Recent progress in Lagrangian mean curvature flow of surfaces

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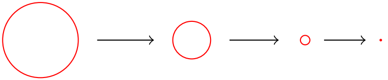
Mean curvature flow

$L^n \hookrightarrow (M^m, g) \rightsquigarrow$ volume functional $\mathrm{Vol}(L)$

- Critical points: minimal $\Leftrightarrow$ mean curvature $H = 0$
- Gradient flow: mean curvature flow (MCF) $\Leftrightarrow \dfrac{\partial L}{\partial t} = H$

Example: $n = 1 \rightsquigarrow$ curves γ

- minimal $\Leftrightarrow$ curvature $\kappa = 0 \Leftrightarrow$ geodesic
- MCF $\leftrightarrow \dfrac{\partial \gamma}{\partial t} = \kappa = \dfrac{\partial^2 \gamma}{\partial s^2}$ (s arclength)
- $\rightsquigarrow$ nonlinear parabolic PDE



in finite time

Lagrangian mean curvature flow

MCF $L^n \hookrightarrow M^m \leftrightarrow$ nonlinear parabolic PDE system

- $m = n + 1$ (hypersurfaces) $\rightsquigarrow$ scalar PDE $\rightsquigarrow$ ✓
- $m > n + 1 \rightsquigarrow$?!

Lagrangian $\rightsquigarrow L^n \hookrightarrow (M^{2n}, \omega)$ symplectic, $\omega|_L \equiv 0$

(Smoczyk 1998) In Kähler–Einstein (M, ω) Lagrangian condition preserved by MCF $\rightsquigarrow$ Lagrangian mean curvature flow (LMCF)

Calabi–Yau $M \Rightarrow$ critical points of LMCF are minima

Example: $F : \mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$

- $\mathrm{Graph}(F) \subseteq \mathbb{R}^{2n}$ Lagrangian $\Leftrightarrow F = \mathrm{grad}\, f$, $f : \mathbb{R}^n \to \mathbb{R}$
- LMCF $\leftrightarrow \dfrac{\partial f}{\partial t} = \displaystyle\sum_{j=1}^{n} \tan^{-1} \lambda_j$ (λ_j eigenvalues of Hess f)
- $\rightsquigarrow$ fully nonlinear parabolic scalar PDE

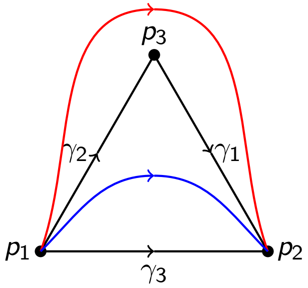
Examples: gravitational instantons

$\mathcal{S}^1$-invariant M^4 gravitational instanton $\Rightarrow$ hyperkähler $\Rightarrow$ Calabi–Yau

Example: Eguchi–Hanson $T^*\mathcal{S}^2 \rightsquigarrow \mathcal{S}^2$ minimal Lagrangian

- infinitely many topological types containing $\mathcal{S}^1$-invariant minimal Lagrangian surfaces L^2

Fibration: $\mathcal{S}^1 \hookrightarrow M \setminus \{p_1, \ldots, p_{k+1}\} \to \mathbb{R}^3 \setminus \{p_1, \ldots, p_{k+1}\}$ with $\{p_1, \ldots, p_{k+1}\} \subseteq \mathbb{R}^2 \rightsquigarrow$ Lagrangian $L^2 \subseteq M \leftrightarrow$ curve $\gamma \subseteq \mathbb{R}^2$

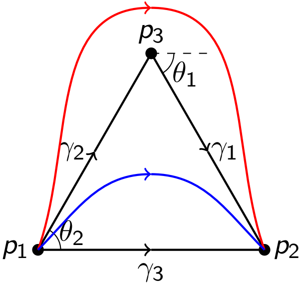


Conjectures

String Theory $\rightsquigarrow$ Mirror Symmetry for Calabi–Yau M $\rightsquigarrow$

Conjecture (Thomas–Yau 2002)

LMCF starting at stable L in M exists for all time and converges



Conjecture (Joyce 2015)

Long-time existence and convergence of LMCF with surgeries $\leftrightarrow$ Bridgeland stability condition on Fukaya category

Singularities

L^n oriented Lagrangian in Calabi–Yau M

$\mathrm{U}(n)/\mathrm{SO}(n)$ oriented Lagrangian Grassmannian

$\rightsquigarrow$ determinant defines $e^{i\theta} : L \to \mathcal{S}^1$ Lagrangian angle

- LMCF converges $\Rightarrow$ L zero Maslov $[\mathrm{d}\theta] = 0$
- L stable $\Rightarrow$ almost calibrated $|\sup\theta - \inf\theta| < \pi$ $\Rightarrow$ $[\mathrm{d}\theta] = 0$

Challenge: L zero Maslov $\Rightarrow$ LMCF singularities are Type II

- tangent flow does not determine singularity model (in general)
- singularity model = ancient solution $(L_t)_{t<0}$ (Type II blow-up)

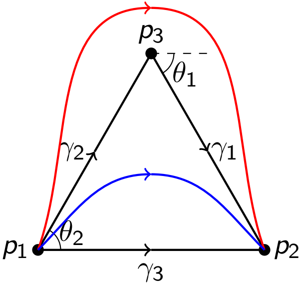
Theorem (Lambert–L.–Schulze, L.–Schulze–Székelyhidi)

Classification for Type II blow-ups of almost calibrated LMCF of surfaces $\rightsquigarrow$ minimal Lagrangians and translators

Thomas–Yau conjecture

Conjecture (Thomas–Yau 2002)

LMCF starting at stable L in M exists for all time and converges



Theorem (L.–Oliveira)

Thomas–Yau conjecture is true for $\mathcal{S}^1$-invariant Lagrangians L^2 in $\mathcal{S}^1$-invariant gravitational instantons M^4