Nielsen Realisation for right-angled Artin groups

Question: Nielsen 1932

Given a finite subgroup $H < MCG(\Sigma)$, can the action be realised by an isometric action on Σ endowed with some hyperbolic metric?

This was the original problem (posed by Jakob Nielsen), which motivates the more general Nielsen Realisation problem.

Question: Nielsen Realisation for G

Given a finite subgroup $H < \mathrm{Out}(G)$, find a metric classifying space X for G, and an isometric action $H \curvearrowright X$ such that

$$H \to \mathrm{Isom}(X) \to \mathrm{Out}(\pi_1(X)) = \mathrm{Out}(G)$$

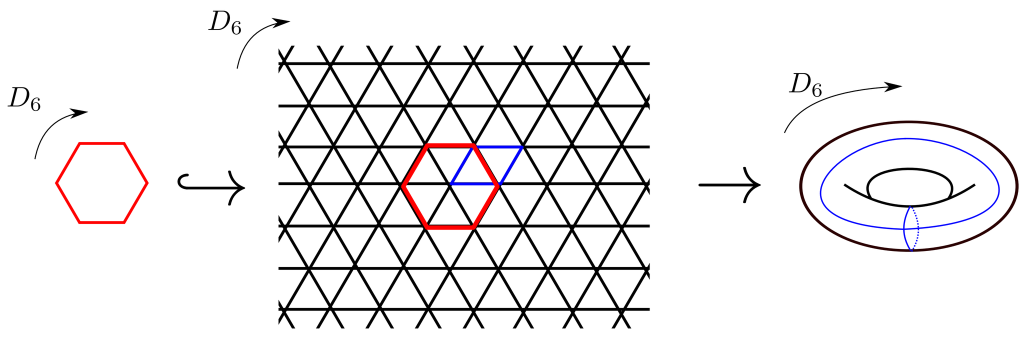
coincides with the given embedding $H < \mathrm{Out}(G)$.

We say that such an X *realises* the action of H.

Theorem

The action of any finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ can be realised by a metric torus.

This amounts to saying that each finite subgroup of $\mathrm{GL}_n(\mathbb{Z}) < \mathrm{GL}_n(\mathbb{R})$ lies in a conjugate of the orthogonal group.

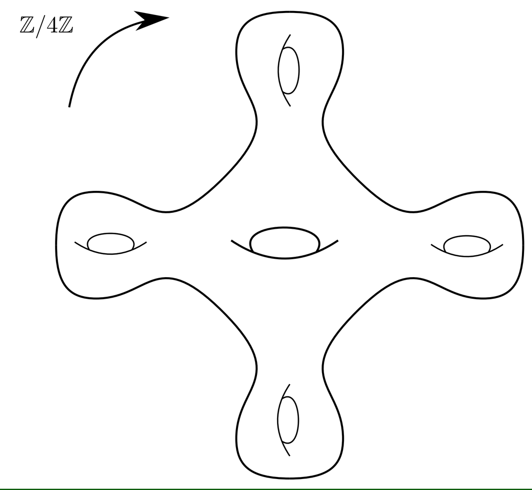


Corollary

Any finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ fixes a point of the symmetric space.

Theorem: Kerckhoff 1983

The action of any finite subgroup of $\mathrm{MCG}(\Sigma)$ can be realised.

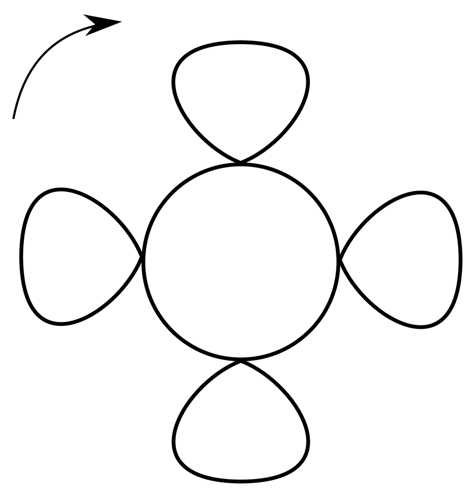


Corollary

Any finite subgroup of $\mathrm{MCG}(\Sigma)$ fixes a point of the Teichmüller space of Σ.

Theorem: Culler; Khramtsov; Zimmermann 1980's

The action of any finite subgroup of $\mathrm{Out}(F_n)$ can be realised by a (metric) graph.



Corollary

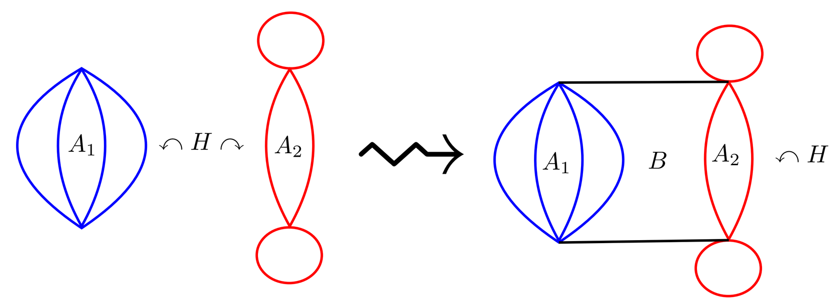
Any finite subgroup of $\mathrm{Out}(F_n)$ fixes a point of the Culler–Vogtmann Outer Space.

Theorem: Adapted Realisation for free groups, Hensel–K.

Suppose that we are given a finite subgroup $H < \mathrm{Out}(F_n)$, and a decomposition

$$F_n = A_1 * \cdots * A_k * B$$

such that each A_i is preserved by H (up to conjugation). Suppose further that each induced action $H \to \mathrm{Out}(A_i)$ is realised by a graph X_i. Then the original action $H < \mathrm{Out}(F_n)$ can be realised by a graph X in such a way that for each i we have an H-equivariant (up to homotopy) embedding $X_i \hookrightarrow X$ inducing $A_i < F_n$.

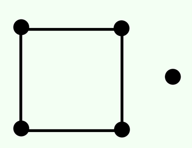
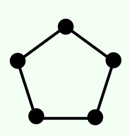


Definition

Given a finite simplicial graph Γ we define the associated *right-angled Artin group* (RAAG)

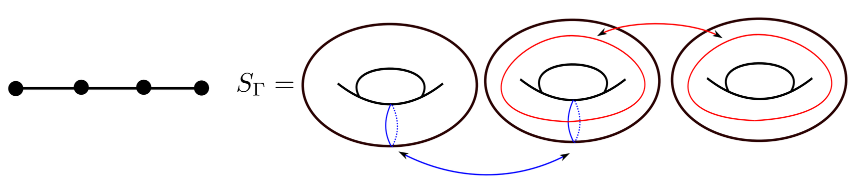
$$A_\Gamma = \langle V(\Gamma) \mid [v, w] \text{ for each } (v, w) \in E(\Gamma) \rangle$$

Examples

- when Γ has no edges, A_Γ is free
- when Γ is complete, A_Γ is free abelian
- $\Gamma =$  $A_\Gamma = F_2 \times F_2 * \mathbb{Z}$
- $\Gamma =$  $A_\Gamma = ?$

Definition: Salvetti complex

Given a finite simplicial graph Γ, the *Salvetti complex* S_Γ is constructed as follows: let $k = |V(\Gamma)|$; we take a k-cube $[0,1]^k$, and make it into a k-torus by identifying the faces in the obvious way. Now our cube complex (the torus) has exactly k edges; we orient and label them with elements of $V(\Gamma)$. Now each face of the cube complex is uniquely determined by the labels of the edges it contains. We remove the interior of each face for which these labels do not span a complete subgraph of Γ.



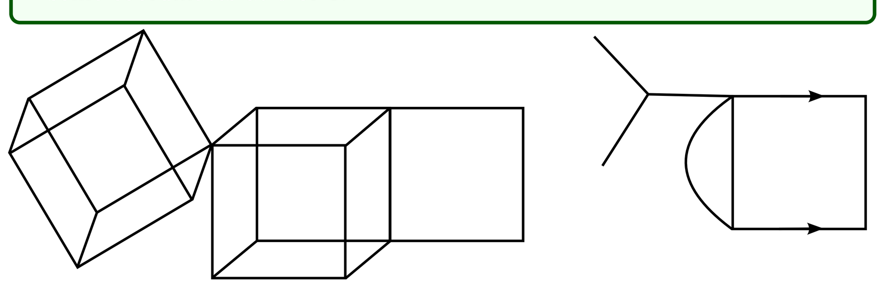
Theorem

The Salvetti complex is a (metric) classifying space for A_Γ.

The fundamental group is isomorphic to A_Γ by construction (it is enough to investigate the 2-skeleton of S_Γ for this). The universal covering of S_Γ is CAT(0), and hence contractible, by Gromov's link condition.

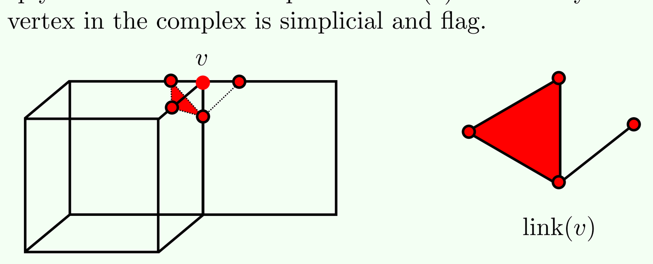
Definition: Cube complex

A *cube complex* is a polyhedral complex, where the polyhedra are Euclidean cubes $[0,1]^n$ of various dimensions, and the allowed gluings are isometries of *faces*, where a face of a cube is a subcube obtained by fixing some coordinates at 0 or 1.



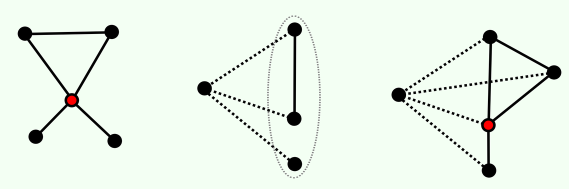
Theorem: Gromov's link condition

A simply-connected cube complex is CAT(0) if and only if the link of each vertex in the complex is simplicial and flag.



Definition: Aconical graph

A simplicial graph is *aconical* when links of vertices are not cones.



Crucial lemma

Suppose that Γ is aconical, and let $v \in \Gamma$ be a vertex. Then

$$A_{\mathrm{lk}(v)} = \langle \mathrm{lk}(v) \rangle < A_\Gamma$$

is preserved up to conjugacy and symmetry of Γ by every automorphism of A_Γ.

Applications

- $\mathrm{Out}(\mathrm{Out}(F_n)) = 1$ [Bridson–Vogtmann]
- Any homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ has image contained in $\mathbb{Z}/2\mathbb{Z}$, provided that $n \geqslant 6$ and $n < m < \binom{n}{2}$ [K.]

Theorem: Nielsen Realisation for RAAGs, Hensel–K.

Suppose that $H < \mathrm{Out}(A_\Gamma)$ is a finite subgroup, and that Γ is aconical and without symmetries. Then there exists a cube complex realising the action of H.

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