Lagrangian mean curvature flow and the Gibbons–Hawking ansatz

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(Joint work with Goncalo Oliveira)

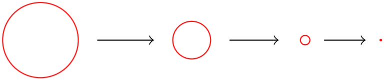
Mean curvature flow

$L^n \hookrightarrow (M^m, g) \rightsquigarrow$ volume functional $\mathrm{Vol}(L)$

- Critical points: minimal $\Leftrightarrow$ mean curvature $H = 0$
- Gradient flow: mean curvature flow (MCF) $\Leftrightarrow \dfrac{\partial L}{\partial t} = H$

Example: $n = 1 \rightsquigarrow$ curves γ

- minimal $\Leftrightarrow$ curvature $\kappa = 0 \Leftrightarrow$ geodesic
- MCF $\leftrightarrow \dfrac{\partial \gamma}{\partial t} = \kappa = \dfrac{\partial^2 \gamma}{\partial s^2}$ (s arclength)
- $\rightsquigarrow$ nonlinear parabolic PDE



in finite time

Lagrangian mean curvature flow

MCF $L^n \hookrightarrow M^m \leftrightarrow$ nonlinear parabolic PDE system

- $m = n+1$ (hypersurfaces) $\rightsquigarrow$ scalar PDE $\rightsquigarrow$ ✓
- $m > n+1 \rightsquigarrow$?!

Lagrangian $\rightsquigarrow L^n \hookrightarrow (M^{2n}, \omega)$ symplectic, $\omega|_L \equiv 0$

(Smoczyk 1998) In Kähler–Einstein (M, ω) Lagrangian condition preserved by MCF $\rightsquigarrow$ Lagrangian mean curvature flow (LMCF)

Example: $F : \mathbb{R}^n \to \mathbb{R}^n \rightsquigarrow$

- $\mathrm{Graph}(F) \subseteq \mathbb{R}^{2n}$ Lagrangian $\Leftrightarrow F = \mathrm{grad}\, f$, $f : \mathbb{R}^n \to \mathbb{R}$
- LMCF $\leftrightarrow \dfrac{\partial f}{\partial t} = \sum_{j=1}^{n} \tan^{-1} \lambda_j \qquad (\lambda_j$ eigenvalues of $\mathrm{Hess}\, f)$
- $\rightsquigarrow$ fully nonlinear parabolic scalar PDE

Main result

Calabi–Yau $M \Rightarrow$ critical points of LMCF are minima

Question

When does LMCF converge?

String Theory $\rightsquigarrow$ Mirror Symmetry $\rightsquigarrow$

Conjecture (Thomas–Yau 2002)

LMCF starting at stable L in M exists for all time and converges

Main result: Proof of (version of) Thomas–Yau conjecture for large class of M^4

Gibbons–Hawking ansatz

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $e^{i\psi} \in \mathcal{S}^1$

- $V : U \subseteq \mathbb{R}^3 \to \mathbb{R}^+$ harmonic function
- $X = (X_1, X_2, X_3) : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ vector field

$$\operatorname{curl} X = \operatorname{grad} V$$

- $\xi = X_1 \mathrm{d}x_1 + X_2 \mathrm{d}x_2 + X_3 \mathrm{d}x_3$ 1-form $\Rightarrow *\mathrm{d}\xi = \mathrm{d}V$

Metric on M^4

$$g = V^{-1}(\mathrm{d}\psi + \xi)^2 + V(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$$

- $\mathrm{Ric}(g) = 0 \leftrightarrow$ "$\Delta_g g = 0$" $\Leftarrow$ hyperkähler $\mathrm{Hol}(g) \subseteq \mathrm{SU}(2)$
- (M^4, g) $\mathcal{S}^1$-invariant
- many ($\mathcal{S}^1$-invariant) Lagrangians $L^2 \hookrightarrow M^4$

Examples: gravitational instantons

$V : U \subseteq \mathbb{R}^3 \to \mathbb{R}^+$ harmonic,
$g = V^{-1}(\mathrm{d}\psi + \xi)^2 + V(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$

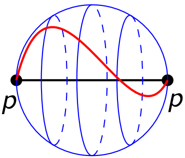
- Natural constraint: V bounded at $\infty \rightsquigarrow V$ has singularities p_i
- (M^4, g) smooth $\rightsquigarrow V \sim \frac{1}{2|x-p_i|}$ near p_i
- At p_i circle shrinks to a point in M^4

Examples

- $V = m > 0 \Rightarrow M = \mathcal{S}^1 \times \mathbb{R}^3$ cylinder
- $V = \frac{1}{2|x|} \Rightarrow M = \mathbb{R}^4$ Euclidean
- $V = m + \frac{1}{2|x|} \Rightarrow M = \mathbb{R}^4$ Taub–NUT
- $V = \frac{1}{2|x-p|} + \frac{1}{2|x+p|} \Rightarrow M = T^*\mathcal{S}^2$ Eguchi–Hanson
- $p_1, \ldots, p_{k+1} \in \mathbb{R}^3 \rightsquigarrow V = m + \sum_{i=1}^{k+1} \frac{1}{2|x-p_i|}$
  multi-Eguchi–Hanson $(m = 0)$ and multi-Taub–NUT $(m > 0)$

Circle-invariant Lagrangians

Eguchi–Hanson $T^*\mathcal{S}^2$: $V = \frac{1}{2|x-p|} + \frac{1}{2|x+p|}$



$\rightsquigarrow$ minimal Lagrangian $\mathcal{S}^2$

Gibbons–Hawking M^4: curves $\gamma \subseteq \mathbb{R}^3 \leftrightarrow \mathcal{S}^1$-invariant $L^2_\gamma \subseteq M^4$

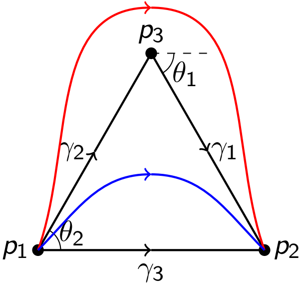
- embedded closed curve in $\mathbb{R}^3 \setminus \{p_i\} \leftrightarrow$ embedded T^2
- embedded arc endpoints p_1, $p_2 \leftrightarrow$ embedded $\mathcal{S}^2$

Lemma

- L_γ *Lagrangian* $\Leftrightarrow \gamma$ *planar*
- L_γ *minimal* $\Leftrightarrow \gamma$ *straight line(s)*

Stability

$\gamma \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \rightsquigarrow \theta$ angle between γ and horizontal



Definition

L_γ compact Lagangian $\rightsquigarrow$ L_γ stable $\Leftrightarrow$

- *almost calibrated* $\Leftrightarrow |\max\theta - \min\theta| < \pi$
- *whenever* $\gamma \sim \gamma_1 \# \gamma_2$, $[\theta_1, \theta_2] \not\subseteq (\min\theta, \max\theta)$

Thomas–Yau conjecture

(M^4, g) multi-EH/TN: $V = m + \sum_{i=1}^{k+1} \frac{1}{2|x-p_i|}$

Theorem (L.–Oliveira)

LMCF starting at stable $\mathcal{S}^1$-invariant Lagrangian in (M^4, g) exists for all time and converges to a minimal Lagrangian

(Neves 2013): Any compact Lagrangian can be perturbed to non-almost calibrated L so that LMCF starting at L develops finite-time singularity (even preserving invariance)

Corollary

LMCF starting at any almost calibrated compact $\mathcal{S}^1$-invariant Lagrangian in Eguchi–Hanson $T^*\mathcal{S}^2$ exists for all time and converges to $\mathcal{S}^2$

Flow of curves

$V = m + \sum_{i=1}^{k+1} \frac{1}{2|x-p_i|}$

L_γ stable $\Rightarrow$ γ embedded arc in $\mathbb{R}^2$ joining p_1, p_2

Lemma

$$LMCF \ \frac{\partial L_\gamma}{\partial t} = H \Leftrightarrow \frac{\partial \gamma}{\partial t} = V^{-1}\kappa = V^{-1}\frac{\partial^2 \gamma}{\partial s^2}$$

$\rightsquigarrow$ flow degenerates at p_1, p_2

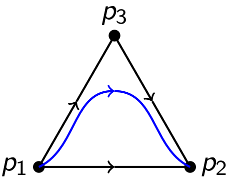
Lemma

$|\max\theta - \min\theta|$ *non-increasing along flow* $\Rightarrow$ *almost calibrated preserved*

Proof: $\frac{\partial\theta}{\partial t} = \Delta\theta \Rightarrow$ apply maximum principle

Singularities

$\frac{\partial \gamma}{\partial t} = V^{-1}\kappa$, γ arc joining p_1, p_2



Suppose, for contradiction, $\exists$ finite-time singularity (p, T)

Lemma

- $p \neq p_i$ *for* $i > 2$
- $\exists$ $(x_k, t_k) \to (p, T)$ *with* $V^{-1}|\kappa|^2 \to \infty$ *as* $k \to \infty$

- Proof: stability + variation of θ non-increasing
- Proof: second fundamental form blows up at (p, T) + almost calibrated $\Rightarrow$ no "winding" around p_1, p_2

Blow-up analysis

$\exists\ (x_k, t_k) \to (p, T)$ singularity with $V^{-1}|\kappa|^2 \to \infty$ as $k \to \infty$

$V \sim \frac{1}{2|x-p_i|}$ near endpoints p_1, p_2 of curve γ, $\frac{\partial\gamma}{\partial t} = V^{-1}\kappa$

Case 1: $p \neq p_1, p_2 \rightsquigarrow$ blow-up analysis for $\frac{\partial\gamma}{\partial t} = \kappa$

- variation of $\theta < \pi \Rightarrow$ no singularity ↯

Case 2: $p = p_1$ and $V^{-1}|\kappa| \to \infty$ ("$x_k \to p$ slowly")

- "scale breaking" $\rightsquigarrow$ flat blow-up limit for LMCF but nonlinear limit curve ↯

Case 3: $p = p_1$ and $V^{-1}|\kappa|$ bounded ("$x_k \to p$ quickly")

- blow-up limit = ancient solution for LMCF with planar asymptotics $\cong \mathbb{R}^2$
- (Lambert–L.–Schulze): classification $\Rightarrow$ must be flat plane ↯

Convergence: $\frac{\partial\theta}{\partial t} = \Delta\theta \rightsquigarrow$ Thomas–Yau conjecture