Geometric flows, G_2-structures and 3-Sasakian geometry

Jason D. Lotay

Oxford

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(Joint work with Aaron Kennon)

Key example: the 7-sphere

Hopf fibration: $\mathcal{S}^3 \to \mathcal{S}^7 \to \mathcal{S}^4$

- round metric $g^{ts} = g_{\mathcal{S}^3} + g_{\mathcal{S}^4}$
- "canonical variation" $g_\kappa = \kappa g_{\mathcal{S}^3} + g_{\mathcal{S}^4}$ for $\kappa > 0$
- g_κ Einstein $\Leftrightarrow$ $g^{ts} = g_1$ or $g^{np} = g_{1/5}$

Octonions: $\mathcal{S}^7 \subseteq \mathbb{O}$

- $\rightsquigarrow$ cross product $\times$
- $\rightsquigarrow$ 3-form

$$\varphi^{ts}(u, v, w) = g^{ts}(u \times v, w) \quad \text{G}_2\text{-structure}$$

- **Fact:** φ^{ts} determines g^{ts}
- $\mathrm{d}\varphi^{ts} = \mu * \varphi^{ts}$ for $\mu > 0$ constant $\leftrightarrow$ nearly parallel
- **Note:** $\mathrm{d} * \varphi^{ts} = 0$

Geometric flows

Ricci flow

$$\frac{\partial g_t}{\partial t} = -2\,\mathrm{Ric}_t$$

- Critical point: Ricci-flat $\mathrm{Ric} = 0$
- $\mathrm{Ric} = \lambda g$, $\lambda > 0 \rightsquigarrow$ shrinker: critical point for rescaled flow

Laplacian flow

$$\frac{\partial \varphi_t}{\partial t} = \Delta_t \varphi_t = (\mathrm{d}\mathrm{d}_t^* + \mathrm{d}_t^*\mathrm{d})\varphi_t$$

Laplacian coflow

$$\frac{\partial *_t \varphi_t}{\partial t} = \Delta_t *_t \varphi_t = \mathrm{d}\mathrm{d}_t^* *_t \varphi_t$$

- Critical point: $\mathrm{d}\varphi = 0 = \mathrm{d}_\varphi^* \varphi \rightsquigarrow$ Ricci-flat with $\mathrm{Hol}(g) \subseteq \mathrm{G}_2$
- Laplacian (co)flow restricted to (co)closed G_2-structure = gradient flow of Hitchin volume functional on $[\varphi_t]$ ($[*_t\varphi_t]$)
- $\mathrm{d}\varphi = \mu * \varphi$, $\mu > 0 \rightsquigarrow$ expander: critical for rescaled flows

Observations and main results

Nearly parallel G_2-structure $\mathrm{d}\varphi = \mu * \varphi$, $\mu > 0$

- $\rightsquigarrow$ Einstein metric $\mathrm{Ric} = \lambda g$, $\lambda > 0$
- G_2-structure more information than metric
- suggests Laplacian flow/coflow finer detail than Ricci flow

$\mathrm{d} * \varphi = 0 \rightsquigarrow$ Laplacian coflow $=$ gradient flow of volume on $[*\varphi]$

- $\rightsquigarrow \dfrac{\partial g_t}{\partial t} = -2\,\mathrm{Ric}_t + Q_t$ where Q_t quadratic in $\mathrm{d}\varphi$
- $\rightsquigarrow$ metric flow $=$ Ricci flow plus lower order terms
- nearly parallel $=$ Laplacian coflow expander but positive Einstein $=$ Ricci flow shrinker
- $\Rightarrow$ lower order terms matter

Main results: compare behaviour of Ricci flow, Laplacian flow & coflow near Einstein metrics/nearly parallel G_2-structures on 3-Sasakian 7-manifolds

G_2-structures on the 7-sphere

Recall: $\mathcal{S}^3 \to \mathcal{S}^7 \to \mathcal{S}^4$, 3-form φ^{ts} inducing round metric g^{ts}

- $\mathcal{S}^3 = \mathrm{SU}(2) \rightsquigarrow$ left-invariant coframe η_1, η_2, η_3
- $\omega_1, \omega_2, \omega_3$ orthogonal self-dual 2-forms on $\mathcal{S}^4$ with length $\sqrt{2}$
- $\rightsquigarrow$ two 3-parameter families of 3-forms for $a, b, c > 0$:

$$\varphi^{\pm} = \pm a^2 b \eta_1 \wedge \eta_2 \wedge \eta_3 - ac^2 \eta_1 \wedge \omega_1 - ac^2 \eta_2 \wedge \omega_2 \mp bc^2 \eta_3 \wedge \omega_3$$

- $\varphi^{\pm}$ induces $g = a^2\eta_1^2 + a^2\eta_2^2 + b^2\eta_3^2 + c^2 g_{\mathcal{S}^4}$
  $\Rightarrow \varphi^+$ and φ^- isometric (and Berger metric on SU(2))

Lemma

- $\mathrm{d} * \varphi^{\pm} = 0$
- $\varphi^{\pm}$ *nearly parallel* $\Leftrightarrow a = b = c \rightsquigarrow \varphi^- = c^3 \varphi^{ts}$ *or*
  $a = b = \frac{1}{\sqrt{5}} c \rightsquigarrow \varphi^+ = c^3 \varphi^{np}$

Note: φ^{np} induces "squashed" Einstein metric g^{np}

3-Sasakian 7-manifolds

Definition

(X^7, g^{ts}) *3-Sasakian* $\Leftrightarrow$ *cone* $(\mathbb{R}^+ \times X^7, g = \mathrm{d}r^2 + r^2 g^{ts})$ *hyperkähler* $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(2)$ $\big(\rightsquigarrow$ *generalizes* $(\mathcal{S}^7, g^{ts}) \big)$

Fact: $\exists$ infinitely many 3-Sasakian 7-manifolds

- $\mathrm{SU}(2)/\Gamma \to X^7 \to Z^4$, Z ASD Einstein
- $a, b, c > 0 \rightsquigarrow g = a^2\eta_1^2 + a^2\eta_2^2 + b^2\eta_3^2 + c^2 g_Z$ induced by

$$\varphi^{\pm} = \pm a^2 b \eta_1 \wedge \eta_2 \wedge \eta_3 - ac^2 \eta_1 \wedge \omega_1 - ac^2 \eta_2 \wedge \omega_2 \mp bc^2 \eta_3 \wedge \omega_3$$

- $a = b = c = 1 \rightsquigarrow \varphi^- = \varphi^{ts}$ nearly parallel inducing g^{ts}
- $\sqrt{5}a = \sqrt{5}b = c = 1 \rightsquigarrow \varphi^+ = \varphi^{np}$ nearly parallel inducing g^{np} "squashed" Einstein metric , cone has $\mathrm{Hol} = \mathrm{Spin}(7)$

Example: Aloff–Wallach space $\big(\mathrm{SU}(3) \times \mathrm{SU}(2)\big)/\big(\mathrm{U}(1) \times \mathrm{SU}(2)\big)$

Ricci flow

$$SU(2)/\Gamma \to X^7 \to Z^4 \rightsquigarrow g_t = a_t^2\eta_1^2 + a_t^2\eta_2^2 + b_t^2\eta_3^2 + c_t^2 g_Z$$

$$\begin{aligned}\frac{\partial g_t}{\partial t} &= -2\,\mathrm{Ric}_t \\ &= -4\left(2 - \frac{b_t^2}{a_t^2} + 2\frac{a_t^4}{c_t^4}\right)(\eta_1^2+\eta_2^2) - 4\left(\frac{b_t^4}{a_t^4} + 2\frac{b_t^4}{c_t^4}\right)\eta_3^2 \\ &\quad - 4\left(6 - \frac{2a_t^2+b_t^2}{c_t^2}\right)g_Z \qquad (*)\end{aligned}$$

$\rightsquigarrow$ ansatz preserved and only critical points for rescaled flow are:

$$a_t = b_t = c_t \leftrightarrow g^{ts} \quad \text{and} \quad \sqrt{5}a_t = \sqrt{5}b_t = c_t \leftrightarrow g^{np}$$

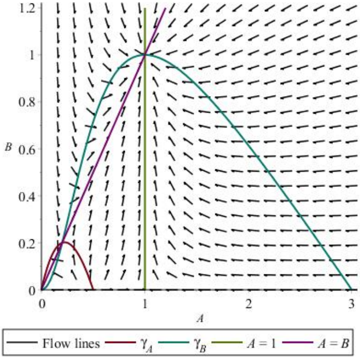
Theorem

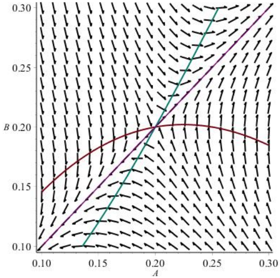
After rescaling, under Ricci flow (*)

- *3-Sasakian Einstein g^{ts} stable*
- *"squashed" Einstein g^{np} unstable (saddle point)*

Dynamics for Ricci flow

$$g = a^2\eta_1^2 + a^2\eta_2^2 + b^2\eta_3^2 + c^2 g_Z \rightsquigarrow A = \frac{a^2}{c^2} \text{ and } B = \frac{b^2}{c^2}$$





- $(A, B) = (0, 0) \leftrightarrow$ Einstein (Z^4, g_Z)
- $(A, B) = (1, 0) \leftrightarrow$ Kähler–Einstein on twistor space Y^6 of Z^4
- $(A, B) = (\frac{1}{2}, 0) \leftrightarrow$ nearly Kähler metric on Y^6

Laplacian coflow

$$*_t\varphi_t^{\pm} = c_t^4 \operatorname{vol}_Z \mp a_t b_t c_t^2 \eta_2 \wedge \eta_3 \wedge \omega_1 \mp a_t b_t c_t^2 \eta_3 \wedge \eta_1 \wedge \omega_2 - a_t^2 c_t^2 \eta_1 \wedge \eta_2 \wedge \omega_3$$

$$\frac{\partial *_t \varphi_t}{\partial t} = \Delta_t *_t \varphi_t = \mathrm{d}\mathrm{d}_t^* *_t \varphi_t \qquad (\dagger)$$

$\rightsquigarrow$ ansatz preserved and only critical points for rescaled flow are:

$$\varphi_t^- : a_t = b_t = c_t \leftrightarrow \varphi^{ts} \quad \text{and} \quad \varphi_t^+ : \sqrt{5}a_t = \sqrt{5}b_t = c_t \leftrightarrow \varphi^{np}$$

Note: for φ_t^-, $a_t = b_t$ not preserved

Theorem

After rescaling, under Laplacian coflow (†)

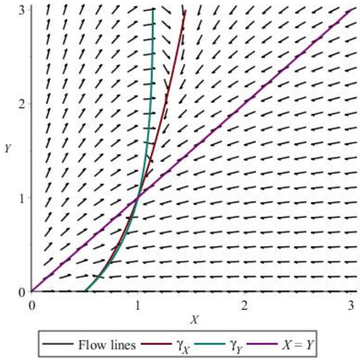
- *nearly parallel φ^{ts} stable and nearly parallel φ^{np} stable*

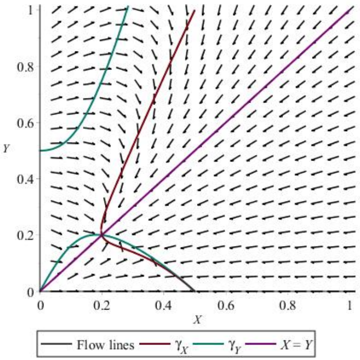
Moreover, any initial $\varphi^{\pm}$ flows to either φ^{ts} or φ^{np}

Dynamics for Laplacian coflow

$a, b, c > 0$ functions of $t \rightsquigarrow X = \dfrac{a^2}{c^2}$ and $Y = \dfrac{ab}{c^2}$

$\rightsquigarrow$ dynamics for φ_t^- and φ_t^+





- $(X, Y) = (0, 0) \leftrightarrow$ volume form vol_Z on Z^4
- $(X, Y) = (\frac{1}{2}, 0) \leftrightarrow$ nearly Kähler structure on twistor space

Laplacian flow

$$\varphi_t^{\pm} = \pm a_t^2 b_t \eta_1 \wedge \eta_2 \wedge \eta_3 - a_t c_t^2 \eta_1 \wedge \omega_1 - a_t c_t^2 \eta_2 \wedge \omega_2 \mp b_t c_t^2 \eta_3 \wedge \omega_3$$

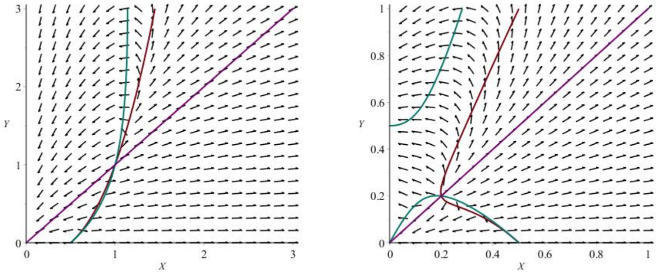
$$\frac{\partial \varphi_t}{\partial t} = \Delta_t \varphi_t = \mathrm{d}_t^* \mathrm{d} \varphi_t \qquad (\ddagger)$$

$\rightsquigarrow$ ansatz (including $\mathrm{d}_t^* \varphi_t = 0$) preserved and same critical points

Theorem

After rescaling, under Laplacian flow $(\ddagger)$

- *nearly parallel* φ^{ts} *and* φ^{np} *both* *unstable* *(sources)*



Conclusion

Results

3-Sasakian 7-manifold $\rightsquigarrow$

- two 3-parameter families of coclosed G_2-structures $\rightsquigarrow$ metrics
- two nearly parallel G_2-structures $\rightsquigarrow$ 3-Sasakian and squashed Einstein metrics
- Ricci flow: 3-Sasakian metric stable, squashed metric unstable
- Laplacian coflow: both nearly parallel G_2-structures stable, all members of family flow to them after rescaling
- Laplacian flow: both nearly parallel G_2-structures unstable, all non-trivial members of family flow away

Questions

- stability of nearly parallel G_2-structures?
- coclosed condition and Laplacian flow?
- short-time and long-time existence of Laplacian coflow?