Prop G f.g. nilpotent has polynomial growth.

Proof Induction on the nilpotency class.

Base case (virtually) free abelian f.g. groups.

$G = \mathbb{Z}^n \oplus$ torsion.

$\gamma_G \simeq \gamma_{\mathbb{Z}^n}$. Take S standard generating set of $\mathbb{Z}^n$. $r_S(m) = m^n$, polynomial.

Inductive Step

$T = \coprod T_i$ finite generating set for G with

$T_i \subseteq G_i$

Let $w \in B_T(1,m)$.

Order T_0: $\{t_1, \ldots, t_k\}$.

$\exists A \in \mathbb{N}\ \forall t \in T$: $t t_i \in t_i t B_{T \setminus T_0}(1, A-5)$

Push t_0 to the left in w.

Get $w_0 = w$ in G, with

$w_0 = t_0^{k_0} w_0'$, $w_0' \in B_{T \setminus \{t_0\}}(1, Am)$.

Repeat for the other t_i's:

$w = w_k$ in G,

$w_k = t_0^{\lambda_0} \cdots t_k^{\lambda_k} w_k'$, $w_k' \in B_{T \setminus T_0}(1, A^{k+1} m)$.

By induction hypothesis, $\exists$ polynomial p st

$|B_{T \setminus T_0}(1, A^{k+1} m)| \leq p(A^{k+1} m) = p'(m)$.

for some polynomial p' (note: k is independent of m).

Also, $|\{ t_0^{\lambda_0} \cdots t_k^{\lambda_k} : \sum \lambda_i \leq m \}| \leq (m+1)^{k+1}$.

So, $\gamma_T(m) \leq (m+1)^{k+1} \cdot p'(m)$ $\square$

Def G is polycyclic • of class 0 iff G is trivial

• of class $n+1$ iff $\exists\, G \twoheadrightarrow$ cyclic with kernel polycyclic of class n

• iff it is polycyclic of some class.

Lemma $A \leq B$, both f.g.

$$\gamma_A \preceq \gamma_B .$$

Proof S finite gen set of A,

$T = S \cup S'$ — " — B.

$$\gamma_A(n) = |B_S(1,n)| \leq |B_T(1,n)| = \gamma_B(n). \quad \square$$

Thm G solvable, f.g. with $\gamma_G^{(n)} \not\asymp e^n$.

G is polycyclic.

Proof Induction on solvability class m of G.

$m=1$ f.g. abelian $\Rightarrow$ polycyclic.

Inductive step $A = G'$ solvable of class $m-1$ subexponential growth.

$B = G$ solvable, f.g., subexponential growth.

$C = G/G'$ f.g. abelian, hence f.p.

$1 \to A \to B \to C \to 1$ exact.

Since C is f.p., $A = \langle\langle a_1, \dots, a_k \rangle\rangle$ in B.

$B = \langle S \rangle$, S finite.

Consider the free monoid M on $\{s, sa_i\}$ for some fixed $s \in S$.

M cannot embed into B, since the distinct $\gamma_B(n) \asymp e^n$. So $\exists$ positive words w, v in 2 variables s.t.

$w(s, sa_i) = v(s, sa_i)$.

Let $s_j = s^j a_i s^{-j}$. We get

$$\prod_{j=1}^{\alpha} s_j^{\varepsilon_j} s^{\alpha} = \prod_{j=1}^{\beta} s_j^{\varepsilon_j'} s^{\beta} \quad \text{in } G,$$

with $\varepsilon_j, \varepsilon_j' \in \{0,1\}$.

So some $s_j \in \langle s_1, \dots, s_{j-1} \rangle$.

Similarly for some $s^{-k} a_1 s^{k}$.

$\therefore$ $\langle\langle a_1 \rangle\rangle$ in G is f.g.

$\therefore$ $\langle\langle A \rangle\rangle$ in G is f.g.

$\cap$

A

By the inductive hypothesis, A is poly-cyclic. So is C, and hence $B = G$. $\square$

We are left with the final case:

Thm Let G be polycyclic, not virt. nilpotent. Then $\gamma_G \simeq e^n$.

This goes in two steps.

Prop Every polycyclic group is virt. nilpotent-by-f.g. free-abelian.

Proof postponed.

Key example

$$G = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}, \quad \alpha: \mathbb{Z} \to GL_n \mathbb{Z}.$$

$\mathbb{Z} = \langle t \rangle$, $\lambda(t) = A$.

1° all e-values of A are on S^1.

$\chi_A(x) \in \mathbb{Z}[x]$, all roots are on the unit circle.

[Kronecker]: these roots are roots of unity.

So $\exists m$: $\mathbb{Z}^n \rtimes m\mathbb{Z} \cong \mathbb{Z}^{n+1}$.

2° $\exists$ e-value λ of A with $|\lambda| \neq 1$.

WLOG $|\lambda| > 1$ is maximal.

$v \in \mathbb{Z}^n \otimes \mathbb{C} = \mathbb{C}^n$ s.t. $|v| = 1$, $Av = \lambda v$.

Replace A by power w/ $|\lambda| \geq 3/_-$.

Find $w \in \mathbb{Z}^n$: $\left| w - \langle v, w \rangle v \right| < \frac{1}{5} |w|$.

Consider free monoid on $\{t, tw\}$.

If it does not contract, then some

$t^j w t^{-j}$ a difference of $\sum_{i=1}^{\lambda} t^{k_i} w t^{-k_i}$

and $\sum_{\ell=1}^{\beta} t^{k_\ell} w t^{-k_\ell}$, with $k_i, k_\ell < j$,

$\{k_i\}, \{k_\ell\}$ without repetitions.

Now: $t^j w t^{-j} = A^j w =$

$= A^j v/\langle v, w \rangle + A^j (w - v \langle v, w \rangle)$

has ~~mean~~ length at least $\lambda^j \langle v, w \rangle - \frac{1}{5} \lambda^j |w|$

$\geq \frac{3}{5} \lambda^j |w|$.

In fact, $\langle v, A^j w \rangle \geq \frac{3}{5}|\lambda|^j |w|$.

The projection of $A^{k_i} w$ onto v is positive, so we ignore it.

We get $\displaystyle\sum_{i=1}^{x} \frac{6}{5}|\lambda|^{k_i} |w| \geq \frac{3}{5}|\lambda|^j |w|$.

$$\wedge\!|$$

$$\frac{6}{5}|w| \frac{1}{(|\lambda|-1)} |\lambda|^j$$

$$\wedge\!|$$

$$\frac{2}{5}|w|\,|\lambda|^j \qquad \#.$$

So $\mathbb{Z}^2 \rtimes \mathbb{Z}$ has exponential growth.

Now we are ready for:

<u>Lemma</u> $\underset{\text{nil}}{N} \hookrightarrow \underset{\text{f.g.}}{G} \twoheadrightarrow \mathbb{Z}^n$

is v. nilpotent or has exponential growth.

<u>Proof</u>

From the key example: $\mathbb{Z}^n \curvearrowright N_i/N_{i+1} \otimes \mathbb{Q}$

has all eigenvalues on S^1, and

hence we may pass to a finite index

subgroup in $\mathbb{Z}^n$ and get $\mathbb{Z}^n \curvearrowright N_i/N_{i+1}$

trivial for all i. So G has center,

and in fact a finite central series. $\square$