Universal L^2-torsion
for ascending HNN extensions of free groups

Dawid Kielak, Bielefeld University dkielak@math.uni-bielefeld.de

SPP 2026 GEOMETRY AT INFINITY

Definition of universal L^2-torsion

Assumptions

Let G be a group such that:

(A1) G admits a finite $K(G,1)$ called X (with universal cover $\widetilde{X}$);

(A2) G is L^2-acyclic, i.e. its L^2-homology vanishes;

(A3) G satisfies the Atiyah conjecture;

(A4) the Whitehead group of G is trivial.

L^2-homology

The (reduced) L^2-homology of G is computed by taking the homology of the chain complex

$$L^2(G)^{d_n} \to L^2(G)^{d_{n-1}} \to \cdots \to L^2(G)^{d_0}$$

obtained from $C_*(\widetilde{X})$ by tensoring it with $L^2(G)$. (When taking the homology, we divide the kernel by the *closure* of the image of the boundary.) The L^2-homology of G does not depend on the choice of X.

Cellular homology

The cellular $\mathbb{Z}$-homology of $\widetilde{X}$ is computed by taking the homology of the cellular chain complex

$$C_n(\widetilde{X}) \to C_{n-1}(\widetilde{X}) \to \cdots \to C_0(\widetilde{X})$$

Observing that the boundary maps are G-equivariant, and that each $C_k(\widetilde{X})$ is a finitely generated free G-module, we can reinterpret the chain complex as a complex of free $\mathbb{Z}G$-modules:

$$(\mathbb{Z}G)^{d_n} \to (\mathbb{Z}G)^{d_{n-1}} \to \cdots \to (\mathbb{Z}G)^{d_0}$$

The ring $\mathcal{D}(G)$

$\mathbb{Z}G$ embeds in the von Neumann algebra of G. The von Neumann algebra in turn embeds in its algebra of affiliated operators. The ring $\mathcal{D}(G)$ is the rational closure of $\mathbb{Z}G$ in this algebra of affiliated operators.

Definition/Theorem

The group G satisfies the *Atiyah conjecture* if and only if $\mathcal{D}(G)$ is a skew-field.

Key properties of $\mathcal{D}(G)$

When G satisfies the Atiyah conjecture, then G is L^2-acyclic if and only if $C_*(G) \otimes_{\mathbb{Z}G} \mathcal{D}(G)$ has trivial homology. We do not have to mention $L^2(G)$ at all!

Let G^{fab} denote the free part of the abelianisation of G, and let $K = \ker(G \to G^{\mathrm{fab}})$. Then $\mathcal{D}(G)$ is the skew-field of fractions (Ore localisation) of $\mathcal{D}(K)G^{\mathrm{fab}}$. In practice, this means that every element in $\mathcal{D}(G)$ can be written as a fraction of elements in $\mathcal{D}(K)G^{\mathrm{fab}}$.

Universal L^2-torsion

Write $C_k(\widetilde{X}) = V_k \oplus W_k$ as G-modules, so that the k^{th} boundary map is trivial on V_k and induces an isomorphism

$$\iota_k \colon W_k \otimes \mathcal{D}(G) \to V_{k-1} \otimes \mathcal{D}(G)$$

We define the *universal L^2-torsion* of G to be

$$\rho_u^{(2)}(G) = \prod_{k=1}^{n} \det(\iota_k)^{(-1)^{k+1}}$$

Thurston norm

M is a closed 3-manifold. By Poincaré duality, every $\phi \in H^1(M;\mathbb{Z})$ is dual to a class in $H_1(M;\mathbb{Z})$ represented by a surface Σ in M.

$$\|\phi\|_T = \min\{\chi_-(\Sigma) \mid \Sigma \text{ is dual to } \phi\}$$

with $\chi_-(\Sigma) = \sum_{\Sigma_0 \in \pi_0(\Sigma)} \max\{-\chi(\Sigma_0), 0\}$

defines the *Thurston norm* $\|\,.\,\|_T$.

Theorem [Friedl–Lück]

Let $G = \pi_1(M)$ be hyperbolic. Then

$$\|\,.\,\|_T = \mathfrak{N}(P(x)) - \mathfrak{N}(P(y))$$

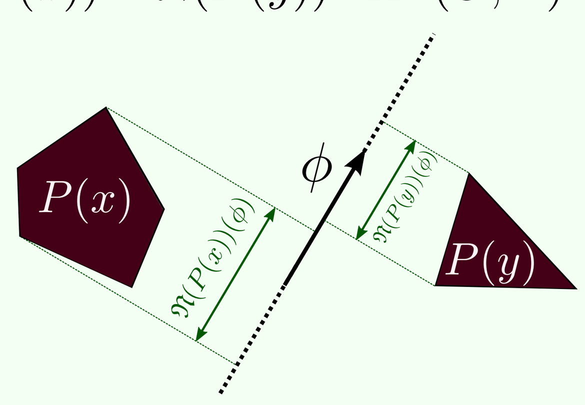
L^2-torsion norm

To the universal L^2-torsion

$$\rho_u^{(2)}(G) = xy^{-1}$$

with $x, y \in \mathcal{D}(K)G^{\mathrm{fab}}$, we associate the function

$$\mathfrak{N}(P(x)) - \mathfrak{N}(P(y)) \colon H^1(G;\mathbb{R}) \to \mathbb{R}$$



From group rings to polytopes

Take $z \in RG^{\mathrm{fab}}$ where R is a ring. The support of z is a finite subset of G^{fab}, which in turn is a subset of the $\mathbb{R}$-vector space $H_1(G;\mathbb{R})$. Taking the convex hull of $\operatorname{supp} z$ in $H_1(G;\mathbb{R})$ gives us a polytope $P(z)$.

From polytopes to norms

Let P be a polytope in $H_1(G;\mathbb{R})$. We define $\mathfrak{N}(P)$ to be a semi-norm $H^1(G;\mathbb{R}) \to [0,\infty)$ given by

$$\mathfrak{N}(P)(\phi) = \max\{\phi(a) - \phi(b) \mid a, b \in P\}$$

Ascending HNN extension

$$F_n *_f = \langle F_n, t \mid t^{-1}xt = f(x), x \in F_n \rangle$$

is an *ascending HNN extension* of F_n provided that f is injective. The group $F_n *_f$ satisfies (A1)–(A4) by the works of Linnell, Lück, and Waldhausen.

Theorem [Funke–K.]

The *Thurston norm* defined by

$$\|\phi\|_T = \mathfrak{N}(P(x)) - \mathfrak{N}(P(y))$$

is a semi-norm.

Theorem [McMullen]

Let M be a closed 3-manifold with the first Betti number at least 2. Then

$$\|\,.\,\|_T \geqslant \|\,.\,\|_A$$

Theorem [Funke–K.]

Let $G = F_n *_f$ with the first Betti number at least 2. Then

$$\|\,.\,\|_T \geqslant \|\,.\,\|_A$$

Alexander polynomial

The map $C_2(\widetilde{X}) \to C_1(\widetilde{X})$ is a $d_1 \times d_2$ matrix over $\mathbb{Z}G$. Applying the map induced by $G \to G^{\mathrm{fab}}$ gives us a $d_1 \times d_2$ matrix over $\mathbb{Z}G^{\mathrm{fab}}$, and the *Alexander polynomial* $\Delta_G \in \mathbb{Z}G^{\mathrm{fab}}$ is the greatest common divisor of the $(d_1 - 1)$-minors of this matrix.

Alexander norm

The *Alexander norm* is defined by

$$\|\,.\,\|_A = \mathfrak{N}(P(\Delta_G))$$