Coassociative conifolds 1

Smoothings of cones

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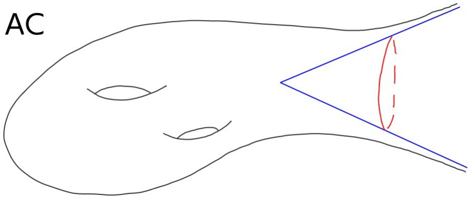
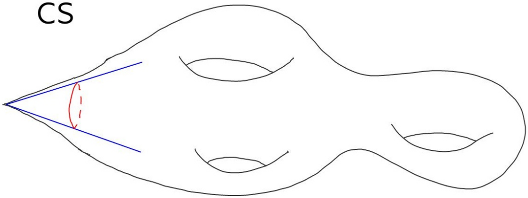
12 August 2013

Introduction

Coassociative 4-folds: $N^4 \subseteq M^7$ volume-minimizing

Conifolds: modelled on cones





Background

Theorem (Morrey 1954)

C^1 minimal $\Rightarrow$ real analytic

Theorem (Lawson–Osserman 1977)

$\exists$ *Lipschitz non-smooth minimal* $C^4 \subseteq \mathbb{R}^7$

$$C = \left\{ r\left(\frac{2}{3}\bar{q}, \frac{\sqrt{5}}{3} q i \bar{q}\right) \in \mathbb{H} \oplus \operatorname{Im}\mathbb{H} \,:\, q \in \mathcal{S}^3 \subseteq \mathbb{H},\, r > 0 \right\}$$

- 3-form $\varphi_0(u, v, w) = g_0(u \times v, w)$
- 4-form $*\varphi_0(e_1, e_2, e_3, e_4) \leq 1 \;\forall$ unit e_1, e_2, e_3, e_4
- $\mathrm{d} * \varphi_0 = 0 \rightsquigarrow *\varphi_0$ calibration
- $*\varphi_0|_C = \mathrm{vol}_C \Leftrightarrow C$ coassociative $\Leftrightarrow \varphi_0|_C = 0$

AC coassociative 4-folds

$q \in \mathcal{S}^3 \subseteq \mathbb{H}$, $(r,s) \in \mathbb{R}^+ \times \mathbb{R}^+ \rightsquigarrow (r\bar{q}, sqi\bar{q}) \in \mathbb{H} \oplus \operatorname{Im}\mathbb{H}$

- $s = \frac{\sqrt{5}}{2} r \leftrightarrow$ cone C
- $s(4s^2 - 5r^2)^2 = \tau$, $4s^2 - 5r^2 > 0 \leftrightarrow N^+ \cong \mathcal{O}_{\mathbb{CP}^1}(-1)$
- $s(4s^2 - 5r^2)^2 = \tau$, $4s^2 - 5r^2 < 0 \leftrightarrow N^- \cong \mathbb{R} \times \mathcal{S}^3$

$N^+ \sim C$ and $N^- \sim C \cup \mathbb{H} \rightsquigarrow N^\pm$ asymptotically conical (AC)

Definition

N AC rate $\lambda < 0$ if $\exists$ cone $C \cong (0,\infty) \times L$, compact K, $R > 0$, diffeomorphism $\Phi : (R,\infty) \times L \to N \setminus K$ such that

$$|\nabla^j(\Phi(r,x) - rx)| = O(r^{\lambda - j}) \quad \text{for all } j \in \mathbb{N} \text{ as } r \to \infty$$

- N AC rate $\lambda_0 \rightsquigarrow N$ AC any rate $\lambda \in [\lambda_0, 0)$

Deformations

Question: How many AC smoothings does C have?

Theorem (L. 2009)

N coassociative AC generic rate $\lambda \in (-2, 0) \Rightarrow$ moduli space $\mathcal{M}(N, \lambda)$ smooth and

$$\dim \mathcal{M}(N, \lambda) = b^2_+(N) + \dim \operatorname{Im}(H^2(N) \to H^2(L)) + \sum_{\mu \in (-2, \lambda)} m_L(\mu)$$

- $b^2_+(N) = \dim\{\alpha \in L^2(\Lambda^2_+ T^* N) \,:\, \mathrm{d}\alpha = 0\}$
- $m_L(\mu) = \dim\{\gamma \in C^\infty(T^* L) \,:\, *\mathrm{d}\gamma = (\mu + 2)\gamma,\ \mathrm{d} * \gamma = 0\}$

$[\alpha], [\beta] \in \operatorname{Im}(H^2_{\text{cpt}}(N) \to H^2(N)) \rightsquigarrow [\alpha] \cup [\beta] = \int_N \alpha \wedge \beta$

- $b^2_+(N) = \dim\{\text{maximal positive subspace w.r.t. } \cup\}$

Application 1

Corollary (L. 2012)

$N^{\pm}$ locally unique

- $N^{+} \cong \mathcal{O}_{\mathbb{CP}^1}(-1)$, $N^{-} \cong \mathbb{R} \times \mathcal{S}^3 \Rightarrow b^2_{+}(N^{\pm}) = 0$
- $L^{+} = \tilde{\mathcal{S}}^3$, $L^{-} = \tilde{\mathcal{S}}^3 \sqcup \mathcal{S}^3 \Rightarrow b^2(L^{\pm}) = 0$
- $m_{\tilde{\mathcal{S}}^3}(\mu) = \begin{cases} 1 & \mu = -\frac{3}{2} \\ 0 & \mu \neq -\frac{3}{2} \end{cases}$ and $m_{\mathcal{S}^3}(\mu) = 0 \ \forall \mu \in (-2, 0)$
- $\Rightarrow \mathcal{M}(N^{\pm}, \lambda) = 1 \ \forall \lambda \in (-\frac{3}{2}, 0)$

Application 2

Corollary (L. 2012)

N AC to $\mathbb{R}^4 \Rightarrow N = \mathbb{R}^4$

- $L = \mathcal{S}^3 \Rightarrow b^2(L) = 0$ and $m_L(\mu) = 0\ \forall \mu \in (-3, 0)$
- $\Rightarrow \mathcal{M}(N, \lambda) = b^2_+(N)\ \forall \lambda \in (-3, 0)$
- $\Rightarrow N$ AC rate $\lambda \leq -3$

Lemma (L. 2009)

N AC rate $\lambda < -2 \Rightarrow N$ is $\mathrm{Aut}(C)$-invariant

- $\mathrm{Aut}(\mathbb{R}^4) = \mathrm{SO}(4) \Rightarrow N$ SO(4)-invariant $\Rightarrow N = \mathbb{R}^4$

Strategy

N coassociative $\Rightarrow$ v normal vector field $\leftrightarrow$ $v \lrcorner \varphi_0$ self-dual 2-form

- $\varphi_0|_{\mathrm{Graph}(v)} = \mathrm{d}(v \lrcorner \varphi_0) + Q(v \lrcorner \varphi_0)$
- $\rightsquigarrow$ Infinitesimal deformations $\leftrightarrow$ $\mathrm{d}\alpha = 0$, α self-dual 2-form
- Graph(v) AC $\leftrightarrow$

$$\alpha \in C^\infty_\lambda = \{\alpha \in C^\infty : \sup_N |r^{j-\lambda}\nabla^j\alpha| < \infty \, \forall j \in \mathbb{N}\}$$

Strategy

- Solve $F(\alpha) = \mathrm{d}\alpha + Q(\alpha) = 0$ for $\alpha \in C^\infty_\lambda$
- $\mathrm{d}F|_0(\alpha) = \mathrm{d}\alpha$
- Show $F^{-1}(0)$ smooth by Implicit Function Theorem

Need Banach X, Y and open $U \subseteq X$ such that $F : U \to Y$ and

- $\mathrm{d}F|_0 : X \to Y$ surjective
- $\operatorname{Ker} \mathrm{d}F|_0$ finite-dimensional

Difficulties

C^∞_λ not Banach $\Rightarrow$ use

$$L^2_{k,\lambda} = \{\alpha \in L^2_{k,\mathrm{loc}} : \sum_{j=0}^{k} \int_N |r^{j-\lambda}\nabla^j \alpha|^2 r^{-4} \mathrm{d}\,\mathrm{vol} < \infty\}$$

- $F : U \subseteq L^2_{k,\lambda}(\Lambda^2_+) \to L^2_{k-1,\lambda-1}(\Lambda^3)$
- $\mathrm{d}F|_0 = \mathrm{d}$ not surjective and not elliptic
- $F(U) \subseteq \mathrm{d}(L^2_{k,\lambda}(\Lambda^2))$ not necessarily Banach
- $(\alpha, \beta) \in \Lambda^2_+ \oplus \Lambda^4 \mapsto \mathrm{d}\alpha + \mathrm{d}^*\beta \in \Lambda^3$ elliptic

Idea: use $G(\alpha, \beta) = F(\alpha) + \mathrm{d}^*\beta$

Completing the proof

Lemma

$G(\alpha, \beta) = 0 \Leftrightarrow F(\alpha) = 0 = \beta$

$G(\alpha, \beta) = 0 \Rightarrow F(\alpha) + \mathrm{d}^*\beta = 0 \Rightarrow \mathrm{d}\mathrm{d}^*\beta = 0 \Rightarrow \beta = 0$ by maximum principle as $\lambda < 0$

Lemma

$\lambda > -2 \Rightarrow \mathrm{d}(L^2_{k,\lambda}(\Lambda^2_+)) = \mathrm{d}(L^2_{k,\lambda}(\Lambda^2))$

- $X = L^2_{k,\lambda}(\Lambda^2_+ \oplus \Lambda^4)$ and $Y = \mathrm{d}(L^2_{k,\lambda}(\Lambda^2_+)) \oplus \mathrm{d}^*(L^2_{k,\lambda}(\Lambda^4))$
- $G : U \subseteq X \to Y$ and $\mathrm{d}G|_0 : X \to Y$ surjective elliptic
- Implicit Function Theorem $\Rightarrow G^{-1}(0) \cong \operatorname{Ker} \mathrm{d}G|_0$ smooth
- Elliptic regularity $\Rightarrow \mathcal{M}(N, \lambda) \cong G^{-1}(0)$
- Dimension count $\leftrightarrow$ index theorem for elliptic operators