Deforming G_2 Conifolds

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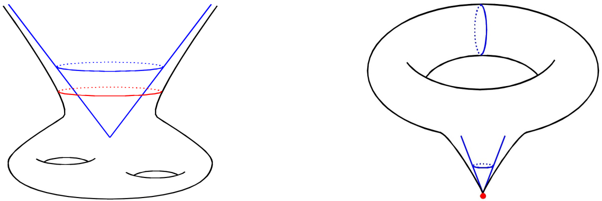
Motivation

Conifolds



- Known examples of G_2 conifolds
- Moduli space of compact G_2 manifolds
- New examples/local uniqueness of holonomy G_2 metrics
- Relevance to M-Theory

Outline

- **Definitions and examples**
- **Deformation theory results**
- **Applications**
- **Sketch proof and key ideas**
- **Open problems**

G_2 conifolds

Cone $C^7 = \mathbb{R}^+ \times \Sigma^6$, $g_C = \mathrm{d}r^2 + r^2 g_\Sigma$

- $\mathrm{Hol}(g_C) \subseteq G_2 \Leftrightarrow \Sigma$ nearly Kähler
- SU(3) structure (g, J, ω, Ω) nearly Kähler if

$$\mathrm{d}\omega = 3\,\mathrm{Re}\,\Omega \quad \text{and} \quad \mathrm{d}\,\mathrm{Im}\,\Omega = -2\omega\wedge\omega$$

- Examples of Σ: $\mathcal{S}^6$, $\mathbb{CP}^3$, $\mathrm{SU}(3)/T^2$, $\mathcal{S}^3\times\mathcal{S}^3$

Definition

M asymptotically conical (AC) if $\exists$ cone C, compact K, $R>0$, diffeomorphism $\Psi : (R,\infty)\times\Sigma \to M\setminus K$ and rate $\nu<0$ such that

$$|\nabla_C^j(\Psi^* g_M - g_C)| = O(r^{\nu-j}) \quad \text{for all } j\in\mathbb{N} \text{ as } r\to\infty$$

- M AC rate $\nu_0 < 0 \rightsquigarrow M$ AC any rate $\nu \in [\nu_0, 0)$

G_2 conifolds

Definition

$\overline{M}$ conically singular (CS) at $z \in \overline{M}$ if $M = \overline{M} \setminus \{z\}$ smooth and $\exists$ cone C, compact K, $\epsilon > 0$, diffeomorphism $\Psi : (0, \epsilon) \times \Sigma \to M \setminus K$ and rate $\nu > 0$ such that

$$|\nabla_C^j(\Psi^* g_M - g_C)| = O(r^{\nu - j}) \quad \text{for all } j \in \mathbb{N} \text{ as } r \to 0$$

- M CS rate $\nu_0 > 0 \rightsquigarrow M$ CS any rate $\nu \in (0, \nu_0]$

Examples

- (Bryant–Salamon 1989) AC holonomy G_2 manifolds
  - $\Lambda^2_-(\mathcal{S}^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ have rate -4, $\Sigma = \mathbb{CP}^3$ and $SU(3)/T^2$
  - $\mathbb{S}(\mathcal{S}^3)$ has rate -3, $\Sigma = \mathcal{S}^3 \times \mathcal{S}^3$
- (Joyce–Karigiannis) Potential method for constructing CS holonomy G_2 manifolds, $\Sigma = \mathbb{CP}^3$

AC deformations

Theorem (Joyce 1996)

M^7 compact G_2 manifold $\Rightarrow$ moduli space of torsion-free G_2 structures is locally a smooth manifold of dimension $b^3(M)$

(Nordström 2009) Asymptotically cylindrical case

Theorem (Karigiannis–L 2012)

M AC G_2 manifold with generic rate $\nu \in (-4, -5/2) \Rightarrow$ moduli space of torsion-free G_2 structures is locally a smooth manifold of dimension

- $b^3_{cs}(M)$ *if* $\nu \in (-4, -3)$
- $b^3_{cs}(M) + \dim \operatorname{Im}\left(H^3(M) \to H^3(\Sigma)\right) + \sum_{\lambda \in (-3,\nu)} m_\Sigma(\lambda)$ *if* $\nu \in (-3, -5/2)$

CS deformations

Theorem (Karigiannis–L 2012)

M CS G_2 manifold with rate ν near $0 \Rightarrow$

- $\exists$ *finite-dimensional vector spaces of forms $\mathcal{I}$ and $\mathcal{O}$*
- $\exists$ *open neighbourhood $\mathcal{U}$ of 0 in $\mathcal{I}$ and smooth map $\pi : \mathcal{U} \to \mathcal{O}$ with $\pi(0) = 0$*

such that the moduli space of torsion-free G_2 structures is locally homeomorphic to $\operatorname{Ker}\pi$ and has expected dimension at least

- $b^3(M) - \dim \operatorname{Im}\left(H^3(M) \to H^3(\Sigma)\right) - \sum_{\lambda \in (-3,0]} m_\Sigma(\lambda)$

- $\mathcal{I}$ is the infinitesimal deformation space
- $\mathcal{O}$ is the obstruction space
- $\mathcal{O} = \{0\} \rightsquigarrow$ smooth moduli space

Applications

1. $M = \Lambda^2_-(\mathcal{S}^4)$ or $\Lambda^2_-(\mathbb{CP}^2)$, AC with rate -4
   - $b^3_{\mathrm{cs}}(M) = b^4(M) = 1,\ b^3(\Sigma) = 0$
   - (Moroianu–Semmelmann 2010) $m_\Sigma(\lambda) = 0$ for $\lambda \in (-3, 0)$
   - $\dim \mathcal{M}_\nu = b^3_{\mathrm{cs}}(M) = 1$ for $\nu \in (-4, 0) \rightsquigarrow$ local uniqueness
2. $M = \mathbb{S}(\mathcal{S}^3)$, AC with rate -3
   - $b^3_{\mathrm{cs}}(M) = 0,\ b^3(M) = 1,\ b^3(\Sigma) = 2$
   - $0 \to H^3(M) \to H^3(\Sigma) \to H^4_{\mathrm{cs}}(M) \to 0$ exact $\rightsquigarrow$ $\dim \mathrm{Im}\left(H^3(M) \to H^3(\Sigma)\right) = 1$
   - $m_\Sigma(\lambda) = 0$ for $\lambda \in (-3, 0)$
   - $\dim \mathcal{M}_\nu = 1$ for $\nu \in (-3, 0) \rightsquigarrow$ local uniqueness

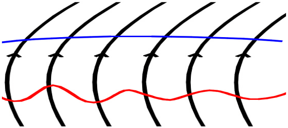
Applications

3. M CS with $\Sigma = \mathbb{CP}^3$ or $\mathcal{S}^3 \times \mathcal{S}^3$
   - $m_\Sigma(0) = 0 \rightsquigarrow \mathcal{O} = \{0\}$
   - $\mathcal{M}_\nu$ smooth, $\dim \mathcal{M}_\nu = b^3(M)$ or $b^3_{\mathrm{cs}}(M)$
4. M CS with $\Sigma = \mathrm{SU}(3)/T^2$
   - $m_\Sigma(0) = 8 \rightsquigarrow \dim \mathcal{O} \leq 8$
   - Smoothness for $\mathcal{M}_\nu \leftrightarrow$ deformations of $\mathrm{SU}(3)/T^2$
5. M CS with cone C and N AC with rate $\nu \leq -3$ to C
   - (Karigiannis 2009) Can desingularize M via gluing with N if topological condition and gauge-fixing condition satisfied
   - Slice theorem $\Rightarrow$ gauge-fixing always holds

Strategy

(M^7, φ) G_2 conifold rate ν

- $\mathcal{T}_\nu = \{\tau \in C^\infty(\Lambda^3_+)$ with rate $\nu : \mathrm{d}\tau = \mathrm{d}^*_\tau \tau = 0\}$
- $\mathcal{D}_\nu = \{$diffeomorphisms with rate ν isotopic to id$\}$
- $\mathcal{M}_\nu = \mathcal{T}_\nu / \mathcal{D}_\nu$



(a) Gauge $\rightsquigarrow$ slice $\mathcal{S}_\nu \ni \varphi$, $\mathcal{S}_\nu \to \mathcal{M}_\nu$ homeomorphism

(b) τ closed, Hodge theory $\rightsquigarrow \exists!$ co-exact β, harmonic γ such that $\tau - \varphi = \mathrm{d}\beta + \gamma$

(c) $\tau \in \mathcal{S}_\nu \Leftrightarrow \Delta_\varphi \beta = \mathrm{d}^*_\varphi F(\mathrm{d}\beta + \gamma)$

(d) Implicit Function Theorem, elliptic regularity $\rightsquigarrow \mathcal{M}_\nu$ locally parametrised by harmonic 3-forms rate ν

Gauge-fixing

(a) $T_\varphi(\varphi\mathcal{D}_\nu) = \{\mathcal{L}_v\varphi\} \rightsquigarrow$ seek τ transverse to all $\mathcal{L}_v\varphi = \mathrm{d}(v\lrcorner\varphi)$

- $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$ with $\Lambda^2_7 = \{v\lrcorner\varphi\}$
- $\langle\tau, \mathrm{d}(v\lrcorner\varphi)\rangle_\varphi = \langle \mathrm{d}^*_\varphi\tau, v\lrcorner\varphi\rangle_\varphi$
- $\rightsquigarrow \pi_7(d^*_\varphi\tau) = 0 \rightsquigarrow$ slice

Analytic framework: weighted Sobolev space $L^2_{k,\nu}$

- $\xi \in L^2_{0,\nu} \Leftrightarrow r^{-\nu-\frac{7}{2}}\xi \in L^2$
- AC $\nu > -\frac{7}{2} \Rightarrow$ not in L^2

$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ with $\Lambda^3_1 = \{f\varphi\}$ and $\Lambda^3_7 = \{v\lrcorner *_\varphi \varphi\}$

- Dirac operator $\not{D}$ acting on $\Lambda^3_1 \oplus \Lambda^3_7$ by
  $f\varphi + v\lrcorner *_\varphi \varphi \mapsto \pi_{1+7}\mathrm{d}(v\lrcorner\varphi) + *_\varphi\mathrm{d}(f\varphi)$
- Surjectivity of $\not{D}$ for AC $\rightsquigarrow$ slice

Key points

(b) Hodge theory not valid in general on $L^2_{k,\nu}$

- AC: decomposition works by choice of rates
- CS: $\tau - \varphi = \mathrm{d}\beta + \gamma + \eta$

(c) Gauge-fixing $\rightsquigarrow \Delta_\varphi \beta = \mathrm{d}^*_\varphi F(\tau - \varphi)$

(d) Maybe $\operatorname{Im} \mathrm{d}^*_\varphi \not\subseteq \operatorname{Im} \Delta_\varphi \rightsquigarrow$ obstructions to applying IFT

- AC: no obstruction
- CS: obstructions $\leftrightarrow \mathcal{O}$

Dimension $\mathcal{K}_\nu = \{\gamma \in L^2_{k,\nu}(\Lambda^3) : \mathrm{d}\gamma = \mathrm{d}^*_\varphi \gamma = 0\}$

- $\dim \mathcal{K}_{-7/2}$ equals $b^3_{\mathrm{cs}}(M)$ if M is AC and $b^3(M)$ if M is CS
- index of $\mathrm{d} + \mathrm{d}^*_\varphi$ on $L^2_{k,\nu}$ "jumps" as ν crosses critical values
- jumps determined by spectrum of elliptic operator on Σ

Open problems

- New examples of nearly Kähler 6-manifolds
- Deformations of nearly Kähler 6-manifolds
- Examples of CS holonomy G_2 manifolds
- Stability for holonomy G_2 cones
- Ricci-flat deformations of G_2 conifolds