\rho_{31}\rho_{21}$ on $E_{1,2}$ (and observing that $\dim E_I \leqslant 1$ for all I) gives us either a diagram for ρ_{21} of the form



or a diagram for ρ_{31} of the form

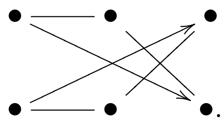


Suppose (for a contradiction) that we are in the latter case. Evaluating $\rho_{21}\rho_{31}$ on E_1 (and observing that the diagrams for ρ_{31} and ρ_{21} are related by conjugation by σ_{23}) yields diagrams for ρ_{21} and ρ_{31} respectively of the form

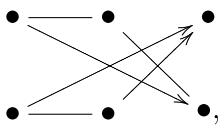
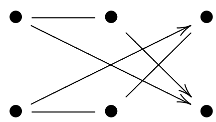
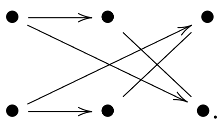
 and 

Now $\rho_{21}\rho_{31}(E_1) = E_3$ and $\rho_{31}\rho_{21}(E_1) = E_2$. But ρ_{31} commutes with ρ_{21}, which yields a contradiction.

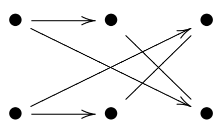
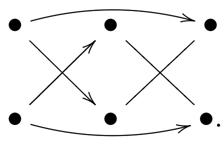
We can repeat the argument after evaluating $\rho_{31}\rho_{21}$ on E_3 and conclude that we have a diagram for ρ_{21} of the form



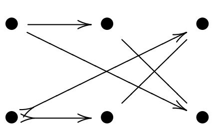
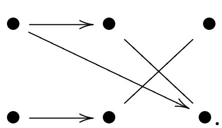
Two diagram chases, starting at E_3 and $E_{1,2}$, show $\rho_{21}^{-1} = \rho_{21}^{\epsilon_1}$ requires ρ_{21} to have a diagram of the form

,  or .

Suppose we are in the third case. We have diagrams for ρ_{21} and ρ_{31} respectively

 and .

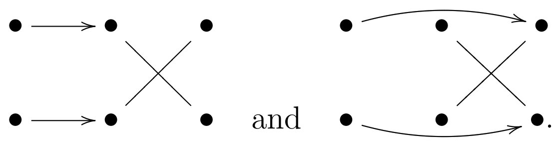
Evaluating $\epsilon_1\sigma_{14} = \rho_{31}\rho_{21}$ on E_2 (and observing that $\epsilon_1\sigma_{14}(V_2) = V_2$) yields a diagram for ρ_{21} of the form

 or .

The first case is impossible, since we would have

$$E_1 = 1(E_1) = \epsilon_1\rho_{21}\epsilon_1\rho_{21}(E_1) \leqslant E_2 \oplus E_{1,2}.$$

After repeating the argument for $E_{2,3}$ we conclude that we have diagrams for ρ_{21} and ρ_{31} respectively as follows

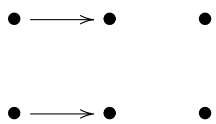


Note that the above diagrams show that $\sigma_{14} = \epsilon_1\rho_{31}\rho_{21}$ preserves both E_2 and E_3, since S_4 preserves V_1 and V_2.

Suppose that σ_{14} preserves each E_i. Then so does $\sigma_{24} = \sigma_{14}^{\sigma_{12}}$. But $\sigma_{24} = \sigma_{12}^{\sigma_{14}}$, and $\sigma_{12}(E_1) = E_2$. This is a contradiction. We can apply an analogous argument to the σ_{14}-action on the subspaces $E_{i,j}$. Now we easily deduce from $\epsilon_1\sigma_{14} = \rho_{31}\rho_{21}$ that

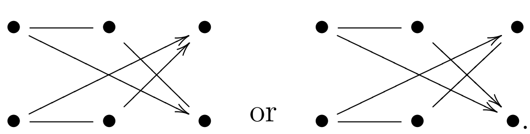
$$\rho_{21}(E_1) \nleqslant E_1 \text{ and } \rho_{21}(E_{2,3}) \nleqslant E_{2,3}.$$

We can now evaluate $\epsilon_1\sigma_{14} = \rho_{31}\rho_{21}$ and $\rho_{21}^{-1} = \rho_{21}^{\epsilon_1}$ on E_1 and $E_{2,3}$ and conclude that we have a diagram for ρ_{21} of the form

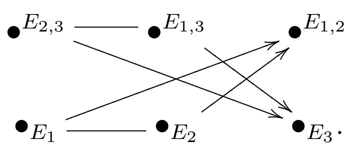


which shows that both V_1 and V_2 are $\mathrm{Out}(F_n)$-invariant.

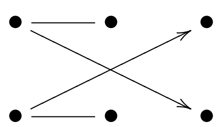
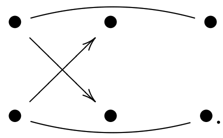
Suppose now that we are in one of the first two cases, namely that there is a diagram for ρ_{21} of the form



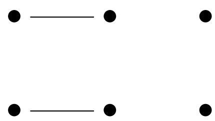
Verifying that $\rho_{31}\rho_{21}$ keeps V_1 and V_2 invariant immediately tells us that in fact we have a diagram for ρ_{21} of the form



The element ρ_{31} keeps E_2 and $E_{1,3}$ invariant, and so, observing that $\epsilon_1\sigma_{14} = \rho_{21}\rho_{31}$ preserves $V_1 \oplus V_3$, we actually have diagrams for ρ_{21} and ρ_{31} respectively

 and 

But, in order for $\rho_{21}\rho_{31}$ to keep V_1 and V_2 invariant, we need to have



as a diagram for ρ_{21}. This finishes the proof. □

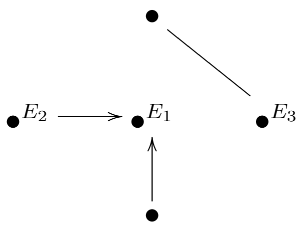
Now let us investigate 5-dimensional representations of $\mathrm{Out}(F_3)$ – we hope to be able to say more in this case!

Proposition 3.2.6. *Let V be a 5-dimensional, $\mathbb{K}$-linear representation of $\mathrm{Out}(F_3)$, where $\mathbb{K}$ is a field of characteristic other than 2 or 3. Suppose that, with the notation of Definition 3.1.1, $V \neq V_0 \oplus V_3$. Then $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ is a decomposition of $\mathrm{Out}(F_3)$-modules, and, as S_4-modules, V_0 is a sum of trivial, V_1 of standard, V_2 of signed standard, and V_3 of determinant representations.*

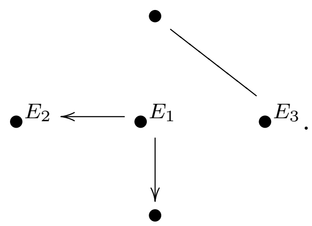
Proof. Since $\dim V = 5$, we have $V_1 = \{0\}$ or $V_2 = \{0\}$. Let us suppose that we have the latter, the other case being entirely similar.

Step 0: We first claim that V_0 is a sum of trivial S_4-modules.

Lemma 3.2.4 gives us two possibilities for a diagram for ρ_{21}, namely



or



The same lemma also tells us that $V/(V_1 \oplus V_3)$ is a representation of $\mathrm{Out}(F_3)$. Its dimension is at most 2 and therefore Lemma 3.2.1 tells us that it is a direct sum of two trivial representations of $\mathrm{Out}(F_3)$ (since we know how ϵ_1 acts), and so the same statement holds for $V/(V_1 \oplus V_3)$ as an S_4-module. Hence it also holds for V_0, since V_0 is an S_4-module isomorphic to $V/(V_1 \oplus V_3)$.

Note that an identical argument shows that V_3 is a sum of determinant S_4-modules in the case when $V_1 = \{0\}$.

Step 1: We now claim that $V_0 \oplus V_1$ is $\mathrm{Out}(F_3)$-invariant. Suppose for a contradiction that it is not the case.

Let U be the projection of $\rho_{21}(E_3)$ onto V_3. Note that $\dim U = 1$ since we have assumed $V_0 \oplus V_1$ not to be $\mathrm{Out}(F_3)$-invariant. Our aim now is to show that U is $\mathrm{Out}(F_3)$-invariant.

If V_3 is $\mathrm{Out}(F_3)$-invariant, then it is an $\mathrm{Out}(F_3)$-module of dimension at most two, and hence we can use Lemma 3.2.2 to conclude that it is in fact a sum of determinant representations. Hence, in particular, U is $\mathrm{Out}(F_3)$-invariant.

Now suppose that V_3 is not $\mathrm{Out}(F_3)$-invariant.

A diagram chase shows that

$$\rho_{21}\rho_{31}(E_3) \leqslant U \oplus E_1 \oplus E_3.$$

But ρ_{21} and ρ_{31} commute, and so $\rho_{31}\big(r_{1,2,3}(\rho_{21}E_3)\big) \leqslant U$, that is $\rho_{31}(U) \leqslant U$. Observe that $\epsilon_1|_U$ is an isomorphism, and hence $\rho_{31}^{-1} = \rho_{31}^{\epsilon_1}$ also preserves U. Therefore $\rho_{31}|_U$ is an isomorphism.

We can repeat the argument above for λ_{23}, since this element commutes with ρ_{21} and

$$\rho_{21}\lambda_{23}(E_3) \leqslant U \oplus E_2 \oplus E_3 \oplus V_0.$$

We conclude that $\lambda_{23}|_U$ is an isomorphism, and hence so is $\rho_{23} = \lambda_{23}^{\Delta}$, as $\Delta|_U$ is an isomorphism as well. The equation $\rho_{21}^{-1} = [\rho_{23}^{-1}, \rho_{31}^{-1}]$ yields that $\rho_{21}|_U$ is an isomorphism.

Note that U is the unique non-trivial invariant subspace of V_3 for each ρ_{21}, ρ_{31} and ρ_{23}, as otherwise V_3 would be invariant under the action of

$$\langle S_3, \rho_{21}\rangle = \langle S_3, \rho_{31}\rangle = \langle S_3, \rho_{23}\rangle = \mathrm{Out}(F_3).$$

Hence U is σ_{23}- and σ_{13}-invariant, and therefore S_3 preserves U. From this we conclude that $\mathrm{Out}(F_3)$ preserves U.

Lemma 3.2.2 informs us that U is a determinant representation of $\mathrm{Out}(F_3)$. Since $\rho_{21}^{-1} = \rho_{21}^{\epsilon_1}$, we must have

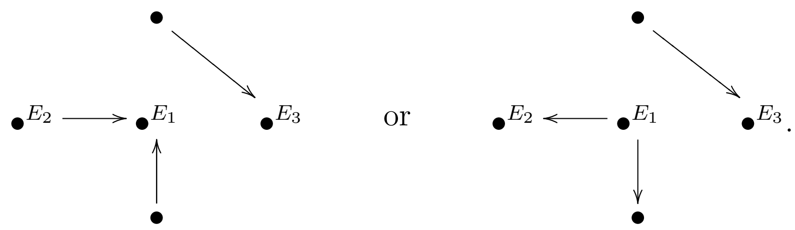
$$\forall v \in E_3 \,:\, \rho_{21}(v) \in v + U.$$

Using similar relations we establish that, when restricted to $E_3 \oplus U$, λ_{21} acts as ρ_{21}, and ρ_{12} acts as $\rho_{21}^{\pm 1}$. Hence, taking $v \in E_3$,

$$v + (2 \mp 1)u = \rho_{21}^{2\mp 1}(v) = \rho_{21}\rho_{12}^{-1}\lambda_{21}(v) = \epsilon_1\sigma_{12}(v) \in E_3$$

where $u = \rho_{21}(v) - v \in U$. This shows that $u = 0$, and hence $V_0 \oplus V_1$ is $\mathrm{Out}(F_3)$-invariant, which is the desired contradiction.

We have thus shown that there is a diagram for ρ_{21} of the form



Step 2: We claim that V_1 is a standard S_4-module.

As an S_3-module, both V_1 and V_2 are sums of one standard with either one trivial or one determinant representation. The branching rule tells us therefore that, as S_4-modules, each of the subspaces can be either a standard or a signed standard representation, or the one corresponding to partition $(2, 2)$. The last case is ruled out by Lemma 3.2.1, since $(V_0 \oplus V_1)/V_0$ is clearly not a sum of trivial and determinant $\mathrm{Out}(F_3)$-modules.

Focusing only on V_1, we have a diagram for ρ_{21} of the form

Note that in both cases these are the minimal diagrams for ρ_{21} when restricted to V_1, since otherwise σ_{12} could not permute E_1 and E_2.

Let us pick vectors $v_i \in E_i$ in such a way that each $v_i - v_j$ is an eigenvector of σ_{ij}. Let us also set $v = v_1 + v_2 + v_3$. The way in which S_4 acts on V_1 in our case is determined by one parameter; we can calculate it by finding $\mu \in \mathbb{C}$ such that $v_1 + \mu v$ is an eigenvector of σ_{14}. The eigenvalue of this eigenvector will also determine the way in which S_4 acts. Let us note that we can also find this parameter μ by computing $\sigma_{14}(v_2 - v_1) = \mu v + v_2$.

In the case of the first diagram for ρ_{21}, we immediately see that

$$\sigma_{14}(v_2 - v_1) = \epsilon_1 \rho_{21} \rho_{31}(v_2 - v_1) \in E_1 \oplus E_2,$$

and hence $\mu = 0$. Now both ρ_{23} and ρ_{31} preserve E_1, and so observing that

$$\rho_{21}^{-1} = [\rho_{23}^{-1}, \rho_{31}^{-1}]$$

yields that ρ_{21} acts trivially on E_1. By an analogous argument so does ρ_{31}. Hence $\sigma_{14}(v_1) = \epsilon_1(v_1)$. In our case this shows that we are dealing with a standard representation; if however V_1 is trivial, ϵ_1 acts as plus one on the appropriate vector, and wee see that V_2 is a signed standard S_4-module.

In the case of the second diagram we immediately see two eigenspaces of σ_{14}, namely E_2 and E_3. These spaces are interchanged by the action of σ_{23} which commutes with σ_{14}, and hence must have the same eigenvalue. In a standard or a signed standard representation of S_4 each σ_{ij} has always exactly two repeated eigenvalues, and it is this eigenvalue that determines the representation. It is enough for us then to find a third eigenvector of σ_{14} and compute its eigenvalue. The vector must have a non-trivial E_1-component, and our diagram tells us that it is enough to check how σ_{14} acts on E_1. By an argument similar to the one above we show that $\epsilon_1\sigma_{14}(v_1) \in v_1 + E_2 \oplus E_3$, and the claim follows.

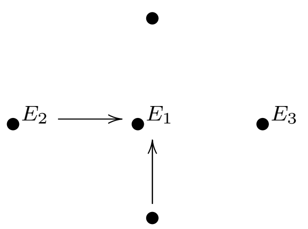
Step 3: We now claim that V_3 is a sum of determinant S_4-representations.

As an S_3-module, V_1 is a sum of one standard, and one trivial representation. Since $V_0 \oplus V_1$ is $\mathrm{Out}(F_3)$-invariant, we can consider $V/(V_0 \oplus V_1)$ as a $\mathrm{Out}(F_3)$-representation. Lemma 3.2.2 tells us that it is a sum of determinant modules. Hence V_3 is a sum of determinant S_3-modules, since $V_3 \cong V/(V_0 \oplus V_1)$ as an S_3-module.

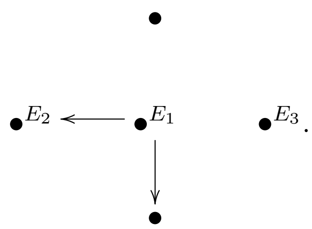
We have already found one standard representation of S_4, and the branching rule tells us that there can only be determinant representations of S_4 left. If at least one of them does not lie entirely in V_3, then it would appear in $(V_1 \oplus V_3)/V_3$ by Schur's Lemma. This is not possible, since $(V_1 \oplus V_3)/V_3$ is a standard representation of S_4. Hence all the other irreducible S_4-modules lie within V_3.

Step 4: Our last claim is that each V_i is $\mathrm{Out}(F_3)$-invariant.

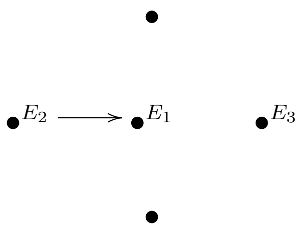
We have already shown this for $V_1 \oplus V_0$. We have just shown that V_3 is S_4 invariant, and so, $\rho_{21}\rho_{31} = \epsilon_1\sigma_{14}$ keeping V_3 invariant yields a diagram for ρ_{21} of the form



or



We have already shown in step 2 that in both cases $\rho_{21}(v) \in v + E_2 \oplus V_0$ for each $v \in E_1$. Also, $(V_0 \oplus V_1)/V_1$ is an $\mathrm{Out}(F_3)$-module of dimension at most 2, and hence is described by Lemma 3.2.2. In particular $\rho_{12}(w) = w + E_1$ for all $w \in V_0$. Analogous statements hold for ρ_{31} and so observing that σ_{23} acts as ± 1 on $E_1 \oplus V_0$ and that $\rho_{21} = \rho_{31}^{\sigma_{23}}$ yields that $\epsilon_1\sigma_{12} = \rho_{31}\rho_{21}(V_0)$ has a non-trivial V_1-component if and only if $\rho_{21}(V_0)$ does, and similarly that $\epsilon_1\sigma_{12} = \rho_{31}\rho_{21}(V_1)$ has a non-trivial V_0-component if and only if $\rho_{21}(V_0)$ does. Hence we have a diagram



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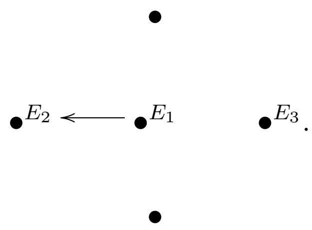

for ρ_{21}, which was what we claimed. □

Lemma 3.2.7. *Suppose V is a $\mathbb{K}$-linear representation of* $\mathrm{Out}(F_3)$*, such that $V_0 \oplus V_2$ and $V_1 \oplus V_3$ are* $\mathrm{Out}(F_3)$*-invariant. Then the representation factors through the natural projection*

$$\pi_3 : \mathrm{Out}(F_3) \to \mathrm{GL}_3(\mathbb{Z}).$$

Proof. Note that $\phi(\Delta)$ lies in the product

$$Z(\mathrm{GL}(V_0 \oplus V_2)) \times Z(\mathrm{GL}(V_1 \oplus V_3))$$

of the centres of the general linear groups of the components $V_0 \oplus V_2$ and $V_1 \oplus V_3$. Therefore we have $\phi(\rho_{ij}) = \phi(\rho_{ij})^{\phi(\Delta)} = \phi(\lambda_{ij})$ for each $i \neq j$, and so ϕ factors as

$$\begin{array}{ccc} \mathrm{Out}(F_3) & \xrightarrow{\quad\phi\quad} & \mathrm{GL}(V) \\ \downarrow & & \uparrow \\ \mathrm{Out}(F_3)/\langle\!\langle \{\rho_{ij}\lambda_{ij}^{-1} \mid i \neq j\} \rangle\!\rangle & \xrightarrow{\cong} & \mathrm{GL}_3(\mathbb{Z}). \end{array}$$

This finishes the proof. □

Observe an immediate consequence of the above.

Lemma 3.2.8. *Suppose V is a $\mathbb{K}$-linear representation of* $\mathrm{Out}(F_3)$ *of dimension at most 5, where the characteristic of $\mathbb{K}$ is not 2 or 3. Then the representation factors through the natural projection* $\pi_3 : \mathrm{Out}(F_3) \to \mathrm{GL}_3(\mathbb{Z})$*.*

Proof. We have $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ as a vector space. Suppose first that $V_1 \oplus V_2$ is trivial. Then Lemma 3.1.3 tells us that $V = V_0 \oplus V_3$ as an $\mathrm{Out}(F_3)$-module.

Supposing that $V_1 \oplus V_2 \neq \{0\}$ allows us to use Proposition 3.2.6, and conclude that each V_i is $\mathrm{Out}(F_3)$-invariant. We can now use Lemma 3.2.7 and finish the proof. □

To conclude this section we will need the following result, proven independently by Cohen–Pakianathan, Farb, and Kawazumi.

Theorem 3.2.9. *Let $n \geqslant 3$ and let $V \cong \mathbb{Z}^n$ denote the standard $\mathrm{GL}_n(\mathbb{Z})$-module. Then the abelianisation of the Torelli subgroup $\mathrm{IA}_n < \mathrm{Aut}(F_n)$ decomposes into*

$$V \oplus \mathbb{S}_{(3,3,2^{n-3},1)}(V)$$

as an $\mathrm{Aut}(F_n)/\mathrm{IA}_n = \mathrm{GL}_n(\mathbb{Z})$-module, where the action is induced by the left conjugation action of $\mathrm{Aut}(F_n)$ on IA_n.

Proof. We offer a sketch of the proof of Kawazumi.

First consider the function $\langle \cdot, \cdot \rangle \colon \mathrm{Aut}(F_n) \times F_n \to F_n$ given by

$$\langle \phi, x \rangle = x^{-1}\phi(x).$$

Note that if we restrict our function to IA_n, we obtain (with a slight abuse of notation)

$$\langle \cdot, \cdot \rangle \colon \mathrm{IA}_n \times F_n \to F_n' = [F_n, F_n].$$

Now consider following the map $\langle \cdot, \cdot \rangle$ with the quotient map $F_n' \to \Gamma = F_n'/[F_n', F_n]$. We obtain

$$\langle \cdot, \cdot \rangle' \colon \mathrm{IA}_n \times F_n \to \Gamma.$$

If we fix $\phi \in \mathrm{IA}_n$, it is easy to verify that the map

$$\langle \phi, \cdot \rangle' \colon F_n \to \Gamma$$

is a homomorphism. We have thus obtained a function

$$\tau' : \mathrm{IA}_n \to \mathrm{Hom}(F_n, \Gamma)$$

defined by $\tau'(\phi) = \langle \phi, \cdot \rangle'$. As Γ is abelian, and the group of homomorphisms between two abelian groups is also abelian, this descends to

$$\tau : \mathrm{IA}_n^{\mathrm{ab}} \to \mathrm{Hom}(V, \Gamma),$$

where V is the abelianisation of F_n, and $\mathrm{IA}_n^{\mathrm{ab}}$ is the abelianisation of IA_n.

Observe that $\langle \phi, x \rangle' = \langle \phi, \psi(x) \rangle'$ whenever $\phi, \psi \in \mathrm{IA}_n$, since $\tau'(\phi)(x)$ only depends on $[x]$, the image of x in V, and this is the same as $[\psi(x)]$. An easy calculation now shows that τ is in fact a homomorphism. Note that this is one of the Johnson homomorphisms.

Note that $\mathrm{IA}_n^{\mathrm{ab}}$ has a structure of an $\mathrm{Aut}(F_n)$-module, where the action is the left conjugation (i.e. an element ψ acts via conjugation by ψ^{-1}). Consider $\phi \in \mathrm{IA}_n$ and $\psi \in \mathrm{Aut}(F_n)$. Then

$$\tau([\psi\phi\psi^{-1}])([x]) = [x^{-1}\psi\phi\psi^{-1}(x)]_\Gamma = [\psi\Big(\psi^{-1}(x^{-1})\phi\big(\psi^{-1}(x)\big)\Big)] = [\psi\big(\langle \phi, \psi^{-1}(x) \rangle\big)]_\Gamma$$

where $[.]$ and $[.]_\Gamma$ denote images in the obvious quotients. This calculation shows that the Johnson homomorphism τ is $\mathrm{Aut}(F_n)$-equivariant, where $\mathrm{Hom}(V,\Gamma)$ is given the natural $\mathrm{Aut}(F_n)$-module structure.

The Magnus embedding gives us the second homomorphism we shall require, namely an $\mathrm{Aut}(F_n)$-equivariant homomorphism

$$\theta : \Gamma \to V \otimes V.$$

For an introduction to the Magnus embedding, see [20].

We combine τ and θ to obtain an $\mathrm{Aut}(F_n)$-equivariant homomorphism

$$\eta\colon \mathrm{IA}_n^{\mathrm{ab}} \to \mathrm{Hom}(V, V \otimes V) \cong V^* \otimes V \otimes V$$

where V^* denotes the dual of V (as an $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(F_n)/\mathrm{IA}_n$-module).

The construction of θ is completely explicit, and allows us to compute the image under η of a finite set of generators of IA_n. We discover that the image of η is isomorphic to $V^* \otimes \mathbb{S}_{(1,1)}(V) \cong V \oplus \mathbb{S}_{(3,3,2^{n-3},1)}(V)$. $\square$

Remark 3.2.10. Note that we can also consider the abelianisation of $\overline{\mathrm{IA}}_n$, the Torelli subgroup of $\mathrm{Out}(F_n)$, as an $GL_n(\mathbb{Z}) = \mathrm{Out}(F_n)/\overline{\mathrm{IA}}_n$-module. We immediately conclude from the above that it is isomorphic to $\mathbb{S}_{(3,3,2^{n-3},1)}(V)$, since we know that the inner automorphisms do not become trivial in the abelianisation of IA_n.

We are now ready for the main result of this section.

Theorem 3.2.11. *Suppose V is a $\mathbb{K}$-linear representation of $\mathrm{Out}(F_3)$ of dimension at most 6, where the characteristic of $\mathbb{K}$ is not 2 or 3. Then the representation factors through the natural projection $\pi_3 : \mathrm{Out}(F_3) \to \mathrm{GL}_3(\mathbb{Z})$.*

Proof. Let $\phi : \mathrm{Out}(F_3) \to \mathrm{GL}(V)$ be our representation. Using the notation of Definition 3.1.1, we have $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ as a vector space. We need to consider a number of cases.

Suppose first that $V_1 \oplus V_2$ is trivial. Then Lemma 3.1.3 tells us that $V = V_0 \oplus V_3$ as an $\mathrm{Out}(F_3)$-module. Suppose now that $V_0 \oplus V_3$ is trivial. Lemma 3.2.5 tells us that $V = V_1 \oplus V_2$ as an $\mathrm{Out}(F_3)$-module. In both situations we can apply Lemma 3.2.7.

We are left with the most general case: suppose that $\dim V_1 \oplus V_2 = 3$. We are going to assume that in fact $V_2 = \{0\}$, the other case being analogous. Applying Lemma 3.2.4 gives us two $\mathrm{Out}(F_3)$-representations $r : \mathrm{Out}(F_3) \to V/(V_1 \oplus V_3)$ and $s : \mathrm{Out}(F_3) \to V/V_0$, where at least one of them occurs as a submodule of V. Also, r and s factor through π_3 by Lemma 3.2.7. If any of these representations has dimension

0, then we are done. In what follows we shall suppose that the dimension of both r and s is non-zero, and thus that V is reducible as an $\mathrm{Out}(F_3)$-module. We can choose a basis for V so that the matrices in $\phi(\mathrm{Out}(F_3))$ are all in a block-upper-triangular form, with diagonal blocks corresponding to the representations r and s.

Let $\overline{\mathrm{IA}}_3 = \ker \pi_3$ be the Torelli subgroup. Our aim is to show that $\overline{\mathrm{IA}}_3$ lies in the kernel of ϕ.

Elements in $\overline{\mathrm{IA}}_3$ map to matrices with identities on the diagonal, and all non-zero off-diagonal entries located in the block in the top-right corner. Hence $\overline{\mathrm{IA}}_3$ maps to an abelian group isomorphic to $\mathbb{K}^m$, where $m \in \{5, 8, 9\}$ depends on the dimension of r.

Note that all products $\epsilon_i \epsilon_j$ lie in the kernel of r, and hence so do all elements

$$\rho_{kj}^2 = (\rho_{kj}^{\epsilon_i \epsilon_j} \rho_{kj}^{-1})^{-1},$$

where we took $k \neq i$. Theorem 2.3.1 now shows that in fact r factors through a finite group: when restricted to $\mathrm{SOut}(F_3) = \pi_3^{-1}(\mathrm{SL}_3(\mathbb{Z}))$, it factors through

$$\mathrm{SOut}(F_3) \to \mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_2).$$

Let A denote the kernel of this map. Note that $\overline{\mathrm{IA}}_3 < A$.

We have shown above that $r|_A$ is trivial, and so A maps to the identity matrix in the block corresponding to r. Note that $\phi(A)$ acts by conjugation on the abelian group of matrices with identity blocks on the diagonal, and a trivial block in the bottom-left corner. As remarked above, this group is isomorphic to $\mathbb{K}^m$. Each row or column (depending on which diagonal block corresponds to r) in the top-right corner corresponds to an A-submodule, and so the group $\mathbb{K}^m$ splits as an A-module into

$$\mathbb{K}^5, 2.\mathbb{K}^4, 3.\mathbb{K}^3, 4.\mathbb{K}^2 \text{ or } 5.\mathbb{K},$$

depending on the dimension of r, where the multiplicative notation indicates the number of direct summands.

Let $T = \overline{\mathrm{IA}}_3/[\overline{\mathrm{IA}}_3, \overline{\mathrm{IA}}_3]$ denote the abelianisation of the Torelli group seen as an $\mathrm{Out}(F_3)/\overline{\mathrm{IA}}_3 = \mathrm{GL}_3(\mathbb{Z})$-module, where the action is the one induced by the conjugation action $\mathrm{Out}(F_3) \curvearrowright \overline{\mathrm{IA}}_3$. The structure of this module is known (see Theorem 3.2.9 and the remark afterwards) – it is the second symmetric power of the dual of the standard $\mathrm{GL}_3(\mathbb{Z})$-module, tensored with the determinant representation. After tensoring T with $\mathbb{K}$, we can apply Proposition 2.3.5, and conclude that $T \otimes_{\mathbb{Z}} \mathbb{K}$ is an irreducible A-module of dimension 6. By Schur's Lemma, if we have an A-equivariant quotient of T, it is either isomorphic to T or equal to $\{0\}$.

Now consider the action of $\phi(A)$ on $\phi(\overline{\mathrm{IA}}_3) \otimes_{\mathbb{Z}} \mathbb{K}$ by conjugation. It is at the same time an equivariant quotient of an irreducible 6-dimensional module and a submodule of

$$\mathbb{K}^5, 2.\mathbb{K}^4, 3.\mathbb{K}^3, 4.\mathbb{K}^2 \text{ or } 5.\mathbb{K}.$$

This implies that the image of $\overline{\mathrm{IA}}_3$ under ϕ is trivial. This finishes the proof. □

3.3 The general case

In what follows, let us fix a field $\mathbb{K}$ of characteristic either 0 or greater than $n+1$.

Proposition 3.3.1. *Suppose V is an m-dimensional $\mathbb{K}$-linear representation*

$$\mathrm{Out}(F_n) \to \mathrm{GL}(V),$$

where $m < n(n-2)$, such that, with the notation of Definition 3.1.1, $V_i = \{0\}$ for all $i \notin \{0, 1, n-1, n\}$. Suppose also that $n \geqslant 6$ or that $n \geqslant 4$ and $\dim V_1 + \dim V_{n-1} = n$. Then V decomposes as an $\mathrm{Out}(F_n)$-module as

$$V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n,$$

where the action of $\mathrm{Out}(F_n)$ on V_0 is trivial, and on V_n is via the determinant map. Moreover, as modules of $S_{n+1} < \mathrm{Out}(F_n)$, V_1 is a sum of standard, and V_{n-1} of signed standard representations.

Proof. We are going to proceed in a number of steps.

Step 0: Let us first prove that $V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n$ as a module of G_n and W_n.

When $n \geqslant 5$, Lemma 3.1.3 tells us that

$$\rho_{ij}(V_0 \oplus V_1) \leqslant V_0 \oplus V_1 \oplus V_2 \oplus V_3 = V_0 \oplus V_1$$

since $V_2 = V_3 = \{0\}$ by assumption.

When $n = 4$ then either $V_0 = \{0\}$ or $V_2 = \{0\}$. In either case

$$\rho_{ij}(V_0 \oplus V_1) \leqslant V_0 \oplus V_1$$

since $V_2 = \{0\}$.

Also, each ϵ_i keeps $V_0 \oplus V_1$ invariant, and therefore so does the entire group $\mathrm{Out}(F_n)$. Similarly $\mathrm{Out}(F_n)$ keeps $V_{n-1} \oplus V_n$ invariant.

The group G_n commutes with Δ, and thus preserves its eigenspaces. These are, depending on the parity of n, $V_0 \oplus V_{n-1}$ and $V_1 \oplus V_n$ or $V_0 \oplus V_n$ and $V_1 \oplus V_{n-1}$. In any case we conclude that G_n preserves each V_i. Clearly so does W_n.

Step 1: We claim that as S_{n+1}-modules, V_1 is a sum of standard representations, and V_{n-1} is a sum of signed standard representations.

Let us look more closely at the way S_{n-1} acts on E_1 and $E_{N \smallsetminus \{1\}}$, where S_{n-1} is the stabiliser of 1 when S_n acts on the indices of $\{\epsilon_1, \dots, \epsilon_n\}$. Note that E_1 and $E_{N \smallsetminus \{1\}}$ are S_{n-1}-invariant, since S_{n-1} commutes with ϵ_1. The dimension of each of these representations is less than $n-2$ (by Lemma 3.1.2 and our assumption on m). If $n \geqslant 6$ then these have to be sums of trivial and determinant representations (see e.g. [23]). If $n \in \{4,5\}$ then

$$\dim E_1 \in \{0,1\} \text{ and } \dim E_{N \smallsetminus \{1\}} \in \{0,1\}$$

by assumption on dimensions of V_1 and V_{n-1}. Hence, as S_{n-1}-representations, E_1 and E_{n-1} are sums of trivial and determinant representations.

Fix a basis $\{b_1, \dots, b_k\}$ of E_1, so that each $\langle b_i \rangle$ is S_{n-1}-invariant. We see that for each i, $\langle \sigma(b_i) \mid \sigma \in S_n \rangle$ is an n-dimensional representation of S_n, which has to be either the permutation or the signed permutation representation (since we know how S_n acts on the spaces $E_1, \dots, E_n$). We immediately conclude, using the branching rule (Proposition 2.1.6), that the representation of S_{n+1} on each V_1 and V_{n-1} is a sum of standard and signed standard representations.

Again we will focus on the subspaces E_1 and $E_{N \smallsetminus \{1\}}$. We shall only discuss the E_1 case, since the other case is analogous. Note that Lemma 3.1.3 gives us

$$\rho_{ij} E_I \leqslant \bigoplus_{I \triangle J \subseteq \{i,j\}} E_J.$$

Hence in particular

$$\rho_{ij} E_1 \leqslant E_1$$

for all $i, j \neq 1$, since $E_{1,i} = E_{1,j} = E_{1,i,j} = \{0\}$, as $V_2 = V_3 = \{0\}$. But each ρ_{ij} is an isomorphism, hence it has to be an isomorphism on E_1. Now the actions of ρ_{23} and ρ_{34} on E_1 are conjugate by the action of $\sigma_{24}\sigma_{34}$, which is trivial on E_1. Hence $\rho_{24} = [\rho_{34}^{-1}, \rho_{23}^{-1}]$ acts trivially on E_1. The same is true for λ_{24} and λ_{42}, and hence $\sigma_{24}\epsilon_4 = \lambda_{24}\lambda_{42}^{-1}\rho_{24}$ acts trivially on E_1. Therefore the representation of S_{n+1} on V_1 is a sum of standard representations, whereas the representation on V_{n-1} is a sum of signed standard representations, which proves the claim.

Note that we have also shown that ρ_{ij} acts as identity on E_k and $E_{N \smallsetminus \{k\}}$ for each $k \notin \{i,j\}$. This fact will turn out to be very useful in the remaining part of the proof.

Step 2: We now claim that V_1 and V_{n-1} are $\mathrm{Out}(F_n)$-invariant.

In fact, we will only prove this claim for V_1, the V_{n-1} case being analogous. We shall comment on any differences of note.

Note that the action of A_n on V_1 gives isomorphisms $\iota_{ij} : E_i \cong E_j$ for each i, j. Let us consider $W \leqslant V_1$, an irreducible representation of S_{n+1}. We have shown that W is a standard representation of S_{n+1}. Our aim now is to find a natural basis for W.

Let $a \in W \cap L$ be a non-zero vector, where L is the (-1)-eigenspace of $\sigma_{1(n+1)}$. Note that $\langle a \rangle = W \cap L$. Let us remark here that if we were considering V_{n-1}, then W would have been a signed standard representation, and we would have taken L to be the $(+1)$-eigenspace of $\sigma_{1(n+1)}$ to the same effect.

We write $a = \sum_{i=1}^n a_i$, where $a_i \in E_i$ for each i. Now $[\sigma_{1(n+1)}, \sigma] = 1$ for each $\sigma \in A_n$ such that σ fixes 1 in the natural action $A_n \curvearrowright \{1, 2, \ldots, n\}$. Therefore, for each such σ, $\sigma(a) \in W \cap L = \langle a \rangle$. But W is a standard representation of A_{n+1}, and hence $\sigma(a) = a$. So $a_j = \iota_{2j}(a_2)$ for each $j > 2$. If $a_1 = \iota_{21}(a_2)$, then in fact $\langle a \rangle \leqslant V_1$ is A_{n+1}-invariant, which is a contradiction, since V_1 is a sum of standard representations of A_{n+1}. Hence $a_1 \neq \iota_{21}(a_2)$

Let $u = \iota_{21}(a_2) + \sum_{i=2}^n a_i \in V_1$ and set $v_1 = a - u$ and $v_j = \iota_{1j}(a_1) - a_j$ for each $j > 1$. Note that $v_i \in E_i$ for each i. Note also that $\langle u \rangle$ is preserved by S_n, and hence A_n fixes u. Now

$$\{v_1 + u, v_2 + u, \ldots, v_n + u\}$$

is a basis for W; it is in fact the natural basis for the standard representation (see Definition 2.1.5). We can conclude that in particular $\sigma_{1(n+1)}(v_i + u) = v_i - v_1$ for each $i > 1$.

We claim that in fact $u = 0$. Let us suppose that $u \neq 0$. The strategy now is to find a trivial representation of S_{n+1} in V_1, which will be a contradiction.

We have, for $i > 1$,

$$\begin{aligned}
\sigma_{1(n+1)}(u) &= \sigma_{1(n+1)}(u + v_i - v_i) \\
&= \sigma_{1(n+1)}(u + v_i) - \sigma_{1(n+1)}(v_i) \\
&= v_i - v_1 - \sigma_{1(n+1)}(v_i).
\end{aligned}$$

But

$$\sigma_{1(n+1)}(v_i) = \epsilon_1 \rho_{i1} \prod_{j \neq i} \rho_{j1}(v_i) = \epsilon_1 \rho_{i1}(v_i) \in V_0 \oplus E_1 \oplus E_i$$

by Lemma 3.1.3. So

$$\sigma_{1(n+1)}(u) = v_i - v_1 - \sigma_{1(n+1)}(v_i) \in V_0 \oplus E_1 \oplus E_i$$

for each $i \neq 1$. Hence $\sigma_{1(n+1)}(u) \in V_0 \oplus E_1$. But also V_1 is S_{n+1}-invariant, and therefore $\sigma_{1(n+1)}(u) = x_1 \in E_1$. Note that $u \neq 0$ and so $x_1 \neq 0$.

Define $x_i = \iota_{1i}(x_1) \in E_i$ and note that, since A_n acts trivially on u, $x_i = \sigma_{i(n+1)}(u)$. Now, for $i \neq 1$,

$$\begin{aligned}\sigma_{1(n+1)}(x_i) &= \sigma_{1(n+1)}\sigma_{i(n+1)}(u)\\ &= \sigma_{1(n+1)}\sigma_{i(n+1)}\sigma_{1i}(u)\\ &= \sigma_{1(n+1)}\sigma_{1i}\sigma_{1(n+1)}(u)\\ &= \sigma_{i(n+1)}(u)\\ &= x_i.\end{aligned}$$

Note that this calculation is slightly different in the case of V_{n-1} due to extra signs occurring, but the conclusion stays the same.

We have shown that $\{u, x_1, x_2, \ldots, x_n\}$ forms a basis of a permutation representation of S_{n+1} within V_1. In particular this implies that $\langle u + \sum_{i=1}^{n} x_i\rangle$ is a one-dimensional representation of S_{n+1} within V_1, which is a contradiction. We conclude that $u = 0$.

We have thus shown that a natural basis for W is given by $\{v_1, v_2, \ldots, v_n\}$, and therefore

$$\begin{aligned}v_2 + v_1 &= \epsilon_1\sigma_{1(n+1)}(v_2)\\ &= \prod_{i=2}^{n}\rho_{i1}(v_2)\\ &= \rho_{21}(v_2).\end{aligned}$$

Also, as $[\epsilon_1\sigma_{1(n+1)}, \rho_{21}] = 1$,

$$\begin{aligned}v_3 + v_1 &= \epsilon_1\sigma_{1(n+1)}(v_3)\\ &= \epsilon_1\sigma_{1(n+1)}\rho_{21}(v_3)\\ &= \rho_{21}\epsilon_1\sigma_{1(n+1)}(v_3)\\ &= \rho_{21}(v_3 + v_1)\\ &= v_3 + \rho_{21}(v_1).\end{aligned}$$

So, combining these two computations with Lemma 3.1.3 shows that $\rho_{21}(W) \leqslant V_1$. The same argument works for any ρ_{ij} and any standard representation $W \leqslant V_1$ of S_{n+1}, and these representations sum up to V_1, so we conclude that ρ_{ij} keeps V_1 invariant for each $i \neq j$. The same is clearly true for each ϵ_i, and therefore $\mathrm{Out}(F_n)(V_1) = V_1$. Analogously $\mathrm{Out}(F_n)(V_{n-1}) = V_{n-1}$.

Now we can quotient these two spaces out and obtain a representation of $\mathrm{Out}(F_n)$ on the direct sum of $\widetilde{V_0} = (V_0 \oplus V_1)/V_1$ and $\widetilde{V_n} = (V_{n-1} \oplus V_n)/V_{n-1}$.

Step 3: We claim further that $\widetilde{V_0} \oplus \widetilde{V_n}$ is a sum of $\mathrm{Out}(F_n)$-modules, and the action of $\mathrm{Out}(F_n)$ on $\widetilde{V_0}$ is trivial, and on $\widetilde{V_n}$ is a sum of determinant representations.

We have shown that $V_0 \oplus V_1$ and $V_{n-1} \oplus V_n$ are $\mathrm{Out}(F_n)$-invariant, and hence both $\widetilde{V_0}$ and $\widetilde{V_n}$ are representations of $\mathrm{Out}(F_n)$. This way we get two maps of the form $\phi : \mathrm{Out}(F_n) \to \mathrm{GL}_\nu(\mathbb{K})$, with $\nu \leqslant m$, each of which sends all elements ϵ_i to either the identity or the minus identity matrix.

Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(F_n) & \xrightarrow{\phi} & \mathrm{GL}_\nu(\mathbb{K}) \\ & \searrow & \downarrow s \\ & & \mathrm{PGL}_\nu(\mathbb{K}), \end{array}$$

where s is the natural projection. All elements ϵ_i are in the kernel of the diagonal map, and hence, using Corollary 2.3.2, we get another commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(F_n) & \longrightarrow & \mathrm{GL}_\nu(\mathbb{K}) \\ \downarrow & \searrow & \downarrow s \\ \mathrm{GL}_n(\mathbb{Z}_2) & \longrightarrow & \mathrm{PGL}_\nu(\mathbb{K}). \end{array}$$

Now we can use Theorem 2.3.3: if $n \geqslant 5$ then the inequality

$$\nu \leqslant m < n(n-2) \leqslant 2^{n-1} - 1$$

allows us to conclude that the bottom map is trivial. If $n = 4$ then we need to additionally use the assumption that $\dim V_1 + \dim V_{n-1} \geqslant 4$. This tells us that

$$\nu \leqslant m - 4 < n(n-2) - 4 \leqslant 2^{n-1} - 1$$

and hence we can apply the theorem.

In either case, the image of $\mathrm{Out}(F_n)$ in $\mathrm{GL}_\nu(\mathbb{K})$ lies in the kernel of s, which is isomorphic to $\mathbb{K}^*$. So ϕ is in fact a sum of identical one-dimensional representations of $\mathrm{Out}(F_n)$, and therefore we see that ϕ is either a sum of trivial or the determinant representations. But we know the image of ϵ_1 under ϕ (depending on whether we are looking at $\widetilde{V_0}$ or $\widetilde{V_n}$), which finishes the proof of this step.

Step 4: It remains to show that in fact both V_0 and V_n are $\mathrm{Out}(F_n)$-invariant.

Let $v \in V_0$. We know that $\epsilon_1\sigma_{1(n+1)}(v) \in V_0$, and therefore in particular its projection onto each of E_i is zero. Now, by Lemma 3.1.3, for $j > 1$, the E_j-component of

$$\epsilon_1\sigma_{1(n+1)}(v) = \prod_{i=2}^{n} \rho_{i1}(v)$$

is non-zero if and only if $\rho_{j1}(v) \neq 0$. Therefore $\rho_{j1}(v) \in V_0 \oplus E_1$ for all $j > 1$.

Using the fact that ρ_{21} acts as identity on V_0, which we proved in Step 3, let

$$\rho_{21}(v) = v + v',$$

where $v' \in E_1$. Hence

$$\rho_{21}^{-1}(v) = \epsilon_1\rho_{21}\epsilon_1(v) = v - v',$$

and so, to ensure that $\rho_{21}\rho_{21}^{-1} = 1$, we need $\rho_{21}(v') = v'$. Now

$$\epsilon_1\sigma_{1(n+1)}(v) = \big(\prod_{i=3}^{n} \sigma_{i2}\rho_{21}\sigma_{i2}\big)\rho_{21}(v) = v + (n-1)v',$$

which belongs to V_0 only if $v' = 0$. This shows that $\rho_{21}(v) = v \in V_0$.

The argument works in an identical manner for all ρ_{ij}, and for V_n. We have therefore finished the proof of this step, and consequently of the proposition. $\square$

Lemma 3.3.2. *Suppose* $\phi\colon \mathrm{Out}(F_n) \to \mathrm{GL}(V)$ *is an* m*-dimensional* $\mathbb{K}$*-linear representation of* $\mathrm{Out}(F_n)$*, where* $n \geqslant 4$ *and* $m < \binom{n+1}{2}$*, such that, with the notation of Definition 3.1.1, at least one of* V_2, V_{n-2} *has non-zero dimension. Then* $\phi(\Delta)$ *lies in the centre of* $\phi(\mathrm{Out}(F_n))$*.*

Proof. Without loss of generality let us assume that $V_2 \neq \{0\}$. Lemma 3.1.2 informs us that

$$m - \dim V_2 < \binom{n+1}{2} - \binom{n}{2} = n,$$

and hence $V_i = 0$ if i is not equal to $0, 2$ or n.

Now, if $n \geqslant 5$, Lemma 3.1.3 shows that each ρ_{ij} preserves $V_0 \oplus V_2$ and V_n. Clearly, this is also true for each ϵ_i, and hence $V_0 \oplus V_2$ and V_n are subrepresentations of

$\mathrm{Out}(F_n)$. This immediately implies that $\phi(\Delta)$ lies in the centre of $\phi(\mathrm{Out}(F_n))$, since it lies in the centre of

$$\mathrm{GL}(V_0 \oplus V_2) \times \mathrm{GL}(V_n).$$

If $n = 4$ then $V = V_0 \oplus V_2 \oplus V_4$, which is precisely the $(+1)$-eigenspace of Δ. Hence, as above, $\phi(\Delta)$ lies in the centre of $\phi(\mathrm{Out}(F_n))$. □

Combining the two results above yields

Theorem 3.3.3. *Let $\mathbb{K}$ be a field of characteristic equal to zero or greater than $n+1$. Suppose $\phi : \mathrm{Out}(F_n) \to \mathrm{GL}(V)$ is an m-dimensional $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$, where $n \geqslant 6$ and $m < \binom{n+1}{2}$. Then ϕ factors through the natural projection $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

Proof. Firstly, Lemma 3.1.2 shows that

$$\forall i \notin \{1, 2, 3, n-2, n-1, n\} \, : \, \dim V_i = 0.$$

We claim that $\phi(\Delta)$ lies in the centre of $\phi(\mathrm{Out}(F_n))$. We shall consider two cases.

Suppose at least one of V_2, V_{n-2} has non-zero dimension. Then we are in the case of Lemma 3.3.2, which asserts the claim.

Suppose now that $V_2 = V_{n-2} = \{0\}$. Let us note that, since $n \geqslant 6$,

$$m < \binom{n+1}{2} < n(n-2).$$

We can therefore apply Proposition 3.3.1 to V and conclude that, as an $\mathrm{Out}(F_n)$-module, $V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n$. Now Δ acts as an element of the centre of each $\mathrm{GL}(V_i)$, and hence $\phi(\Delta)$ commutes with $\phi(x)$ for all $x \in \mathrm{Out}(F_n)$. The claim is thus proven.

The relation $\phi([\Delta, x]) = 1$ for all $x \in \mathrm{Out}(F_n)$ in particular holds for $x = \rho_{ij}$, and shows that $\phi(\rho_{ij}) = \phi(\rho_{ij}^{\Delta}) = \phi(\lambda_{ij})$. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(F_n) & \xrightarrow{\phi} & \mathrm{GL}(V) \\ {\scriptstyle \pi_n}\downarrow & \nearrow & \uparrow \\ \mathrm{Out}(F_n)/\langle\langle\{\rho_{ij}\lambda_{ij}^{-1} \mid i \neq j\}\rangle\rangle & \xrightarrow{\cong} & \mathrm{GL}_n(\mathbb{Z}) \end{array}$$

which finishes the proof. □

In a similar vein we obtain

Theorem 3.3.4. *Let $\mathbb{K}$ be a field of characteristic equal to zero or greater than 5. Suppose $\phi\colon \mathrm{Out}(F_n) \to \mathrm{GL}(V)$ is an m-dimensional $\mathbb{K}$-linear representation of $\mathrm{Out}(F_n)$, where $n \in \{4,5\}$ and $m < 2n+1$. Then ϕ factors through the natural projection $\pi_n\colon \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

Proof. First let us suppose that $\dim V_2 + \dim V_{n-2} > 0$. Then we apply Lemma 3.3.2, which asserts our claim.

If $V_2 = V_{n-2} = \{0\}$ then either we satisfy the hypothesis of Proposition 3.3.1, in which case we proceed just as in the proof above, or we have $\dim V_1 + \dim V_{n-1} = 2n$. In the latter case, if $n = 4$, then $V = V_1 \oplus V_3$ and so $\phi(\Delta)$ commutes with $\phi(\mathrm{Out}(F_n))$. If $n = 5$, then $V = V_1 \oplus V_4$. Lemma 3.1.3 tells us that both V_1 and V_4 are $\mathrm{Out}(F_n)$-invariant, and hence in particular $\phi(\Delta)$ lies in the centre of $\phi(\mathrm{Out}(F_n))$. □

To put our theorems in context, let us mention the work of Potapchik and Rapinchuk [22]. They study complex linear representations of $\mathrm{Aut}(F_n)$ in dimension at most $2n-2$. By using the fact that every representation of $\mathrm{Out}(F_n)$ is also a representation of $\mathrm{Aut}(F_n)$ via the natural projection $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$, we obtain the following corollary of one of their theorems.

Theorem 3.3.5 (Potapchik, Rapinchuk [22, Theorem 3.1])**.** *Let $\phi : \mathrm{Out}(F_n) \to \mathrm{GL}_m(\mathbb{C})$ be a representation, where $n \geqslant 3$ and $m \leqslant 2n-2$. Then ϕ factors through the natural projection $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

Theorem 3.3.3 is a strengthening of the above for large n. In the spirit of the work of Potapchik and Rapinchuk we can rephrase it in the following manner.

Corollary 3.3.6. *Let $\phi : \mathrm{Aut}(F_n) \to \mathrm{GL}_m(\mathbb{K})$ be a representation over a field $\mathbb{K}$ with characteristic either equal to zero or greater than $n+1$, where $n \geqslant 6$ and $m < \binom{n+1}{2}$. Then either ϕ factors through the natural projection $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$, or it does not vanish on the inner automorphisms of F_n.*

Proof. Suppose that ϕ does vanish on the inner automorphisms of F_n. Then it factors as

$$\begin{array}{ccc} \mathrm{Aut}(F_n) & \xrightarrow{\phi} & \mathrm{GL}_m(\mathbb{K}) \\ \downarrow & \nearrow & \\ \mathrm{Out}(F_n) & & \end{array}$$

and the result follows by an application of Theorem 3.3.3. □

3.4 Representations not factoring through π_n

In this section we look at the two known ways of obtaining linear representations $\mathrm{Out}(F_n) \to \mathrm{GL}(V)$ with infinite image which do not factor through

$$\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}).$$

3.4.1 Bridson–Vogtmann construction

The first known example follows from the construction of Bridson–Vogtmann [8], which we will look more closely at in Chapter 4, Section 4.5. They give an embedding $\phi : \mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$, where $m = k^n(n-1)+1$, for all natural numbers k coprime to $n-1$. In particular, when n is odd, we get

$$m \geqslant 3^n(n-1)+1.$$

We can compose ϕ with π_m and then $i_m^{\mathbb{K}}$ to get a $\mathbb{K}$-linear representation

$$\mathrm{Out}(F_n) \to \mathrm{GL}_m(\mathbb{K}).$$

We easily see that, given any field $\mathbb{K}$, the element $[\rho_{12}, \rho_{13}] \in \overline{\mathrm{IA}}_n$ acts non-trivially on $H_1(F_m, \mathbb{K})$, and hence our representation cannot factor through π_n, as $\ker \pi_n = \overline{\mathrm{IA}}_n$.

3.4.2 A new construction

Consider S, the set of all epimorphisms $F_n \to \mathbb{Z}_2$, with $F_n = \langle a_1, a_2, \ldots, a_n \rangle$. Note that $|S| = 2^n - 1$, and that $\mathrm{Aut}(F_n)$ acts transitively on S. Let $G < \mathrm{Aut}(F_n)$ be the stabiliser of $f : F_n \to \mathbb{Z}_2$, where

$$f(a_i) = \begin{cases} 1 & \text{if} \quad i = n \\ 0 & \text{if} \quad i \neq n \end{cases}.$$

Note that G is of index $2^n - 1$ in $\mathrm{Aut}(F_n)$.

Let R_n be the n-rose with a fixed isomorphism $\pi_1(R_n) = F_n$, such that the i^{th} petal b_i corresponds to the letter a_i. Observe that G contains exactly those based homotopy equivalences of R_n which lift to based homotopy equivalences of a based 2-sheeted covering $X \to R_n$, where X has two vertices joined by lifts of b_n, and all the other edges are loops – see Figure 3.4.1.

This way we get a map $G \to \mathrm{Aut}(F_{2n-1})$. We can compose it with the natural maps

$$\mathrm{Aut}(F_{2n-1}) \xrightarrow{p_{2n-1}} \mathrm{Out}(F_{2n-1}) \xrightarrow{\pi_{2n-1}} \mathrm{GL}_{2n-1}(\mathbb{Z}) \xrightarrow{\iota_{2n-1}} \mathrm{GL}_{2n-1}(\mathbb{C})$$

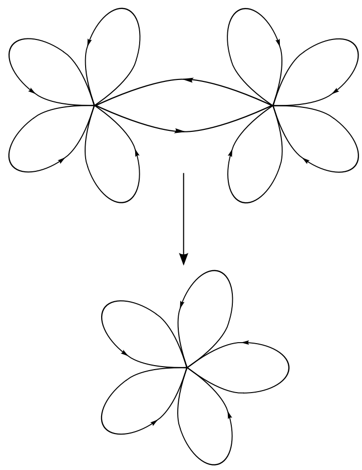

Figure 3.4.1: The 2-sheeted covering $X \to R_5$

to obtain

$$\psi : G \to \mathrm{GL}_{2n-1}(\mathbb{C}).$$

Since the covering $X \to R_n$ is regular, the action of G on $H_1(X, \mathbb{C})$ commutes with the action of τ, the non-trivial deck transformation of X. Let V denote the (-1)-eigenspace of τ, generated by $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$, where each α_i can be represented by the difference of the two loops in X which project to b_i. We now have

$$\psi' : G \to \mathrm{GL}(V) \cong \mathrm{GL}_{n-1}(\mathbb{C}).$$

The group $\mathrm{Inn}(F_n)$ of inner automorphisms of F_n is generated by elements

$$c_{a_i} : w \to a_i^{-1} w a_i,$$

where $w \in F_n$. We immediately see that $\mathrm{Inn}(F_n) < G$ and that

$$\psi'(c_{a_i}) = \begin{cases} \mathrm{I} & \text{if} \quad i \neq n \\ -\mathrm{I} & \text{if} \quad i = n \end{cases}.$$

We can project $\mathrm{GL}(V) \to \mathrm{GL}(V)/\langle -\mathrm{I} \rangle$ to obtain

$$\phi : G/\mathrm{Inn}(F_n) \to \mathrm{GL}(V)/\langle -\mathrm{I} \rangle \cong \mathrm{GL}_{n-1}(\mathbb{C})/\langle -\mathrm{I} \rangle.$$

Note that $|\mathrm{Out}(F_n) : G/\mathrm{Inn}(F_n)| = 2^n - 1$.

Let $\overline{\mathrm{IA}}_n$ be the kernel of $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}(H_1(F_n, \mathbb{Z}))$. It is well known that $\overline{\mathrm{IA}}_n$ is generated by partial conjugations $\rho_{ij}\lambda_{ij}^{-1}$ and commutators $[\rho_{ij}, \rho_{ik}]$. We have

$$\psi'(\rho_{ij}\lambda_{ij}^{-1})(\alpha_l) = \begin{cases} \alpha_l & \text{if} \quad j \neq n \quad \text{or} \quad l \neq i \\ -\alpha_i & \text{if} \quad j = n \quad \text{and} \quad l = i \end{cases},$$

and

$$\psi'([\rho_{ij},\rho_{ik}])(\alpha_l) = \begin{cases} \alpha_l & \text{if} \quad n \notin \{j,k\} \quad \text{or} \quad l \neq i \\ \alpha_i - 2\alpha_k & \text{if} \quad j = n \quad \text{and} \quad l = i \\ \alpha_i + 2\alpha_j & \text{if} \quad k = n \quad \text{and} \quad l = i \end{cases}.$$

In particular $\psi'(\overline{\mathrm{IA}}_n)$ is infinite.

Consider μ, a partition of an even number, and let $\mathbb{S}_\mu$ be the associated Schur's functor. Then $U = \mathbb{S}_\mu V$ is a representation of $\mathrm{GL}_{n-1}(\mathbb{C})$ factoring through

$$\mathrm{GL}_{n-1}(\mathbb{C}) \to \mathrm{GL}(V)/\langle -\mathrm{I}\rangle.$$

Thus U is a representation of $G/\mathrm{Inn}(F_n)$, and we can induce it to a representation $\theta : \mathrm{Out}(F_n) \to \mathrm{GL}_m(\mathbb{C})$ of dimension $m = (2^n-1)\dim U$. Note that if U is not 1-dimensional, then $\overline{\mathrm{IA}}_n \not\leqslant \ker\theta$, and hence θ does not factor through

$$\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}).$$

When $n \geqslant 4$, the smallest m for which U is non-trivial is obtained when

$$\mu = (1,1).$$

Then U is the second exterior power of V, its dimension is $\binom{n-1}{2}$, and so

$$m = (2^n-1)\binom{n-1}{2}.$$

When n is odd this is smaller then the dimension of the smallest Bridson–Vogtmann representation, and hence the smallest known.

When $n = 3$, we need to take $\mu = (2)$, since the second exterior power of U is isomorphic to the determinant representation in this case. We get

$$m = (2^n-1)\binom{n}{2} = 21,$$

which is again smaller that the dimension of the smallest Bridson–Vogtmann representation, which is 55 for $n = 3$.

Chapter 4

Free representations of $\mathrm{Out}(F_n)$

In this chapter we will investigate the free representation theory of $\mathrm{Out}(F_n)$, that is the structure of homomorphisms

$$\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

for a range of values n and m.

Our strategy here will be based on the approach devised by Bridson–Vogtmann in [8]: we will realise the image of the finite group G_n in $\mathrm{Out}(F_m)$ as a subgroup of an automorphism group of a finite admissible graph X of rank m. We will then apply our results on the low-dimensional representations of $\mathrm{Out}(F_n)$ to the action on $H_1(X, \mathbb{C})$, and use this new knowledge to analyse the structure of X. The extensive use of linear representations allows us to explore a greater range of ranks of X than was previously possible.

We would like to mention a result of Bridson–Vogtmann [8] here, which stands in a stark contrast with our results.

Theorem 4.0.1 (Bridson–Vogtmann [8]). *For all positive integers n and d, there exists a subgroup of finite index $G < \mathrm{Out}(F_n)$ and a monomorphism $G \hookrightarrow \mathrm{Out}(F_m)$, where $m = d(n-1)+1$.*

The above statement shows that in order to obtain results like Theorem 4.4.7, we cannot merely use the methods of coarse geometry. Hence the dependence of our proofs on the structure of torsion subgroups of $\mathrm{Out}(F_n)$ is, at the very least, justifiable.

4.1 Graphs realising finite subgroups of $\mathrm{Out}(F_n)$

Definition 4.1.1 (Admissible graphs)**.** Let X be a connected graph with no vertices of valence 2, and suppose we have a group G acting on it. We say that X is *G-admissible* if and only if there is no G-invariant non-trivial (i.e. with at least one edge) forest in X. We also say that X is *admissible* if and only if it is $\mathrm{Aut}(X)$-admissible.

Lemma 4.1.2. *Let X be a graph with no separating edges. Suppose e is an edge of X with an endpoint x such that*

$$\forall f \in E(X) \,:\, e \cap f = \{x\} \Rightarrow m(f) \neq m(e),$$

where $m(f)$ is the length of a shortest simple loop containing an edge f. Then X is not admissible.

Proof. Let $G = \mathrm{Aut}(X)$. Suppose that X is admissible. Then in particular $G.e$ is not a forest. Let l be a simple loop in this orbit. There exists $g \in G$ such that $e \in g.l$. Hence $g.l$ has to contain an edge f of X intersecting e at x. But this implies that there exists $h \in G$ such that $h.e = f$. This is a contradiction, since then $m(e) = m(h.e) = m(f)$. □

Since we will be dealing with homology of finite graphs quite frequently in this section, let us observe the following.

Lemma 4.1.3. *Let X be a finite, oriented graph. Recall that Definition 1.3.1 gives us two maps $\iota, \tau : E(X) \to V(X)$. We have the following identification*

$$H_1(X, \mathbb{C}) \cong \Big\{ f : E(X) \to \mathbb{C} \mid \forall v \in V(X) : \sum_{\iota(e)=v} f(e) = \sum_{\tau(e)=v} f(e) \Big\}.$$

We will often refer to each such function f as a *choice of weights* of edges in X.

Before proceeding any further, we need to introduce the concept of *collapsing maps* of graphs.

Definition 4.1.4. Let $q : X \to X'$ be a surjective morphism of graphs X and X'. We say that q is a *collapsing map* if and only if for any point $p \in X'$ the preimage $q^{-1}(p)$ is connected.

Note that this is a slight generalisation of the idea of 'collapsing forests', which is present in literature.

Remark 4.1.5. Let us observe two facts:

1. For a graph X, giving a subset of $E(X)$ which will be collapsed specifies a collapsing map π (up to isomorphism);

2. Any collapsing map $\pi : X \to X'$ induces a surjective map on homology.

The following theorem is due to Marc Culler [9], Dmitri Khramtsov [15] and Bruno Zimmermann [26] (each independently).

Theorem 4.1.6 (Culler [9]; Khramtsov [15]; Zimmermann [26]). *Suppose*

$$G \hookrightarrow \mathrm{Out}(F_m)$$

is a monomorphism, where G is a finite group. Then there exists a finite G-admissible graph X of rank m (with a fixed isomorphism $\pi_1 X \cong F_m$), so that the composition

$$G \to \mathrm{Aut}(X) \to \mathrm{Out}(F_n)$$

is the given embedding.

Proof. Let $p_m : \mathrm{Aut}(F_m) \to \mathrm{Out}(F_m)$ be the natural projection. Note that

$$\ker p_m = \mathrm{Inn}(F_m) \cong F_m.$$

Let $H = p_m^{-1}(G)$. Then $\ker p_m \leqslant H$ is a subgroup of finite index, since G is finite.

Now a result of Karras–Pietrowski–Solitar [14] states that every finitely generated virtually free group H (i.e. a group with a subgroup of finite index isomorphic to a free group) is the fundamental group of a finite connected graph of finite groups $\mathcal{G}$. Bass–Serre Theory tells us that we can unfold the underlying graph of $\mathcal{G}$ to get a tree T with an action $T \curvearrowleft H$, such that $\mathcal{G}$ is the quotient graph of groups $T /\!\!/ H$. Therefore in particular the stabilisers of edges and vertices are finite, and hence the intersection of these stabilisers with $F_m \cong \ker p_m \trianglelefteq H$ is trivial. This implies that the action of F_m is free. Let X' be the quotient of T by the action of F_m. Note that X' is a connected graph with the fundamental group isomorphic to F_m, where the isomorphism $\pi_1(X') \cong F_m$ is induced by the action $T \curvearrowleft F_m$. Now the action of H on T induces an action of $G \cong F_m \backslash H$ on X', which in turn induces an action of G on the conjugacy classes of F_m. Note that this is the same action as the one given by $G < \mathrm{Out}(F_m)$, and hence in particular it is faithful. Therefore we get $G \to \mathrm{Isom}(X') \to \mathrm{Out}(F_m)$.

Now we replace X' by its core X (the minimal connected subgraph with the same fundamental group), and further alter X by collapsing a maximal G-invariant forest. This way we obtain a G-admissible graph X as required. □

We can now use the machinery of this section to give a proof (which the author has learned from Martin Bridson) of the following result.

Proposition 4.1.7 (Baumslag–Taylor [3])**.** *Let* $\overline{\mathrm{IA}}_n < \mathrm{Out}(F_n)$ *denote the Torelli subgroup, that is the kernel of* $\pi_n \colon \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$*, where* $n \geqslant 2$*. Then* $\overline{\mathrm{IA}}_n$ *is torsion free.*

Proof. Let $t \in \overline{\mathrm{IA}}_n$ be a torsion element. Let $T = \langle t \rangle < \mathrm{Out}(F_n)$. We use Theorem 4.1.6 to construct a T-admissible graph X with $\pi_1(X) = F_n$ on which T acts in such a way that the action $T \curvearrowright H_1(X, \mathbb{Z})$ is trivial. After tensoring with $\mathbb{C}$ we can assume that $T \curvearrowright H_1(X, \mathbb{C})$ is trivial.

Let e be any edge of X, on which T acts non-trivially. Let l be a simple loop containing e. Our identification of elements in $H_1(X, \mathbb{C})$ with choices of weights on edges of X immediately tells us that T has to preserve l setwise, since otherwise it would not be acting trivially on $H_1(X, \mathbb{C})$. Hence the orbit $T.e$ is a subset of l. But X is T-admissible, and so $T.e = l$. Since T acts trivially on homology of X, but non-trivially on e, it has to act on l by a non-trivial rotation.

Choose a path p connecting two vertices on l such that p intersects l only at these two vertices. Such a p exists since the rank of X is at least 2. Now concatenate p with a simple path p' in l joining the two endpoints of p. If the endpoints coincide we choose p' to be one of them. The concatenation pp' is a loop, and hence T acts trivially on it. Hence T acts trivially on p', a proper subset of l. But this implies that T acts trivially on l, which is a contradiction. □

4.2 The case of $\mathrm{Out}(F_3)$

First let us define the following.

Definition 4.2.1. Let G be a group acting on a graph X, and let e be an edge of X. We define X_e to be the graph obtained from X by collapsing all edges *not contained* in the G-orbit of e.

Note that the action of G on such an X_e is edge-transitive.

Lemma 4.2.2. *Suppose X is a G-admissible graph of rank at most 5, where G is a group. Let e be any edge of X. Then X_e has no vertices of valence 1 or greater than 10 and satisfies*

$$8 \geqslant 2v_2 + v_3 + 2v_4 + 3v_5 + 4v_6 + 5v_7 + 6v_8 + 7v_9 + 8v_{10}, \qquad (*)$$

where v_i is the number of vertices of valence i in X_e.

Proof. First note that there are no vertices of valence 1 in X_e, since they could only occur if there were separating edges in X. But X is admissible, and so there are no such edges.

A simple Euler characteristic count yields

$$2(\text{rank}(X_e) - 1) \geqslant \sum_{i=3}^{\infty}(i-2)v_i,$$

and hence in particular $v_i = 0$ for all $i > 10$, as X_e has rank at most 5.

Since X is admissible, each vertex of X_e of valence two comes from collapsing a subgraph of X which is not a tree, hence

$$\text{rank}(X_e) \leqslant 5 - v_2$$

and the result follows. □

We will now consider graphs satisfying $(*)$ with a transitive action of $W_3 \cong G_3$ yielding particular representations on the $\mathbb{C}$-homology of the graph.

Proposition 4.2.3. *Let X be a graph of rank 5 on which $G \in \{W_3, G_3\}$ acts so that the representation of G on $V = H_1(X, \mathbb{C})$ induced by the action decomposes as $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$, where $\dim V_1 \oplus V_2 > 0$, and where*

- *if $G = W_3$ then the decomposition is the one described in Definition 3.1.1, and, as S_3-modules, V_1 is a sum of permutation, V_2 a sum of signed permutation, V_0 a sum of trivial and V_3 a sum of determinant representations;*
- *if $G = G_3$ then Δ acts as identity on $V_0 \oplus V_2$ and as minus the identity on $V_1 \oplus V_3$, and as S_4-modules, V_1 is a sum of standard, V_2 a sum of signed standard, V_0 a sum of trivial and V_3 a sum of determinant representations.*

Then, there is a subgraph $Y \leqslant X$ isomorphic to a 3-rose, on which G acts in such a way that, as an S_3-module (where $S_3 < W_3 \cap G_3$), $H_1(Y, \mathbb{C})$ contains the standard representation.

Proof. Let $v \in V$ be a vector belonging to a standard representation of $S_3 < G$. It is represented by a choice of weights on edges of X. Let e be an edge with a non-zero weight. Then the image of v in $H_1(X_e, \mathbb{C})$ is non-trivial, and hence Schur's Lemma informs us that $H_1(X_e, \mathbb{C})$ contains a standard S_3-module.

Let $Z = X_e$. Lemma 4.2.2 tells us that Z satisfies $(*)$. Also, since G acts transitively on edges of Z, there are at most two vertex-orbits of this action, and

Figure 4.2.4: Case table

Case number	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	edges	rank
(1)	4									4	1
(2)	3	2								6	2
(3)	3									3	1
(4)	2		1							4	2
(5)	2									2	1
(6)	1									1	1
(7)		8								12	5
(8)		6								9	4
(9)		4								6	3
(10)		2			1					6	4
(11)		2								3	2
(12)			4							8	5
(13)			3							6	4
(14)			2							4	3
(15)			1							2	2
(16)				2						5	4
(17)					2					6	5
(18)					1					3	3
(19)							1			4	4
(20)									1	5	5

hence in particular at most two values v_i can be non-zero. Let us list all possible values of v_i, noting that $iv_i = jv_j$ if there are vertices of valence i and j in Z, and that v_i must be even if i is odd and there are only vertices of valence i in Z. All possible cases are summarised in Figure 4.2.4.

Now, in order to have a standard representation of S_3, we need at least 3 edges in Z, and the rank of Z has to be at least 2. We can therefore immediately rule out cases $(1), (3), (5), (6)$ and (15). Also, since the action of G on the edges of Z is transitive, their number has to divide $|G| = 48$. Hence we can additionally rule out cases $(8), (16)$ and (20). We are left with the cases listed in Figure 4.2.5.

We will need to deal with these cases one by one:

Case (2): Here we have three vertices of valence two, on which S_3 has to act transitively. Each of these comes from collapsing a graph of non-zero rank in X, hence the sum of homologies of these graphs contains another standard representation of S_3. This contradicts our assumptions.

Case (4): Here Z is a subdivided 2-rose, so we cannot get a standard representation of S_3 on the homology of this graph.

Case (7): There are four graphs with an edge-transitive group action with at most 8

Figure 4.2.5: Reduced case table

Case number	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	edges	rank
(2)	3	2								6	2
(4)	2		1							4	2
(7)		8								12	5
(9)		4								6	3
(10)		2			1					6	4
(11)		2								3	2
(12)			4							8	5
(13)			3							6	4
(14)			2							4	3
(17)					2					6	5
(18)					1					3	3
(19)							1			4	4

vertices each of valence 3, namely a 3-cage, the 1-skeleton of a tetrahedron, the complete bipartite graph $K(3,3)$, and the 1-skeleton of a cube. Clearly, only the last one has the required number of vertices. An S_3-action yielding a standard representation has to be the one given by fixing two vertices and permuting 3 edges incident at one of them in a natural way. This however yields two copies of the standard representation when acting on homology, which contradicts our assumptions.

Case (9): In this case we are dealing with an edge-transitive G-action on the 1-skeleton of a tetrahedron. Such an action has to also be vertex-transitive, and therefore each vertex comes from collapsing an isomorphic subgraph. Therefore the rank of X is either 3 (the rank of Z) or at least 7. None of these cases is possible.

Case (10): In this case Z is obtained by taking a wedge of two 3-cage graphs, C_1 and C_2, say. Since the action of G is edge-transitive, for any edge e in Z we have

$$|\mathrm{Stab}_G(e)| = 8$$

and hence each 3-cycle acts freely. Since a 3-cycle cannot swap C_1 and C_2, it must act in the natural way on edges of both. Hence there have to be two copies of the standard representation of S_3 in $H_1(\mathbb{Z}, \mathbb{C})$, which is not the case.

Case (11): In this case Z is a 3-cage. If $G = G_3$, then $S_4 < G_3$ cannot act on Z yielding the desired standard or signed standard representation. Suppose now that $G = W_3$. If ϵ_1 preserves exactly one edge, then so do ϵ_2 and ϵ_3; these edges are distinct, as otherwise we would have some ϵ_i and ϵ_j acting in the same way where $i \neq j$, and so $H_1(Z, \mathbb{C}) \leqslant V_0 \oplus V_3$, where S_3 cannot have a standard representation.

Since the edges are distinct, ϵ_1 and ϵ_2 cannot commute. This shows that each ϵ_i preserves all edges of Z, and hence $H_1(Z, \mathbb{C}) \leqslant V_0 \oplus V_3$, which is a contradiction.

Case (12): The graph Z is a bipartite graph with 4 vertices and exactly zero or two edges connecting each pair of vertices. Hence Z admits a G-equivariant quotient map to a square (i.e. a single cycle made of 4 edges) which is a 2-to-1 map on edges. Each 3-cycle acts trivially on the square; moreover it cannot act non-trivially on the preimages of edges of the square. We conclude that each 3-cycle acts trivially, which is a contradiction.

Case (13): We easily check that the graph Z consists of three vertices, each of which has exactly two edges connecting it to each of the other two. Since the 3-cycle in S_3 acts non-trivially, it has to act transitively on vertices, and so either each vertex in Z comes from collapsing a subgraph which was not a tree in X, or none of them does. Neither of these two cases is possible, since the rank of X is 5.

Case (14): In this case Z is a 4-cage. Each edge in Z has a corresponding edge in X, and the fact that X is G-admissible implies that these edges do not form a forest. Hence they can form either a single simple loop, a pair of simple loops, or a 4-cage in X. The first two cases are impossible, since they would yield a trivial action of the 3-cycle of S_3 on Z. Hence X contains Z as a subgraph.

Our assumption on the representations of G tells us that either Δ or each transposition in S_3 has to flip Z, and so X is a 4-cage with a loop of length one attached to each vertex. Let a and b be two vectors in $H_1(X, \mathbb{C})$, each given by putting a weight 1 on exactly one of the loops.

If Δ flips the graph, then $a+b$ and $a-b$ span two one-dimensional eigenspaces of Δ, one with eigenvalue $+1$, and one with eigenvalue -1. Hence transpositions in S_3 have to map one of these vectors to itself, and the other to minus itself; this is only possible if they flip the graph, which contradicts our assumptions.

A similar argument works if the transpositions in S_3 flip X.

Case (17): In this case $Z = X$ is a 6-cage. As before we have

$$|\mathrm{Stab}_G(e)| = 8$$

for any edge e in Z. Hence each 3-cycle in S_3 acts freely and so we have two copies of the standard S_3-representation, which is a contradiction.

Case (18): In this case Z is a 3-rose. If Z is actually a subgraph of X, then we are done. Suppose it is not.

As Z only has one vertex, there is a connected subgraph X' of X that we collapsed when constructing Z. Since X' is of rank 2, after erasing vertices of valence 2 (in

X'), we are left with two cases: a 2-rose or a 3-cage. Since Z is not a subgraph of X, and the preimages of edges of Z in X cannot form a forest, they either form a simple loop (of length three), or a disjoint union of three loops (each of length one). In any event, we have three vertices on which the 3-cycle in S_3 acts transitively. Hence X' has to be a 3-cage, with the three vertices lying on the three edges of the cage. But then we get two standard representations of S_3 inside the homology of X, which is a contradiction.

Case (19): In this case Z is a 4-rose. The 3-cycle in S_3 acts by permuting three petals, and fixing one; let us call this fixed edge f. We easily check that f is preserved setwise by S_3, and hence also by Δ, since Δ commutes with S_3.

If $G = W_3$ then the one-dimensional subspace in $H_1(Z, \mathbb{C})$ spanned by a vector corresponding to f is contained either in V_0 or in V_3, and hence f has to be preserved by all elements in G. This contradicts transitivity of the action of G on Z.

Suppose $G = G_3$. Note that there is only one way (up to isomorphism) in which S_4 can act on a set of four elements transitively. Therefore, as Δ commutes with S_4, Δ acts as plus or minus the identity on $H_1(Z, \mathbb{C})$. Now $H_1(Z, \mathbb{C})$ as an S_4-representation is a sum of standard and trivial or signed standard and determinant representations. In particular, our hypothesis tells us that Δ cannot act as either plus or minus the identity. This is a contradiction. ☐

Lemma 4.2.6. *Suppose $\phi : \mathrm{Out}(F_3) \to \mathrm{Out}(F_5)$ is a homomorphisms. Note that $\psi = \iota_5^{\mathbb{C}} \circ \pi_5 \circ \phi$ gives a representation of $\mathrm{Out}(F_3)$ on $V = H_1(F_5, \mathbb{C})$. If, as a W_n-module, V splits as $V_0 \oplus V_3$ (with the notation of Definition 3.1.1) then the image of ϕ is finite.*

Proof. The fact that $V = V_0 \oplus V_3$ as a W_n-module implies that $\psi(\epsilon_i \Delta) = 1$ for each i. Now Proposition 4.1.7 tells us that the kernel of π_5 is torsion-free, and so $\phi(\epsilon_i \Delta) = 1$. But this means that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Out}(F_3) & \xrightarrow{\phi} & \mathrm{Out}(F_5) \\ \downarrow & \nearrow & \uparrow \phi' \\ \mathrm{Out}(F_3)/\langle\langle \{\rho_{ij} = \lambda_{ij} : i \neq j\} \rangle\rangle & \xrightarrow{\cong} & \mathrm{GL}_3(\mathbb{Z}) \end{array}$$

This allows us to use a result of Bridson and Farb [6], who have shown that such a ϕ' necessarily has finite image. Therefore the image of ϕ is finite. ☐

Theorem 4.2.7. *Suppose $\phi : \mathrm{Out}(F_3) \to \mathrm{Out}(F_5)$ is a homomorphism. Then the image of ϕ is finite.*

Proof. As above, the composition $\eta = \iota_5^{\mathbb{C}} \circ \pi_5 \circ \phi$ gives us a 5-dimensional complex linear representation $\eta : \mathrm{Out}(F_3) \to \mathrm{GL}_5(\mathbb{C})$.

Suppose first that, with the notation of Definition 3.1.1, V satisfies

$$V = V_0 \oplus V_3.$$

Then Lemma 4.2.6 yields the result.

Now suppose that $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ where $V_1 \oplus V_2 \neq \{0\}$. We apply Proposition 3.2.6 to η.

We will now apply Theorem 4.1.6 to two finite subgroups of $\mathrm{Out}(F_5)$, namely $\phi(G_3)$ and $\phi(W_3)$, to obtain two graphs X and Y respectively, on which the groups G_3 and W_3 act. Note that $H_1(X, \mathbb{C}) \cong H_1(Y, \mathbb{C}) \cong V$, and the representations of G_3 and W_3 induced by the actions of the groups on homology of the respective graphs are isomorphic to the ones given by restricting η. Hence the conclusions of Proposition 3.2.6 apply to these representations, and so we can apply Proposition 4.2.3 to the actions $G_3 \curvearrowright X$ and $W_3 \curvearrowright Y$.

We conclude that both X and Y have a subgraph, preserved by the action of the respective group, isomorphic to a 3-rose. We also know that we can label the petals as e_1, e_2, e_3, so that S_3 acts on this rose by permuting petals in the natural way, with the transpositions potentially also flipping all petals.

Knowing that $V_1 \oplus V_2 \neq \{0\}$ implies that in the W_3 case, either each ϵ_i flips e_i and leaves the other petals fixed, or each ϵ_i fixes e_i and flips the other petals. In the G_3 case, we see that there is only one way in which S_4 can act on the 3-rose inducing a standard or a signed standard representation. Each σ_{i4} has to interchange the two petals with labels different than e_i and preserve the third one; additionally, it either flips e_i and keeps some orientation of the other two fixed, or it flips the other two and fixes e_i. These two cases depend on the action of σ_{ij} for $i, j \leqslant 3$.

In any case, we have

$$\phi(\sigma_{14}) = \phi(\sigma_{23}\epsilon_2\epsilon_3)$$

and so

$$\phi(\lambda_{21}) = \phi(\lambda_{21}^{\sigma_{14}\epsilon_1}) = \phi(\lambda_{21})^{\phi(\Delta\sigma_{23})} = \phi(\rho_{31}).$$

Therefore

$$1 = \phi([\rho_{23}^{-1}, \lambda_{21}^{-1}]) = \phi([\rho_{23}^{-1}, \rho_{31}^{-1}]) = \phi(\rho_{21})^{-1}.$$

It follows that all Nielsen moves (which generate an index 2 subgroup of $\mathrm{Out}(F_3)$) lie in the kernel of ϕ, and so the image of ϕ is of size at most 2, generated by $\phi(\epsilon_1)$. □

4.3 Alternating groups and graphs

Definition 4.3.1. For convenience let us set the following notation for some elements of $\mathrm{Out}(F_n)$:

$$\xi = \begin{cases} \Delta & \text{if } n \text{ is even} \\ \Delta\sigma_{12} & \text{if } n \text{ is odd} \end{cases}$$

and $B_n = \langle A_{n+1}, \xi \rangle \leqslant G_n$. We also set A to be either A_{n-1}, the pointwise stabiliser of $\{1, 2\}$ when A_{n+1} acts on $\{1, 2, \ldots, n+1\}$ in the natural way (in the case n is odd), or A_{n+1} (in the case n is even).

Lemma 4.3.2. *Let X be a connected, oriented, non-trivial graph. Let $n \geqslant 6$. Suppose that B_n acts on X and the action satisfies the following:*

(i) B_n acts transitively on the set of (unoriented) edges of X;

(ii) if A acts non-trivially on an edge e, then ξ flips each edge in $A.e$ (i.e. it maps the edge to itself, but reverses the orientation);

(iii) A acts non-trivially on X.

Then X is either a rose or a cage.

Proof. Let e be an edge of X such that $A.e \neq e$ as sets (if there was no such e, then A would act trivially, since it is perfect). Suppose e is a loop (i.e. is homeomorphic to a circle). Then X is a rose, since it is connected and B_n acts on its edges transitively.

Suppose e is not a loop. Suppose further that there exists an edge f which has only one endpoint in common with e. Then f cannot be flipped by ξ, and in turn must be fixed by A, by (ii). This implies that in particular its endpoints are fixed by A, and hence also one of the endpoints of e is. Therefore all edges in $A.e$ share a vertex, and, since they all are flipped by ξ, they form a cage C. Let $\sigma \in A_{n+1}$ be an element taking e to f. Then σ takes C to a different cage (containing f), which is pointwise fixed by A, again by (ii). So, A^σ has to fix C pointwise.

We claim that $A \cap A^\sigma \neq \{1\}$, and thus the action $A \curvearrowright C$ has a non-trivial kernel. The group A is simple and hence the action is then forced to be trivial, which gives us a contradiction. If n is even then $A = A_{n+1} = A^\sigma$. If n is odd, then it satisfies $n \geqslant 7$, and so $A \cap A^\sigma$ contains at least one 3-cycle.

We conclude, using the connectedness of X, that every edge in X has both endpoints incident with e, and therefore X is a cage. □

In our considerations the following result will be most helpful.

Theorem 4.3.3 ([10, Theorem 5.2A]). *Suppose $n \geqslant 7$, and let $T < A_n$ be a subgroup of index smaller than $\binom{n+1}{2}$. Then T is perfect.*

We are now able to prove the Rose Lemma.

Proposition 4.3.4 (Rose Lemma). *Suppose A_{n+1} acts on a rose X of rank less than $\binom{n+1}{2}$, where $n \geqslant 6$. Then there exists an A_{n+1}-invariant choice of orientation of edges of X. Moreover, for any field $\mathbb{K}$, the multiplicity of the trivial representation of A_{n+1} in $V = H_1(X, \mathbb{K})$ is equal to the number of A_{n+1}-orbits of unoriented edges of X.*

Proof. Let e be an edge in X, and let T be its setwise stabiliser. Then, by the Orbit-Stabiliser Theorem, $|A_{n+1} : T| < \binom{n+1}{2}$. Apply Theorem 4.3.3 to T and conclude that it is perfect.

Now, the action of T on e as an oriented edge yields a homomorphism

$$T \to \mathbb{Z}_2.$$

Since T is perfect, this homomorphism has to be trivial, and therefore T preserves some (and hence any) orientation of e. We can extend this orientation A_{n+1}-equivariantly to the orbit of e. We can also put weight 1 on each oriented edge in the orbit, and put weight zero on all other edges of X. This way we obtain a non-zero vector $v_e \in H_1(X, \mathbb{K})$, which is A_{n+1}-invariant.

We can repeat the above procedure for each edge in X, and conclude the existence of an A_{n+1}-invariant orientation on the edges of X.

It is clear that $\langle v_f \rangle$ is a trivial representation of A_{n+1} for each edge $f \in E(X)$. Suppose $v \in H_1(X, \mathbb{K})$ is a vector spanning a trivial representation of A_{n+1}. Let $e_1, e_2, \dots e_k$ be a collection of representatives of the edge-orbits of the action of A_{n+1} on X. Since v is invariant, it has equal weights on edges in the same orbit. Hence

$$v = \sum_{i=1}^{k} v(e_i) v_{e_1},$$

where $v(e_i)$ is the weight of v on e_i (with respect to the fixed orientation). Hence $v \in \langle v_{e_1}, v_{e_2}, \dots, v_{e_k} \rangle$, and this establishes the equality between number of edge-orbits of A_{n+1} and the multiplicity of the trivial representation in $H_1(X, \mathbb{K})$. □

Let us use similar ideas to prove the following.

Lemma 4.3.5. *Suppose that A_n acts on a non-trivial cage X, where $n \geqslant 5$. Then the multiplicity of the trivial representation of A_n in $H_1(X, \mathbb{C}) = V$ is equal to the number of orbits of edges of the cage minus one.*

Moreover, if there are at least two edge-orbits, for each edge $e \in E(X)$ we can find an A_n-invariant vector v to which e contributes (i.e. the weight of v on e is non-zero).

Proof. First let us note that A_n has to act on the vertex set of X, which gives us a homomorphism $A_n \to \mathbb{Z}_2$. But A_n is perfect, and hence this map has to be trivial. So A_n fixes both vertices of X, and therefore preserves the orientation given by choosing one of the vertices to be the image under τ of all edges.

Suppose that A_n acts transitively on the edges of X, and let v be a vector spanning a one-dimensional module in homology. This module has to be a trivial representation of A_n, and so A_n fixes v. Therefore v is represented by giving the same weight to each edge. But the sum of weights of outgoing edges has to equal that of ingoing edges at each vertex; in this case it forces the weights to be zero, and therefore $v = 0$. This proves our claim in the case when A_n acts transitively on edges of X.

Suppose there are at least two orbits of edges in $A_n \curvearrowright X$. Let us label the orbits as $C_0, C_1, \ldots, C_k$. Let us now define vectors $v_i \in H_1(X, \mathbb{C})$ for $i = 1, \ldots, k$ by saying that v_i is represented by giving each edge in C_i weight $|C_0|$, each edge in C_0 weight $-|C_i|$, and each edge in C_j weight 0 for $j \neq 0, i$. Note that each v_i spans a trivial A_n-module, and that the vectors v_i are linearly independent. Now let v be a vector in $H_1(X, \mathbb{C})$ fixed by A_n. It necessarily has equal weights on edges in the same orbit; let λ_i be the weight of edges in C_i. Then we easily verify (using the condition on sums of outgoing and ingoing weights at vertices) that

$$|C_0| v = \sum_{i=1}^{k} \lambda_i v_i.$$

Note that for each edge $e \in E(X)$ there exists an i such that $e \in C_i$ and hence e contributes to v_i. □

Lemma 4.3.6 (Cage Lemma)**.** *Suppose A_{n+1} (with $n \geqslant 4$) acts on an m-cage X, so that the action on $V = H_1(X, \mathbb{C})$ is a sum of standard representations. Assume also that A_{n+1} acts transitively on the edges of X. Then in fact $m = n + 1$.*

Proof. Let us fix a standard copy of A_n in A_{n+1}, i.e. the stabiliser of an element when A_{n+1} acts in a natural way on a set of size $n + 1$. We know from the branching rule (Proposition 2.1.6) and our assumption about the representation of A_{n+1}, that the multiplicity of the trivial representation of A_n when acting on V is equal to that of the standard representation.

Suppose that A_n does not fix any edge. Then each orbit gives rise to at least one standard representation of A_n. But then, by Lemma 4.3.5, we have more standard representations than trivial representations of A_n, which is a contradiction.

Suppose A_n fixes more than one edge. Let e and e' be such edges. Let $\sigma \in A_{n+1}$ be an element sending e to e'. Then in particular $\sigma \notin A_n$ and A_n^σ has to fix e. Hence $A_{n+1} = \langle A_n, A_n^\sigma \rangle$ fixes e, which is a contradiction.

Let e be the unique edge fixed by A_n, and let f be any other edge of X. There exists $\sigma' \in A_{n+1}$ taking f to e. So f is the unique fixed edge of $A_n^{\sigma'}$, which is a conjugate of A_n. We have therefore shown that there is a bijection between edges of X and subgroups in the conjugacy class of A_n. There are exactly $n+1$ distinct subgroups of A_{n+1} in the conjugacy class of A_n, and hence $m = n+1$. □

4.4 The general case

In this section we combine the representation theory approach with the graph-theoretic lemmata to prove the main theorem.

Definition 4.4.1. Let $B = A_n \times \mathbb{Z}_2$ for some $n \geqslant 5$, and let $\xi \in B$ denote the element generating the centre of B. We say that a representation V of B *admits a convenient split for* B if and only if there exists a decomposition $V = U \oplus W$ of B-modules, such that, as an A_n-module, U is a sum of trivial representations, and such that ξ acts on W as minus the identity (the actions of ξ on U and of A_n on W are not prescribed).

Lemma 4.4.2. *Let $B = A_n \times \mathbb{Z}_2$ for some $n \geqslant 5$, and let ξ be the generator of the centre of B. Suppose that B acts on a graph X so that $A_n < B$ acts non-trivially on each edge of X, and such that the action of B on homology admits a convenient split as $H_1(X, \mathbb{C}) = V = U \oplus W$. Then in fact ξ flips each simple loop in X.*

Proof. If X does not contain any simple loops then the result is vacuously true.

Suppose there exists a simple loop l in X, and let v be the corresponding vector in homology. We claim that $\xi(v) = -v$, or equivalently that ξ flips l.

Suppose for a contradiction that this is not the case. Then $v + \xi(v) \neq 0$, and, as the vector is ξ-invariant, it lies in U, where A_n acts trivially. So $v + \xi(v)$ is B-invariant.

Thus, if $l = \xi.l$ as sets, then l has to be A_n invariant. But A_n cannot act non-trivially on a loop, and hence it fixes each edge. This contradicts our assumption.

Suppose now that we have an edge $f \subseteq l \smallsetminus \xi.l$. In this case we can observe that $A_n.f \subseteq l \cup \xi.l$, since $v + \xi(v)$ is A_n-invariant. Note that $A_n.f \subseteq l \cup \xi.l \smallsetminus (l \cap \xi.l)$. Define a collapsing map $q : X \to X_f$ by collapsing all edges not contained in the B-orbit of f. Note that B acts on X_f and q is B-equivariant. This allows us to use Schur's Lemma (Lemma 2.0.3) to conclude that $H_1(X_f, \mathbb{C})$ admits a convenient split.

We declare the images in X_f of edges of l to be white and images of edges of $\xi.l$ to be black; the action of ξ on X_f will pair up exactly one white edge with exactly one black edge. We claim that X_f has the structure of a daisy-chain graph, where the white edges form a single simple loop, and so do the black edges; see Figure 4.4.3, where the grey lines represent white edges.

Let l' be a shortest loop in X_f containing only white edges; we can obtain such a loop since there will be one in the image of l. Let v' be the vector corresponding to l' in $H_1(X_f, \mathbb{C})$. The vector $v' + \xi v'$ is B-invariant as before. Moreover, it is not zero, as v' has non-zero weights only on white edges, and $\xi(v')$ has non-zero weights only on black edges. We conclude that l' contains all white edges (since B acts transitively on edges of X_f, and $\xi.l$ contains only black edges). We also see that any choice of orientation of l' (i.e. a choice of orientation of its edges such that putting equal weights on each gives a vector in homology) is B-invariant; let us fix one such orientation. We can extend it using the action of ξ to a B-invariant orientation on the entire graph.

The graph X_f is connected, so there is a vertex of l' from which at least one black edge emanates. But all black edges form a simple loop $\xi.l'$ (since white edges form a simple loop l'), and hence in fact we have exactly two black edges emanating from the vertex. The action of B acts transitively on the vertex set of X_f (since it acts transitively on the edge set and preserves the orientation fixed above), so each vertex of l has two white and two black edges emanating from itself. But there are only as many black edges as white, and hence there is a black edge b connecting some two vertices of l'. Let l'' be a loop formed by b and a shortest subpath of l' connecting the endpoints of b; let v'' be the corresponding vector in homology. The vector $v'' + \xi v''$ is again B-invariant.

Suppose $v'' \neq -\xi v''$. Then, on one hand, $l'' \cup \xi.l''$ contains at most half of all white edges plus one, however on the other hand, being B-invariant, it has to contain all white edges. This shows that we have at most two white edges in l', and so at most four edges in X_f. But then we would have a non-trivial action of A_n, with $n \geqslant 5$, on a set of size 4. This is impossible.

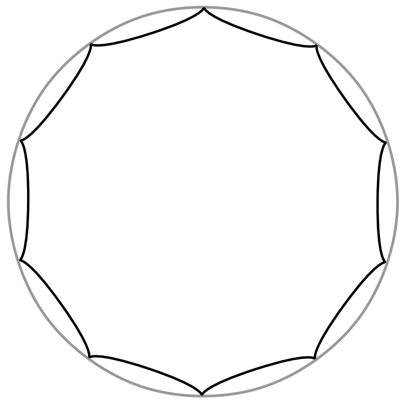

Figure 4.4.3: A daisy-chain graph

We have shown that $v'' = -\xi v''$, and so in particular l'' has length two and contains exactly one white and one black edge. Therefore each black edge in X_f shares both endpoints with a unique white edge. This proves that X_f is a daisy-chain graph as claimed.

Identifying each pair of edges sharing both endpoints gives us an A_n-action on a simple loop. Such an action must be trivial, and hence A_n acts on X_f by permuting white and black edges within each pair. This gives us a homomorphism $A_n \to \mathbb{Z}_2^k$ for some k. But A_n is perfect, and so each such map must be trivial. Therefore A_n acts trivially on X_f. This is a contradiction.

We have therefore shown that ξ sends each simple loop l in X to itself with the opposite orientation. Note that $\mathbb{C}$-linear combinations of simple loops of X span V, and so ξ has to act as minus identity on V. □

Lemma 4.4.4. *Let X be a connected non-trivial graph, on which $\mathbb{Z}_2 = \langle \xi \rangle$ acts in such a way that it flips each simple loop. Then $X = D \cup D'$ as a topological space, where D has the structure of a tree, $D' = \xi.D$, and $D \cap D' = \mathrm{Fix}(\xi)$.*

Proof. Let $F = \mathrm{Fix}(\xi)$ be the fixed point set of ξ in X (where we treat X as a topological space). Let $X' = X \setminus F$.

Firstly, we claim that components of X' are simply connected. Suppose there is a simple loop l in one of the components of X'. Since $X' \subseteq X$, l is a simple loop in X. As ξ flips all such loops, it flips l, and therefore there are two ξ-fixed points in l. So $l \cap F \neq \emptyset$. This is a contradiction, and therefore each component of X' is simply connected.

We now note that the action of ξ pairs up components of X', and so we can write $X' = \bigsqcup_{i=1}^{k}(T_i \sqcup T_i')$ for some k, where each T_i is a connected component of X', and $\xi(T_i) = T_i'$. Let $D = \bigsqcup T_i \sqcup F$. Note that D has a structure of a graph: its vertices are vertices of X contained in D together with all points in F which are midpoints of edges in X; the edge set is induced by $E(X)$ in an obvious manner.

We now claim that D is in fact a tree. To prove this we will use the following fact: let $p : I \to X$ be a path from x to y, where $x, y \in D$. We define a path $p' : I \to D$ as follows

$$p'(t) = \begin{cases} p(t) & \text{if} \quad p(t) \in D \\ \xi.p(t) & \text{if} \quad p(t) \notin D \end{cases}.$$

Note that p' is a path in D connecting x to y. Hence the connectedness of D follows directly from the connectedness of X.

Suppose we have a simple loop l in D. Then, since $\xi.l = l$ as sets, $l \cap T_i = \emptyset$ for each i, and therefore $l \subseteq F$. But then ξ fixes l, which contradicts our assumption on ξ flipping all simple loops. So D is a tree.

Define $D' = \bigsqcup T_i' \sqcup F = \xi D$, and note that $F = D \cap D'$ as required. □

Lemma 4.4.5. *Let X be a connected non-trivial graph, on which $B = A_n \times \mathbb{Z}_2$ acts (with $n \geqslant 5$) in such a way, that there are no A_n-fixed edges in X. Suppose that ξ, the generator of the centre of B, flips each simple loop in X. Suppose also that all vertices which are not fixed by A_n have valence at least 3. Then in fact all vertices have valence at least 3, and A_n fixes at most two vertices.*

Proof. Firstly let us apply Lemma 4.4.4 to X, and conclude that (using notation of the lemma) $X = D \cup D'$. Since A_n commutes with ξ, A_n acts on $X/\xi \cong D$. We know that D is a finite tree, and therefore A_n has to fix d, its centre (this is a standard fact, see e.g. [24]). Now let d' be the centre of D'. Note that it is possible that d and d' are the same point. Our group A_n acts on $\{d, d'\}$, and since it is perfect, it has to fix d and d'.

Suppose that A_n fixes another point, x say. Without loss of generality assume that $x \in D$, and take p to be the unique simple path in D from x to d. Now the action of A_n on p can potentially send each subpath of p connecting two points in $F = D \cap D'$ to a subpath lying in D' connecting the same points. Hence the action of A_n on the orbit of p gives a homomorphism $A_n \to \mathbb{Z}_2^k$ for some $k \in \mathbb{N}$. But A_n is perfect, and therefore such a map must be trivial. This implies that A_n fixes p, and as $x \neq d$, it has to fix at least one edge. This contradicts our assumption.

We have therefore shown that there are at most two fixed points of A_n, namely d and d'. Now, if any of these points were of valence less than 3, then, again as A_n is perfect, each of the edges emanating from it would have to be fixed by A_n. This is however impossible, and the proof is finished. $\square$

We are now ready to prove

Proposition 4.4.6. *Let* $n, m \in \mathbb{N}$ *be distinct,* $n \geqslant 6$, $m < \binom{n+1}{2}$, *and let*

$$\phi : \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

be a homomorphism. Suppose that the representation

$$\mathrm{Out}(F_n) \to \mathrm{GL}(H_1(F_m, \mathbb{C})) = \mathrm{GL}(V)$$

induced by ϕ *satisfies*

$$V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n,$$

with the notation of Definition 3.1.1. Then the image of ϕ *is contained in a copy of* $\mathbb{Z}_2$*, the finite group of order two.*

Proof. Before proceeding with the proof, let us recall Definition 4.3.1: if n is even, $A = A_{n+1}$ and $\xi = \Delta$; if n is odd, $A = A_{n-1}$ and $\xi = \Delta\sigma_{12}$; we also set

$$B_n = \langle A_{n+1}, \xi \rangle < G_n,$$

and $B = \langle A, \xi \rangle \leqslant B_n$.

First let us use Theorem 4.1.6 for $\phi(B_n)$ to obtain a finite B_n-admissible graph X, with an identification $\pi(X) \cong F_m$, such that the action on the conjugacy classes of F_m induced by the action of B_n on X agrees with that given by ϕ.

The general strategy of this proof will be to first use the results about representation theory of $\mathrm{Out}(F_n)$ to produce obstructions on the way B_n can act on X. Then we will apply the results of this section (dealing with convenient splits), and finally those of Section 4.3, to conclude that $A < B_n$ has to act trivially on X, and hence on the conjugacy classes of elements of its fundamental group. First let us suppose that this last statement is true, and let us deduce the result from there.

Step 0: Suppose that $A < B_n$ acts trivially on X. We claim that in this case ϕ factors through $\mathbb{Z}_2$.

Since A acts trivially on X, it acts trivially on the fundamental group of X, and hence it lies in the kernel of ϕ. But $A \leqslant A_{n+1}$, which is simple, and therefore A_{n+1} lies in the kernel of ϕ. Hence, as A_n acts transitively on the set $\{\rho_{ij} \mid i \neq j\}$,

$\phi(\rho_{ij}) = \phi(\rho_{jk})$, and using $[\rho_{ij}^{-1}, \rho_{jk}^{-1}] = \rho_{ik}^{-1}$ we see that each ρ_{ij} (and similarly λ_{ij}) lies in the kernel of ϕ. This implies that ϕ factors through $\mathbb{Z}_2 \cong \langle \epsilon_1 \rangle$.

In what follows let us suppose, for a contradiction, that A does not act trivially on X.

Step 1: We claim that V admits a convenient split for $B = \langle A, \xi \rangle$.

We know that $\mathrm{Out}(F_n)$ acts via ϕ on $H_1(F_m, \mathbb{C}) = V$. Note that also

$$V \cong H_1(X, \mathbb{C}) \cong \mathbb{C}^m.$$

Since $n \geqslant 6$, we have the inequality

$$m < \binom{n+1}{2} \leqslant n(n-2),$$

and so we can apply Proposition 3.3.1 to V. If n is even, the A_{n+1}-modules $V_0 \oplus V_n$ and $V_1 \oplus V_{n-1}$ satisfy the definition. If n is odd, the sum of all standard representations of $A = A_{n-1}$ is a subspace of $V_1 \oplus V_{n-1}$. Since we chose A_{n-1} to be the stabiliser of 1 and 2 when A_{n+1} acts on $\{1, 2, \ldots, n+1\}$, this subspace intersects

$$E_1 \oplus E_2 \oplus E_{N \smallsetminus \{1\}} \oplus E_{N \smallsetminus \{2\}}$$

trivially. This guarantees that ξ (which equals $\Delta \sigma_{12}$ in this case) acts on this subspace as minus the identity. This proves the claim.

Let us firstly investigate some of the structure of X. We construct a graph Y by collapsing all edges in X which are fixed by A pointwise. Note that, by our assumption, Y is non-trivial (i.e. has at least one edge), and is connected. Since A commutes with ξ, we get a B-action on Y; note that the collapsing map $X \to Y$ is B-equivariant. By Schur's Lemma (Lemma 2.0.3), the $\mathbb{C}$-homology of Y admits a convenient split for B. Hence we apply Lemma 4.4.2, and conclude that ξ flips all simple loops in Y.

Note that if we take a vertex x in Y which is not fixed by A, then we know that this vertex does not come from collapsing a subgraph of X, since we only collapse subgraphs which are A-fixed. Therefore such an x comes from a vertex in X, and so its valence is at least 3. This shows that the graph Y (together with the action of B on it) satisfies all conditions of Lemmata 4.4.4 and 4.4.5.

Using the notation of the former lemma, we have $Y = D \cup D'$ and

$$F = D \cap D' = \mathrm{Fix}(\xi).$$

Let y be a point in ∂D, i.e. an endpoint of a leaf of D. Since its valence (as a vertex of the graph $D \cup D'$) is 2, we see that y is not a vertex of Y; it is therefore a midpoint of an edge of Y.

Also, the maximal subgraph of Y not containing the A-fixed points (d and d') is actually a subgraph of X, since any edge collapsed by the map $X \to Y$ yields an A-fixed point in Y.

Step 2: We claim that D is the union of its leaves.

Suppose (for a contradiction) that this is not the case. Let z be a farthest (with respect to the graph metric on D) vertex of D from d, which is not in ∂D. We have just assumed that such a vertex is not d. Note that z is a vertex of Y and that it cannot be fixed by A, as it is neither d nor d'. Let e be an edge of Y emanating from z, such that its midpoint does not belong to ∂D.

Suppose $z \notin F$. Then all edges in Y emanating from z, except for e, contain as midpoints points in ∂D. There are at least two such edges (since the valence of z is at least 3), and therefore each such edge belongs to a loop of length 2. Also, neither of these edges forms a loop, since $z \notin F$, so the shortest loop through any of them is of length 2. This however cannot be true for e, since it would require both its endpoints to be in F, which is not the case. All of this holds in X as well as Y, and we can therefore apply Lemma 4.1.2 to X and arrive at a contradiction, since we have assumed that X was B-admissible, and hence in particular admissible.

We have thus shown that $z \in F$. But then there exists an edge f in Y emanating from z, which is in fact a loop. Note that f is also a loop in X. Now consider X_f, a graph obtained from X by collapsing all edges but those in $B_n.f$. Note that B_n acts on X_f, and the collapsing map $X \to X_f$ is B_n-equivariant.

Since f is a loop, X_f is a rose. Also, its rank is at most $m < \binom{n+1}{2}$. We can therefore apply Proposition 4.3.4 (the Rose Lemma), and obtain an A_{n+1}-invariant orientation of edges in X_f. By putting equal weight 1 on each edge we obtain an A_{n+1}-invariant vector $v \in H_1(X_f, \mathbb{C})$.

Schur's Lemma (Lemma 2.0.3) tells us that the image of $V_0 \oplus V_n$ in $H_1(X_f, \mathbb{C})$ is the sum of all trivial representations of A_{n+1} in $H_i(X_f, \mathbb{C})$, and also that the entire group B_n acts trivially on this subspace. Hence v must lie in the image of $V_0 \oplus V_n$, and so $\xi \in B_n$ has to act trivially on it. But ξ flips f, which contributes to this vector. This is the required contradiction.

Step 3: We claim that X is in fact an $(n+1)$-cage.

We have shown that all edges in D are leaves, and hence are flipped by ξ. Hence, in X, all edges which are not fixed by A are flipped by ξ. Let f be an edge of X which is not collapsed by the map $X \to Y$, and let X_f be the graph obtained from X by collapsing all edges not contained in $B_n.f$, as before. Note that A acts non-trivially on f, since it only fixes one point in D. We can now apply Lemma 4.3.2 to X_f, which shows that X_f is either a rose or a cage.

The graph X_f cannot be a rose, since if it were, we could construct an A_{n+1}-invariant vector $v \in H_1(X_f, \mathbb{C})$ as in the previous step, on which ξ acts trivially, but to which f (which is flipped by ξ) contributes.

So X_f is a cage. Since ξ flips f, it has to permute the two vertices of X_f. Also, as A_{n+1} is perfect, it has to fix each of these two vertices. These vertices have potentially come from non-trivial subgraphs of X. Suppose there exists a simple loop in one of these subgraphs, l say. Let v be a corresponding vector in homology.

Let us assume first that n is odd. We have shown that ξ permutes the vertices of X_f – in fact this is true for all $\Delta\sigma_{ij}$, since these elements are related by conjugating by elements of A_{n+1}. So each $\Delta\sigma_{ij}$ maps l to a loop disjoint from it. So $v + \Delta\sigma_{ij}(v)$ has to be fixed by A^σ, where $\sigma \in A_{n+1}$ is an element such that A^σ commutes with $\Delta\sigma_{ij}$. But each A^σ is a simple alternating group, and such groups cannot act on disjoint unions of two circles non-trivially. Hence all A^σ fix l pointwise, and therefore so does A_{n+1}.

When n is even, A_{n+1} acts trivially on $v + \xi(v)$, and so on a disjoint union of two simple loops $l \cup \xi.l$ as above. So A_{n+1} fixes l pointwise, just as in the odd case. Then, in both cases, $v \in V_0 \oplus V_n$ and hence the action of ξ on v has to be trivial. This is however not the case.

Hence the only subgraphs of X we collapsed when constructing X_f were trees. We have however taken X to be B_n-admissible, and therefore these trees have to be trivial, i.e. consist of one vertex each.

So X is in fact a cage. Suppose that the action of A_{n+1} on the edge set of X is not transitive. Then, by Lemma 4.3.5, there is an A_{n+1}-invariant vector w to which f contributes. As w is A_{n+1}-invariant, it lies in the image of $V_0 \oplus V_n$, and hence is B_n-invariant. But ξ flips f, which is a contradiction. We have thus shown that the action of A_{n+1} on $E(X)$ is transitive. We can apply the Cage Lemma (Lemma 4.3.6) and conclude that X is an $(n+1)$-cage. But this means that $m + 1 = n + 1$, which is a contradiction, since we have assumed that n and m are distinct. □

The proposition immediately leads to

Theorem 4.4.7. *Let $n, m \in \mathbb{N}$ be distinct, $n \geqslant 6$, $m < \binom{n}{2}$, and let*

$$\phi : \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

be a homomorphism. Then the image of ϕ is contained in a copy of $\mathbb{Z}_2$, the finite group of order two.

Proof. As above, let $V = H_1(F_m, \mathbb{C})$ be a representation of $\mathrm{Out}(F_n)$ induced by ϕ. Since $m < \binom{n}{2}$, application of Lemma 3.1.2 yields

$$V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n,$$

with the notation of Definition 3.1.1 as usual. Hence we can apply Proposition 4.4.6, which proves the claim. □

We can utilise our main tool, Proposition 4.4.6, together with a special case of a result of Bridson and Farb [6] to obtain a result reaching a little further. First let us state the required theorem.

Theorem 4.4.8 (Bridson, Farb [6])**.** *Suppose $\phi : \mathrm{PGL}_n(\mathbb{Z}) \to \mathrm{Out}(F_m)$ is a homomorphism, where $n, m \geqslant 2$. Then the image of ϕ is finite.*

Now we can prove

Theorem 4.4.9. *Let $n, m \in \mathbb{N}$ be distinct, with n even and at least 6. Let*

$$\phi : \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

be a homomorphism. Then the image of ϕ is finite, provided that

$$\binom{n}{2} \leqslant m < \binom{n+1}{2}.$$

Proof. Let $V = H_1(F_m, \mathbb{C})$ be a representation of $\mathrm{Out}(F_n)$ induced by ϕ as before. Lemma 3.1.2 shows that either

$$V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n$$

or

$$V = V_0 \oplus V_2 \oplus V_{n-2} \oplus V_n.$$

We will proceed by investigating the two cases.

If $V = V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n$, then we can apply Proposition 4.4.6, which asserts the claim.

If $V = V_0 \oplus V_2 \oplus V_{n-2} \oplus V_n$, then, as n is even, Δ acts as identity. This means that $\phi(\Delta) \in \overline{\mathrm{IA}}_m$. But Proposition 4.1.7 tells us that $\overline{\mathrm{IA}}_m$ is torsion free, and so $\phi(\Delta) = 1$. This yields the following commutative diagram:

$$\begin{CD} \mathrm{Out}(F_n) @>\phi>> \mathrm{Out}(F_m) \\ @VVV @AAA \\ \mathrm{Out}(F_n)/\langle\!\langle \Delta \rangle\!\rangle @>\cong>> \mathrm{PGL}_n(\mathbb{Z}) \end{CD}$$

and now an application of Theorem 4.4.8 finishes the proof. □

4.5 Positive results

In this section we will explore some positive results on embeddings of outer automorphism groups of finitely generated free groups. Firstly let us mention the result of Khramtsov [15]: he has shown that there exists an embedding $\mathrm{Out}(F_2) \hookrightarrow \mathrm{Out}(F_m)$ for all $m \geqslant 4$. This case however is extremely special, as $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$ is virtually free.

Now we shall look at two other results in more detail. The first one is due to Aramayona–Leininger–Souto [1], the second due to Bogopol'skii–Puga [5]. We will however discuss the proof due to Bridson–Vogtmann [8] of the latter.

Both proofs start with an observation that $\mathrm{Out}(F_n)$ can be identified with the group of free homotopy equivalences of a given graph Y with fundamental group F_n, where we identify free homotopy equivalences g and g' if and only if there exists a free homotopy $h : g \sim g'$ such that $x \mapsto h(x,t)$ is a free homotopy equivalence for all t. We will denote this group by $HE(Y)$ and note that the isomorphism $HE(Y) \cong \mathrm{Out}(F_n)$ is induced by $\pi_1(Y) \cong F_n$.

Suppose we are given a covering map $\kappa : X \to R_n$, where R_n is the n–rose. We construct a short exact sequence

$$1 \to D \to HE^*_\kappa(X) \to HE_\kappa(R_n) \to 1, \qquad (*)$$

where the notation is as follows: $HE_\kappa(R_n)$ is the group of all free homotopy equivalences $g : R_n \to R_n$, such that there exists $g' : X \to X$ with $\kappa \circ g' = g$ (i.e. such that g has a lift), up to homotopy (where the homotopy for any fixed time t gives us a free homotopy equivalence $R_n \to R_n$ which lifts); and $HE^*_\kappa(X)$ is the group of lifts of $HE_\kappa(R_n)$ up to homotopy (where the homotopy for any fixed time t gives us a lift of a free homotopy equivalence $R_n \to R_n$); D is the group of deck transformations of $\kappa : X \to R_n$.

The fact that this sequence is exact, as well as the facts that $HE_\kappa(R_n) \leqslant HE(R_n)$, follows from elementary homotopy theory. We also have

$$HE^*_\kappa(X) \leqslant HE(X)$$

provided that the centraliser of $\pi_1(X)$ in $\pi_1(R_n) \cong F_n$ is trivial (which amounts to requiring that $\pi_1(X)$ is not trivial or cyclic). For details see [8, Appendix].

In both papers the authors pick coverings in a way which guarantees that

$$HE_\kappa(R_n) = HE(R_n).$$

Then Aramayona, Leininger and Souto pick a covering with a trivial deck transformation group D, and hence obtain

$$\mathrm{Out}(F_n) \cong HE(R_n) = HE_\kappa(R_n) \cong HE^*_\kappa(X) \leqslant HE(X) \cong \mathrm{Out}(F_m)$$

for some $m > n$. Bridson and Vogtmann pick a covering for which the short exact sequence $(*)$ splits, and obtain the desired injection this way.

Let us now focus on the details of the first result.

Theorem 4.5.1 (Aramayona–Leininger–Souto [1])**.** *Let $n \geqslant 2$ be fixed. Then there exists $m > n$ and a monomorphism $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$.*

Proof. We sketch the proof from [1].

We will start by procuring a finite group G with two special properties. Consider a surjective homomorphism $\phi_1 : F_n \to S_3$. Note that $\mathrm{Aut}(F_n)$ acts on the set of all such homomorphisms, and hence we can consider $\{\phi_1, \dots, \phi_k\}$, the orbit of ϕ_1 under this action. Let $S = (S_3)^k$, the k^{th} direct power of S_3. Define $G = \mathrm{im}\,\phi$, where $\phi = \phi_1 \times \ldots \times \phi_k : F_n \to S$. Note that G projects surjectively onto each factor of S. Now define $H < G$ to be a Sylow 2-subgroup of G (which is a subgroup of a Sylow subgroup of S, which is in turn isomorphic to $\mathbb{Z}_2^k$). The fact that G projects onto each factor of S implies in particular that G contains elements of order divisible by 3, and therefore H is a proper subgroup. The fact that H is a subgroup of a group isomorphic to $\mathbb{Z}_2^k$, generated by a single transposition in each factor, implies that

$$N_G(H) = H,$$

where $N_G(H)$ is the normaliser of H in G.

Sylow's theorem tells us that

$$\mathrm{Aut}(G).H = \mathrm{Inn}(G).H,$$

where $\mathrm{Inn}(G)$ denotes the inner automorphisms of G. The construction also implies that $\ker\phi$ is a characteristic subgroup of F_n.

Let $\kappa : X \to R_n$ be the covering associated to $\phi^{-1}(H) < F_n$. Note that this is a finite-degree non-trivial covering.

The free-homotopy lifting property tells us that a homotopy equivalence

$$g : R_n \to R_n$$

lifts to $g' : X \to X$ if and only if it satisfies $g_*\kappa_*(\overline{\pi_1 X}) \leqslant \kappa_*(\overline{\pi_1 X})$, where g_*, κ_* are the induced maps on the conjugacy classes of elements in $\pi_1 X$ (the set of which is denoted by $\overline{\pi_1 X}$). This statement will be true for all g provided we can show that

$$\mathrm{Aut}(F_n).\phi^{-1}(H) = \mathrm{Inn}(F_n).\phi^{-1}(H),$$

since $g_*\kappa_*(\overline{\pi_1 X}) = g.\overline{\pi_1 X} < \overline{F_n}$, where we use the fact that $g \in HE(R_n) \cong \mathrm{Out}(F_n)$. Suppose $h \in \mathrm{Aut}(F_n)$. We get a commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow{h} & F_n \\ \phi\downarrow & & \downarrow\phi \\ G & \dashrightarrow & G \end{array}$$

since the existence of the dotted automorphism is ensured by the fact that ϕ is surjective and $\ker\phi$ is characteristic. Hence we can use $\mathrm{Aut}(G).H = \mathrm{Inn}(G).H$ to conclude that

$$\mathrm{Aut}(F_n).\phi^{-1}(H) = \mathrm{Inn}(F_n).\phi^{-1}(H).$$

Therefore $HE_\kappa(R_n) = HE(R_n)$.

It is a well known fact that the group of deck transformations satisfies

$$D \cong N_{F_n}\big(\phi^{-1}(H)\big)/\phi^{-1}(H).$$

But we know that $N_G(H) = H$ and hence D is trivial. This in turn implies that the short exact sequence $(*)$ becomes $HE_\kappa(R_n) \cong HE^*_\kappa(X)$. The result follows. □

The question of the exact cardinality of m remains unanswered – the authors of the above paper state that their best result in this area is a doubly exponential upper bound.

Let us now look at the result of Bridson and Vogtmann.

Theorem 4.5.2 (Bogopol'skii–Puga [5]; Bridson–Vogtmann [8]). *Let $n \geqslant 1$. Then for each natural number k coprime to $n-1$ we get an embedding $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$, where $m = 1 + (n-1)k^n$.*

Proof. We sketch the proof from [8].

Let us first choose the covering space $\kappa : X \to R_n$ to be the covering corresponding to $L = [F_n, F_n]F_n^{(k)} < F_n$, where $F_n^{(k)}$ is the subgroup generated by the k^{th} powers of the elements of F_n. Note that L is a characteristic subgroup of F_n and therefore the free-homotopy lifting property immediately implies that $HE_\kappa(R_n) = HE(R_n)$. Also, since characteristic subgroups are normal, we get $F_n/L \cong \mathbb{Z}_k^n$. Note that, after fixing a basepoint, X is the standard Cayley graph for $\mathbb{Z}_k^n$.

Our aim now is to construct a splitting of the short exact sequence

$$1 \to D \to HE_\kappa^*(X) \to HE(R_n) \to 1.$$

Note that there is a standard lift of any element $g \in \mathrm{Aut}(F_n)$ to a homotopy equivalence of X; we define such a lift g' firstly by its action on the vertices of X. Since X is the Cayley graph of $\mathbb{Z}_k^n$, each vertex v corresponds to an element $x_v \in \mathbb{Z}_k^n$. We define $g'(v)$ to be the vertex corresponding to $g.x_v$, where we use the action $\mathrm{Aut}(F_n) \curvearrowright F_n/L \cong \mathbb{Z}_k^n$. Secondly, each edge e, emanating from some vertex v, is sent to the (unique) lift of $g.\kappa(e)$ starting at $g'(v)$.

Note that this lift does not descend to a lift of $\mathrm{Out}(F_n)$, since the lift of $\prod_{i \neq j} \rho_{ij}\lambda_{ij}^{-1}$ for any fixed j is not homotopy equivalent to the trivial map. It is in fact sending every generator e_i of $\mathbb{Z}_k^n$ $(i \neq j)$ to a $\sqcup$-shaped path starting at the origin, and spelling out the word $e_j^{-1}e_ie_j$. However, we can remedy this by using a different action on the vertices of X.

Let us define a map $\eta : \mathrm{Aut}(F_n) \to \mathrm{GL}_{n+1}(\mathbb{Z})$. Since k and $(n-1)$ are coprime, we can find a natural number s satisfying $s(n-1) \equiv 1 \pmod{k}$. Define η by

$$\begin{aligned}
\eta(\rho_{ij}) &= \left(\begin{array}{c|c} \pi_n p_n(\rho_{ij}) & \\ \hline & 1 \end{array}\right), \\
\eta(\lambda_{ij}) &= \left(\begin{array}{c|c} \pi_n p_n(\lambda_{ij}) & -s\underline{v}_j \\ \hline & 1 \end{array}\right), \\
\eta(\epsilon_i) &= \left(\begin{array}{c|c} e_i & s\underline{v}_i \\ \hline & 1 \end{array}\right),
\end{aligned}$$

where $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis for $\mathbb{Z}^n$, and where the blank spaces indicate zero vectors. One can check (using Gersten's presentation) that this map is in fact a homomorphism. Note that the way η is set up, its image is actually contained in $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$. We can therefore define an affine action of $\mathrm{Aut}(F_n)$ (via η) on $\mathbb{Z}^n$: if $\eta(g) = (\underline{v}, M) \in \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ then $\underline{u}^{\eta(g)} = M\underline{u} + \underline{v}$ for all $\underline{u} \in \mathbb{Z}^n$. By postcomposing with the natural projection $\mathbb{Z}^n \to \mathbb{Z}_k^n$ we get an action on vertices of X.

Now we are ready to define the splitting $\phi : \mathrm{Out}(F_n) \to HE^*_\kappa(X)$. Each element $g \in \mathrm{Out}(F_n)$ will send an edge e emanating from a vertex v to a path spelling the appropriate word in the generators of $\mathbb{Z}_k^n$ (given by the standard lift of $\mathrm{Aut}(F_n)$ described above), and emanating from $\eta(g).v$. This gives

$$\eta(\prod_{i \neq j} \rho_{ij}\lambda_{ij}^{-1}) = \left(\begin{array}{c|c} \mathrm{I} & s(n-1)\underline{v}_j \\ \hline & 1 \end{array}\right) \simeq \left(\begin{array}{c|c} \mathrm{I} & \underline{v}_j \\ \hline & 1 \end{array}\right),$$

where the congruence is mod k. Therefore, $\phi(\prod_{i \neq j} \rho_{ij}\lambda_{ij}^{-1})$ is homotopy equivalent to the trivial map. $\square$

Chapter 5

Finite index subgroups of $\mathrm{Out}(F_n)$

In this chapter we investigate some aspects of the structure of finite index subgroups in $\mathrm{Out}(F_n)$. In particular we are interested in the open problem of deciding whether $\mathrm{Out}(F_n)$ (for $n \geqslant 4$) possesses a finite index subgroup with infinite abelianisation.

5.1 Rank of abelianisations of finite index normal subgroups

Definition 5.1.1. Given a finitely generated abelian group G we define the *rank* of G to be the rank of its maximal free abelian subgroup.

Let us consider the following.

Proposition 5.1.2. *Let $N \trianglelefteq \mathrm{Out}(F_n)$ be a normal subgroup of finite index, where $n \geqslant 6$. Suppose that N^{ab}, the abelianisation of N, is infinite. Then*

$$\mathrm{rank}\,(N^{\mathrm{ab}}) \geqslant \binom{n+1}{2}.$$

Proof. Since N is a normal subgroup, we have a homomorphism $\mathrm{Out}(F_n) \to \mathrm{Aut}(N)$ induced by the conjugation action of $\mathrm{Out}(F_n)$ on N. Since the commutator subgroup $[N, N]$ is characteristic (it is a verbal subgroup), we also have

$$\mathrm{Out}(F_n) \to \mathrm{Aut}(N) \to \mathrm{Aut}(N^{\mathrm{ab}}).$$

The abelian group N^{ab} is finitely generated (since N is), and so it is of the form $N^{\mathrm{ab}} \cong \mathbb{Z}^r \times T$, where $r = \mathrm{rank}\,(N^{\mathrm{ab}})$ and T is a torsion group. The group T is again characteristic, and so we obtain

$$\mathrm{Out}(F_n) \to \mathrm{Aut}(N^{\mathrm{ab}}) \to \mathrm{GL}_r(\mathbb{Z}).$$

An embedding $\mathbb{Z} \hookrightarrow \mathbb{C}$ induces $\phi : \mathrm{Out}(F_n) \to \mathrm{GL}_r(\mathbb{C}) = \mathrm{GL}(V)$, an r-dimensional $\mathbb{C}$-linear representation of $\mathrm{Out}(F_n)$.

Suppose we have $0 < \operatorname{rank}(N^{\mathrm{ab}}) < \binom{n+1}{2}$. Then, by Theorem 3.3.3, the representation ϕ has to factor as shown below.

$$\begin{array}{ccc} \mathrm{Out}(F_n) & \xrightarrow{\phi} & \mathrm{GL}_r(\mathbb{C}) \\ {\scriptstyle \pi_n}\downarrow & & \uparrow \\ \mathrm{Out}(F_n^{\mathrm{ab}}) & \xrightarrow{\cong} & \mathrm{GL}_n(\mathbb{Z}) \end{array}$$

Note that $N \leqslant \ker\phi$, and so ϕ factors through a finite group $\mathrm{Out}(F_n)/N$.

After restricting to $\mathrm{SOut}(F_n)$ we get the following commutative diagram.

$$\begin{array}{ccc} \mathrm{SOut}(F_n) & \xrightarrow{\phi|_{\mathrm{SOut}(F_n)}} & \mathrm{SL}_r(\mathbb{C}) \\ {\scriptstyle \pi_n|_{\mathrm{SOut}(F_n)}}\downarrow & \nearrow{\scriptstyle \phi'} & \\ \mathrm{SL}_n(\mathbb{Z}) & & \end{array}$$

The congruence subgroup property tells us that each representation of $\mathrm{SL}_n(\mathbb{Z})$ with finite image factors through some $\mathrm{SL}_n(\mathbb{Z}_k)$.

So ϕ' factors through some $\mathrm{SL}_n(\mathbb{Z}_k)$. The Chinese remainder theorem tells us that $\mathrm{SL}_n(\mathbb{Z}_k) \cong \prod_{i=1}^{\beta} \mathrm{SL}_n(\mathbb{Z}_{p_i^{\alpha_i}})$ where $k = \prod_{i=1}^{\beta} p_i^{\alpha_i}$ is a prime decomposition of k.

Consider V as a representation of $\mathrm{SL}_n(\mathbb{Z}_{p_1^{\alpha_i}})$. Theorem 2.3.4 tells us that $\mathrm{SL}_n(\mathbb{Z}_{p_i^{\alpha_i}})$ has only one irreducible $\mathbb{C}$-linear representation in dimension $r < 2^{n-1} - 1$, namely the trivial one. Hence, noting that

$$r < \binom{n+1}{2} < 2^{n-1} - 1,$$

we conclude that ϕ' is a sum of trivial representations. Hence

$$\mathrm{SOut}(F_n) \leqslant \ker\phi.$$

Now let us consider the homomorphism

$$\eta \colon \mathrm{SOut}(F_n) \to \mathbb{Z}^r \wr F = \bigoplus_{f \in F} \mathbb{Z}^r \rtimes F$$

where

$$\zeta \colon \mathrm{SOut}(F_n)/\big(N \cap \mathrm{SOut}(F_n)\big) = F$$

induced by the composition $\theta \colon N \cap \mathrm{SOut}(F_n) \hookrightarrow N \to \mathbb{Z}^r$ as follows. Let

$$\{f_1, f_2, \ldots, f_{|F|}\}$$

be a set of right coset representatives of $N \cap \mathrm{SOut}(F_n)$ in $\mathrm{SOut}(F_n)$. Then we define

$$\eta(x) = \big(z_1, z_2, \ldots, z_{|F|}, \zeta(x)\big)$$

with

$$z_i = \theta(n_i)$$

where $n_i \in N \cap \mathrm{SOut}(F_n)$ satisfies

$$f_i x = n_i f_{j_i}.$$

Since $\mathbb{Z}^r$ is abelian, we can construct a homomorphism $\psi : \mathbb{Z}^r \wr F \to \mathbb{Z}^r$ by declaring $\psi\big((z_1, \ldots, z_{|F|}, f)\big) = z_1 + \ldots + z_{|F|}$. Composing η and ψ yields a homomorphism $\mathrm{SOut}(F_n) \to \mathbb{Z}^r$. But $\mathrm{SOut}(F_n)^{\mathrm{ab}} = 1$, and so the composition $\psi \circ \eta$ is trivial, and so in particular it vanishes on $N \cap \mathrm{SOut}(F_n)$. But, since $\phi|_{\mathrm{SOut}(F_n)}$ is trivial,

$$\eta(n) = \big(a(n), \ldots, a(n), 1\big)$$

for each $n \in N \cap \mathrm{SOut}(F_n)$, where $a : N \to N^{\mathrm{ab}}$ is the natural projection. This yields $\psi \circ \eta(n) = |F|a(n) = 0$, and so a is the trivial map. This is a contradiction which finishes the proof. □

The following (unpublished) result is due to Martin Bridson.

Proposition 5.1.3 (Bridson)**.** *Let $N \trianglelefteq \mathrm{Out}(F_n)$ be a normal subgroup of finite index, where $n \geqslant 4$. Suppose that N^{ab}, the abelianisation of N, is infinite. Then*

$$\mathrm{rank}\,(N^{\mathrm{ab}}) \geqslant n.$$

Proof. Suppose for a contradiction that $\mathrm{rank}\,(N^{\mathrm{ab}}) = r < n$.

Consider the representation $\phi : \mathrm{Out}(F_n) \to \mathrm{GL}_r(\mathbb{C})$ as before. The group $\mathrm{Out}(F_n)$ contains a subgroup A_{n+1} abstractly isomorphic to the alternating group of degree $n+1$. Since $n \geqslant 4$, A_{n+1} is simple. However, A_{n+1} does not have non-trivial representations in dimensions smaller than n, and so, since $r < n$, $A_{n+1} \leqslant \ker \phi$. In this case the image of ϕ is of order at most two (see [7] by Bridson–Vogtmann), in particular we have $\mathrm{SOut}(F_n) \leqslant \ker \phi$. Now we can use the last paragraph of the proof above. □

Bridson's original proof of Proposition 5.1.3 involved the Serre spectral sequence. The above proof uses only elementary representation theory of alternating groups.

5.2 Minimal quotients of $\mathrm{Out}(F_n)$

In this section we use the results of Chapter 4 to improve known lower bounds on the cardinality of finite quotients of $\mathrm{Out}(F_n)$ which do not factor through

$$\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}).$$

The best known lower bound is $2^n\, n! = |W_n|$, and comes from the following observation of Bridson and Vogtmann [7].

Proposition 5.2.1. *Let $\phi : \mathrm{Out}(F_n) \to G$ be a homomorphism. Then either ϕ factors through $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ or ϕ maps $W_n < \mathrm{Out}(F_n)$ injectively.*

Proof. Note that the result is trivial when $n = 2$, since $\mathrm{Out}(F_2) = \mathrm{GL}_2(\mathbb{Z})$.

Let $K = \ker\, \phi \cap W_n$. Suppose K contains an element $(\epsilon, \sigma) \in \mathbb{Z}_2^n \rtimes S_n = W_n$ with $\sigma \neq 1$. We claim that in this case ϕ factors through

$$\det \circ \pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to \mathbb{Z}_2.$$

If $n = 3$ or $n \geqslant 5$ then the projection of K onto $S_n < W_n$ has to contain A_n, and so in particular all 3-cycles. Moreover, since 3-cycles are of order 3 and $\mathbb{Z}_2^n$ is a 2-group, K itself contains all 3-cycles. In particular $\sigma_{12}\sigma_{13} \in K$ and so $\rho_{23} = \rho_{12}^{\sigma_{12}\sigma_{13}} \in \rho_{12}^K$. Therefore

$$\phi(\rho_{13}^{-1}) = \phi([\rho_{12}^{-1}, \rho_{23}^{-1}]) = [\phi(\rho_{12}^{-1}), \phi(\rho_{12}^{-1})] = 1.$$

If $n = 4$ then the projection of K onto $S_4 < W_4$ contains V_4. Arguing like in Proposition 3.2.1, and possibly conjugating by ϵ_3, we find

$$\phi(\rho_{21}) = \phi(\lambda_{31})^{-1}$$

and conclude that $\phi(\rho_{31}^{-1}) \in \ker\, \phi$. This proves the claim.

Suppose now that $\{1\} < K \leqslant \mathbb{Z}_2^n < W_n$. Note that

$$\forall x \in \mathbb{Z}_2^n : x = \prod_{i=1}^{n} \epsilon_i^{\alpha_i(x)}$$

where $\alpha_i(x) \in \{0, 1\}$ for all i. Let $x \in \mathbb{Z}_2^n$ be an element such that $\alpha(x) = \sum_{i=1}^n \alpha_i(x)$ is maximal. If $\alpha(x) = 1$ then there exists $\sigma \in S_n$ such that $x \neq x^\sigma \in K$ (as K is normal in W_n) and so $\alpha(x^\sigma x) = 2$, which is a contradiction. Hence $\alpha(x) \geqslant 2$. Now there exists $\sigma \in S_n$ such that $\alpha_1(x^\sigma) = \alpha_2(x^\sigma) = 1$ and so

$$\rho_{12}^{(x^\sigma)} = \lambda_{12}.$$

Hence ϕ factors through π_n. The only case left is when $K = \{1\}$, and then $\phi|_{W_n}$ is injective. □

We can now prove the following.

Proposition 5.2.2. *Let $n \geqslant 6$. Suppose $\phi : \mathrm{Out}(F_n) \to F$ is an epimorphism with a finite image which does not factor through π_n. Then*

$$|F| \geqslant 2^n\, n! \binom{n}{2}.$$

Proof. The previous result tells us that we have $W_n \leqslant F$. Let $[F : W_n] = k$, and let $d \colon W_n \to \mathbb{Z}_2$ be the determinant map.

Firstly note that if $k = 1$, then $F = W_n$ and so in particular $\phi(\Delta)$ lies in the centre of $\operatorname{im} \phi$. This implies that ϕ factors through

$$\mathrm{Out}(F_n)/\langle\langle\{[x, \Delta] \mid x \in \mathrm{Out}(F_n)\}\rangle\rangle = \mathrm{GL}_n(\mathbb{Z}).$$

This contradicts our assumptions, and so $k > 1$.

We will let F act on a $2k$-rose X with oriented edges $\{e_{i,j} \mid (i, j) \in \mathbb{Z}_k \times \mathbb{Z}_2\}$. We choose a set $\{f_0, \ldots, f_{k-1}\}$ of left coset representatives of W_n in F, with $f_0 = 1$. An element $f_i w \in F$ (where $w \in W_n$) will act in the following way:

$$f_l w.e_{i,j} = e_{i+l,j+d(f_i^{-1} w f_i)}.$$

Note that this is the action of F on a $2k$-rose induced from the edge-transitive action of W_n on a 2-rose via d.

This action gives us a homomorphism

$$\psi : F \to \mathrm{Out}(\pi_1(X)) = \mathrm{Out}(F_{2k})$$

with image of cardinality higher than 2 Precomposing ψ with ϕ, and applying Theorem 4.4.7, tells us that in particular $k \neq n$.

Now we will construct an action of F on a k-rose: take the action on the $2k$-rose X, and consider the quotient k-rose Y obtained from X by identifying each edge $e_{j,0}$ with the edge $e_{j,1}$ running in the opposite direction. Note that this is, similarly to the above, the action of F induced from the non-trivial action of W_n on a single petal via d. Again as above, we get a homomorphism $\psi : F \to \mathrm{Out}(F_k)$ with image of cardinality higher than 2. After precomposing ψ with ϕ an application of Theorem 4.4.7 yields that $k < \binom{n}{2}$ (since we have shown that $k \neq n$), and the result follows. ☐

Note that there do exist smaller finite quotients of $\mathrm{Out}(F_n)$, for example $\mathrm{PGL}_n(\mathbb{Z}_2)$. They all however need to factor through $\pi_n : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.

5.3 Lower central series of the Torelli subgroup

In this section we give a condition preventing finite index subgroups of $\mathrm{Out}(F_n)$ from having infinite abelianisations.

We first need the following standard result on nilpotent groups.

Lemma 5.3.1. *Let G be a finitely generated nilpotent group, and let $H \leqslant G$ be a subgroup of finite index. Then the index $|G' : H'|$ is finite, where H' and G' are the derived subgroups of H and G respectively.*

Proof. Let $N = G/H'$. Then N is a finitely generated, virtually abelian, nilpotent group. Every finitely generated nilpotent group has a torsion free quotient with a finite kernel (due to a theorem of Hirsch, see e.g [2, Theorem 2.1]). Let M be such a quotient of N. We claim that M is in fact abelian.

Our assumptions give us a short exact sequence

$$A \longrightarrow M \xrightarrow{p} F \ ,$$

where A is torsion free abelian and F is finite. Let us take F to be minimal with respect to cardinality.

We are going to proceed by induction on m, the nilpotency class of M. If $m = 1$ the result is trivial. Suppose $m = 2$. Then M' is normal, torsion free and abelian. Hence so is $\langle A, M' \rangle$. Minimality of $|F|$ tells us that in fact $M' \leqslant A$. This implies that F is abelian.

Now let $x \in M \smallsetminus A$. Then there exists an exponent α such that $x^\alpha \in A$. Let $a \in A$. Then

$$[x, a] \in M'$$

and so in particular $xax^{-1} = ab$, where $b \in M'$. But M' lies in the centre of M, and so

$$a = x^\alpha a x^{-\alpha} = ab^\alpha.$$

As M is torsion free, $b = 0$. Hence x commutes with all elements in A. Now $\langle x, A \rangle$ is an abelian, torsion free subgroup of M. It is also normal, since F is abelian. This contradicts maximality of F. Hence no such x exists, that is $A = M$. The group M is abelian.

To prove the inductive step, we quotient M by the last non-trivial term in its lower central series, use the inductive hypothesis to show that in fact the nilpotency class m satisfies $m = 2$. This proves our claim.

Since M is abelian, we have $N' \leqslant \ker(N \to M)$. But this kernel is finite, and so N' is finite. But

$$|G' : H'| = |N'| < \infty. \qquad \square$$

The following extends a result of Bogopolski and Vikentiev [4, Theorem 4.1].

Theorem 5.3.2. *Let $\{X_i\}$ be the lower central series of $\overline{\mathrm{IA}}_n$, the Torelli subgroup of $\mathrm{Out}(F_n)$, with $\overline{\mathrm{IA}}_n = X_0$. Let $N \trianglelefteq \mathrm{Out}(F_n)$ be a normal subgroup of finite index. If there exists j such that $X_j \leqslant (N \cap X_0)' = [N \cap X_0, N \cap X_0]$, then the abelianisation of N is finite.*

Proof. Let $G = N \cap X_0$, which is a finite-index subgroup of X_0. Let $q : X_0 \to X_0/X_i$ be the quotient map, where $i = \max\{1, j\}$. We are going to use the (standard) notation of writing H' to denote the commutator subgroup $[H, H] \leqslant H$.

Note that $q(X_0)$ is a nilpotent group, and hence $q(G)'$ is a finite index subgroup of $q(X_0)' = q(X_1)$ by Lemma 5.3.1. Therefore

$$\langle G', X_i \rangle \leqslant X_1$$

is a subgroup of finite index. But $X_i \leqslant G'$ by assumption, and hence $G' \leqslant X_1$ is of finite index.

Now, by Theorem 3.2.9 (and the remark afterwards), X_0/X_1 as an $\mathrm{Out}(F_n)$-module does not contain a submodule on which $\mathrm{Out}(F_n)$ acts with a kernel of finite index. Therefore (by Schur's Lemma) the same is true in any $\mathrm{Out}(F_n)$-equivariant quotient of any $\mathrm{Out}(F_n)$-invariant subgroup of X_0/X_1. One such is

$$G/\langle\langle N' \cap X_0, N \cap X_1 \rangle\rangle.$$

But this group is also a quotient of a subgroup of N/N', on which $\mathrm{Out}(F_n)$ acts via $\mathrm{Out}(F_n)/N$, which is finite. Hence

$$G/\langle\langle N' \cap X_0, N \cap X_1 \rangle\rangle = \{1\}.$$

Since all subgroups involved are normal, we have $G = (N' \cap X_0) \cdot (N \cap X_1)$. We have shown above that $G' = N' \cap X_1$ is a finite index subgroup of X_1, and so

$$(N' \cap X_0) \cdot (N \cap N' \cap X_1) = N' \cap X_0$$

is a finite index subgroup of G.

Consider the following commutative diagram, where rows and columns are exact.

$$\begin{array}{ccccc}
X_0 \cap N' & \longrightarrow & N' & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
G = X_0 \cap N & \longrightarrow & N & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & N/N' & \longrightarrow & D
\end{array}$$

Note that B is a finite-index subgroup of $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{Out}(F_n)/X_0$, and hence its abelianisation is finite (this is a classical fact, and follows easily from the observation that $[\pi_n(\rho_{ij}^{-p}), \pi_n(\rho_{jk}^{-p})] = \pi_n(\rho_{ik}^{-p^2})$ and Theorem 2.3.1). But D is a quotient of N/N', which is abelian, and so it has to be finite. Also, we have just shown that C is finite. These two facts imply that N/N' is finite. □

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