3. The axioms of set theory, expressed formally

For those who are curious, or concerned, to see how it is possible, we indicate here how to express the axioms of set theory in the language of set theory.

Notice that this is an infinite list of axioms. One can prove that (using the usual language of set theory, and assuming ZF to be consistent!*) there is no finite set of statements equivalent to ZF (or to ZFC).

Axiom of extensionality *Two sets are equal if and only if they have the same elements:* this becomes

$$\forall x \forall y\,(x = y \leftrightarrow \forall z\,(z \in x \leftrightarrow z \in y)).$$

Empty set axiom *The empty set $\varnothing$ exists.*

$$\exists x \forall y\,(\neg\, y \in x).$$

Axiom of Pairs *If a and b are sets, then so is $\{a, b\}$.*

$$\forall a \forall b \exists x \forall y\,(y \in x \leftrightarrow (y = a \vee y = b)).$$

Axiom of Unions *Suppose A is a set. Then so is the union $\bigcup A$ of its elements.*

$$\forall A \exists x \forall y (y \in x \leftrightarrow \exists z\,(y \in z \wedge z \in A)).$$

Subset axiom scheme *Suppose A is a set, $b_1, \ldots, b_n$ are also sets, and $\phi(x, y_1, \ldots, y_n)$ is a statement in the language of set theory. Then*

$$\{x \in A : \phi(x, b_1, \ldots, b_n)\}$$

is a set. [Here we are making explicit that we are allowing the statement ϕ to refer to entities other than x.]

$$\forall A \forall b_1 \ldots \forall b_n \exists y \forall x\,(x \in y \leftrightarrow (x \in A \wedge \phi(x, b_1, \ldots, b_n))).$$

Notice that this is a collection of axioms, one for each $\phi(x)$.

Foundation axiom *Suppose A is a non-empty set. Then A has an $\in$-minimal element; that is, there exists $m \in A$ such that $m \cap A = \varnothing$.*

$$\forall A\,((\exists x\, x \in A) \rightarrow (\exists m (m \in A \wedge \forall x (x \in A \rightarrow \neg\, x \in m)))).$$

Powerset axiom *Let X be a set. Then $\wp X$ is a set.*

$$\forall X \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in X)).$$

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* If ZF is inconsistent, then ZF is equivalent to $0 = 1$. But also if ZF is inconsistent, then it is very bad news for set theory.

Axiom of Infinity *There is a successor set.*

We will show how to express this in several steps. We can express "$\varnothing \in x$" thus:

$$\exists y\,(y \in x \wedge \forall z\, \neg\, z \in y).$$

We can express "$y \cup \{y\} \in x$" thus:

$$\exists z\,\bigl(z \in x \wedge \forall w\,(w \in z \leftrightarrow (w = y \vee w \in y))\bigr).$$

The Axiom of Infinity is:

$$\exists x(\varnothing \in x \wedge \forall y(y \in x \to y \cup \{y\} \in x)).$$

Replacement Axiom Scheme *Given a set X, sets $b_1, \ldots, b_n$, and a rule which associates, with each element x of X and each sequence of sets $w_1, \ldots, w_n$, a unique set $\Phi(x, w_1, \ldots, w_n)$,*

$$\{y : \exists x \in X\; y = \Phi(x, b_1, \ldots, b_n)\}$$

is a set.

We first need to express the concept of a rule precisely. To understand what follows, imagine that the statement "$y = \Phi(x, w_1, \ldots, w_n)$" stands for a formula $\psi(y, x, w_1, \ldots, w_n)$ having the property that for all $x \in X$ and $w_1, \ldots, w_n$, there is a unique set y making it true.

This is a restriction on ψ which we express by the formula $\psi!(X)$ which I define to be the following:

$$\forall x \forall w_1 \ldots \forall w_n (x \in X \to \exists y(\psi(y, x, w_1, \ldots, w_n) \wedge \forall z(\psi(z, x, w_1, \ldots, w_n) \to z = y))).$$

Then the Replacement Scheme is the set containing, for each formula $\psi(y, x, w_1, \ldots, w_n)$ of the language of set theory, the formula

$$\forall X\,(\psi!(X) \to \forall b_1 \ldots \forall b_n \exists z \forall y\,(y \in X \leftrightarrow \exists x\,(x \in X \wedge \psi(y, x, b_1, \ldots, b_n)))).$$

Axiom of Choice (AC) *Let $\mathscr{A}$ be a non-empty set of disjoint non-empty sets. Then there exists a set B such that for all $A \in \mathscr{A}$, $|A \cap B| = 1$.*

We can express $|A \cap B| = 1$ as

$$\exists x\,((x \in A \wedge x \in B) \wedge \forall y\,((y \in A \wedge y \in B) \to y = x)).$$

Now we can express the Axiom of Choice as:

$$\begin{aligned}\forall \mathscr{A}((\exists x\,(x \in \mathscr{A}) \wedge \forall x\,(x \in \mathscr{A} \to \exists y\,(y \in x))\\ \wedge \forall x \forall y((x \in \mathscr{A} \wedge y \in \mathscr{A} \wedge \exists z\,(z \in x \wedge z \in y)) \to x = y))\\ \to \exists B \forall A\,(A \in \mathscr{A} \to |A \cap B| = 1)).\end{aligned}$$