9. Circumstances where axioms are necessary*

These lists are very far from being exhaustive. The techniques of proof of the statements made in this handout are beyond the scope of the course.

9.1. You need the Axiom of Infinity to prove that there are any infinite sets. (Not surprisingly.)

9.2. You need the Powerset Axiom to prove that there are any uncountable sets.

9.3. You need the Replacement Scheme to prove:

1. that there exist uncountable ordinals,
2. that the ordinal $\omega + \omega$ exists,
3. that there are uncountably many cardinal numbers,
4. that there are infinite cardinal numbers other than $\aleph_0$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, ...
5. that there are infinite cardinal numbers other than $\aleph_n$ for $n \in \omega$.

9.4. You need the Axiom of Choice to prove:
in set theory:

1. that if X is not finite, then $|X| \geq \aleph_0$,
2. a countable union of countable sets is countable,
3. a countable union of finite sets is countable,
4. a countable union of two-element sets is countable,
5. a countable union of countable ordinals is countable,
6. if $f : X \to Y$ is onto, then there exists $g : Y \to X$ which is one-to-one,
7. if κ and λ are infinite cardinal numbers, then $\kappa + \lambda = \kappa.\lambda = \max(\kappa, \lambda)$ (but then, if AC is false, $\max(\kappa, \lambda)$ might not be well-defined),
8. if λ is an infinite cardinal number and $\kappa < \lambda$, then $\kappa + \lambda = \kappa.\lambda = \lambda$,
9. if κ is an infinite cardinal number, then $\kappa = \kappa + \kappa = \kappa.\kappa$,

in algebra, logic, and order theory:

10. every vector space has a basis,
11. for every set X, there is a group operation $*$ on X,
12. every set can be totally ordered,
13. the Löwenheim-Skolem Theorem: let L be a countable language of first-order predicate calculus, and let T be a set of sentences of L such that T has an infinite model. Then T has a model of every infinite cardinality. (As we have seen, we need AC to prove this even if T is the axioms of group theory.)

There are many other famous theorems that depend on, or even are equivalent to, the Axiom of Choice, but which we are not yet in a position to state precisely. Among these are:

14. Tychonoff's Theorem: any product of compact topological spaces is compact,
15. The Boolean Prime Ideal Theorem: any ideal on a Boolean algebra can be extended to a maximal ideal. (A Boolean algebra is a partially ordered set with operations

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analogous to the connectives $\wedge$, $\vee$ and $\neg$ from propositional calculus, with properties and structure designed to formalise this correspondence. If a Boolean algebra is a set of truth values which we might assign to some statements (such as "true", "false", and various kinds of "doubtful" or "not sure"), an ideal is, intuitively, a subset of the Boolean algebra which might correspond to a notion of falsehood; perhaps some new evidence has come in and we can see that some statements that we thought were "doubtful" or "possible but not certain" are now just false. A maximal ideal would then correspond to a situation in which we have, for every statement, decided either that it is true or that it is false.)

One very frequently encounters a theorem which is stated in some lecture course with a hypothesis of smallness, such as (in algebra) countability, or (in topology) separability. Very often, this hypothesis is unnecessary, but in order to drop it, we need some form of the Axiom of Choice.