Deformed G_2-instantons

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(Joint work with Goncalo Oliveira)

Overview

Deformed G_2-instantons

- special connections in 7 dimensions
- "mirror" to calibrated cycles $\rightsquigarrow$ enumerative invariants?
- critical points of Chern–Simons-type functional $\mathcal{F}$

Results

- first non-trivial examples
- examples detect different G_2-structures (including nearly parallel and isometric)
- deformation theory: obstructions and topology of moduli space
- relation to $\mathcal{F}$

Key example: the 7-sphere

Hopf fibration: $\mathcal{S}^3 \to \mathcal{S}^7 \to \mathcal{S}^4$

- round metric $g^{ts} = g_{\mathcal{S}^3} + g_{\mathcal{S}^4}$
- "canonical variation" $g_t = t^2 g_{\mathcal{S}^3} + g_{\mathcal{S}^4}$ for $t > 0$
- g_t Einstein $\Leftrightarrow g^{ts} = g_1$ or $g^{np} = g_{1/\sqrt{5}}$

Octonions: $\mathcal{S}^7 \subseteq \mathbb{O}$

- $\rightsquigarrow$ cross product $\times$
- $\rightsquigarrow$ 3-form

$$\varphi^{ts}(u, v, w) = g^{ts}(u \times v, w) \quad \text{G}_2\text{-structure}$$

- **Fact:** φ^{ts} determines g^{ts}
- $\mathrm{d}\varphi^{ts} = \lambda * \varphi^{ts}$ for $\lambda > 0$ constant $\leftrightarrow$ nearly parallel
- **Note:** $\mathrm{d} * \varphi^{ts} = 0$

Deformed G_2-instantons

X^7, φ G_2-structure, $\mathrm{d} * \varphi = 0$, connection A on bundle over X

Definition (J.-H. Lee–N.C. Leung)

A deformed G_2-instanton (dG_2) $\Leftrightarrow$ curvature F_A satisfies

$$F_A \wedge *\varphi + \frac{1}{6} F_A^3 = 0$$

$\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$,
$\varphi(u, v, w) = g_{\mathbb{R}^7}(u \times v, w) \rightsquigarrow$ **"mirror branes"**

- $A = i(a_j(x)\mathrm{d}x_j + u_k(x)\mathrm{d}y_k)$ dG_2-instanton $\Leftrightarrow$
  $L = \mathrm{Graph}(u)$ associative $\varphi|_L = \mathrm{vol}_L$ and $i(a_j \mathrm{d}x_j)$ flat
- $A = i(v_j(y)\mathrm{d}x_j + b_k(y)\mathrm{d}y_k)$ dG_2-instanton $\Leftrightarrow$
  $M = \mathrm{Graph}(v)$ coassociative $*\varphi|_M = \mathrm{vol}_M$ and
  $B = i(b_k \mathrm{d}y_k)$ anti-self-dual (ASD) $F_B = - * F_B$

$\rightsquigarrow$ primary interest in U(1)-connections

Lower dimensions

4 dimensions: $\pi : X^7 \to Z^4$, Z ASD Einstein, connection B on Z

Lemma

$\pi^* B$ dG_2*-instanton* $\Leftrightarrow$ B *ASD*

- ASD-instantons on $S^4 \rightsquigarrow$ dG_2-instantons on S^7

6 dimensions: $\pi : X^7 \to Y^6$, Y Calabi–Yau 3-fold Hol $\subseteq \mathrm{SU}(3)$
ω Kähler form on Y, connection B on Y

Lemma

$\pi^* B$ dG_2*-instanton* $\Leftrightarrow$ B *deformed Hermitian–Yang–Mills*

$$F_B^{(0,2)} = 0 \quad \textit{and} \quad \mathrm{Im}\left(\left(\omega + F_B\right)^3\right) = 0.$$

- Conjecture: existence of dHYM $\leftrightarrow$ stability condition

G_2-structures on the 7-sphere

Recall: $\mathcal{S}^3 \to \mathcal{S}^7 \to \mathcal{S}^4$, 3-form φ^{ts} inducing round metric g^{ts}

- $\mathcal{S}^3 = \mathrm{SU}(2) \rightsquigarrow$ left-invariant coframe η_1, η_2, η_3
- $\omega_1, \omega_2, \omega_3$ orthogonal self-dual 2-forms on $\mathcal{S}^4$ with length 2
- $\rightsquigarrow$ two 1-parameter families of 3-forms for $t > 0$:

$$\varphi_t^{\pm} = \pm t^3 \eta_1 \wedge \eta_2 \wedge \eta_3 - t\eta_1 \wedge \omega_1 - t\eta_2 \wedge \omega_2 \mp t\eta_3 \wedge \omega_3$$

- $\varphi_t^{\pm}$ induces $g_t = t^2 g_{\mathcal{S}^3} + g_{\mathcal{S}^4} \Rightarrow \varphi_t^+$ and φ_t^- isometric

Lemma

- $\mathrm{d} * \varphi_t^{\pm} = 0$
- $\varphi_t^{\pm}$ *nearly parallel* $\Leftrightarrow \varphi^{ts} = \varphi_1^-$ *or* $\varphi^{np} = \varphi^+_{1/\sqrt{5}}$

Note: φ^{np} induces "squashed" Einstein metric g^{np}

3-Sasakian 7-manifolds

Definition

(X^7, g^{ts}) *3-Sasakian* $\Leftrightarrow$ *cone* $(\mathbb{R}^+ \times X^7, g = \mathrm{d}r^2 + r^2 g^{ts})$ *hyperkähler* $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(2)$ $\big(\rightsquigarrow$ *generalizes* $(\mathcal{S}^7, g^{ts}) \big)$

Fact: $\exists$ infinitely many 3-Sasakian 7-manifolds

- $V^3 \to X^7 \to Z^4$, $V = \mathrm{SU}(2)/\Gamma$, Z ASD Einstein
- $\rightsquigarrow g_t = t^2 g_V + g_Z$ and

$$\varphi_t^{\pm} = \pm t^3 \eta_1 \wedge \eta_2 \wedge \eta_3 - t\eta_1 \wedge \omega_1 - t\eta_2 \wedge \omega_2 \mp t\eta_3 \wedge \omega_3$$

- $\varphi^{ts} = \varphi_1^-$ nearly parallel inducing g^{ts}
- $\varphi^{np} = \varphi^+_{1/\sqrt{5}}$ nearly parallel inducing g^{np} "squashed" Einstein metric, cone has $\mathrm{Hol} = \mathrm{Spin}(7)$

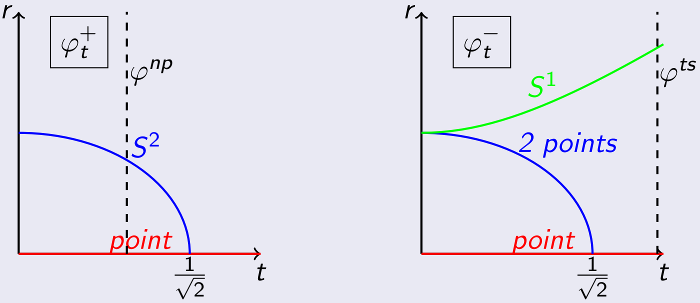
Example: Aloff–Wallach space $\big(\mathrm{SU}(3) \times \mathrm{SU}(2)\big) / \big(\mathrm{U}(1) \times \mathrm{SU}(2)\big)$

Non-trivial examples

- (X^7, g^{ts}) 3-Sasakian, $\varphi_t^{\pm}$ inducing g_t
- $a = (a_1, a_2, a_3) \in \mathbb{R}^3 \rightsquigarrow A = i(a_1\eta_1 + a_2\eta_2 + a_3\eta_3)$ connection on trivial line bundle, $r = |a|$ "distance to trivial connection"

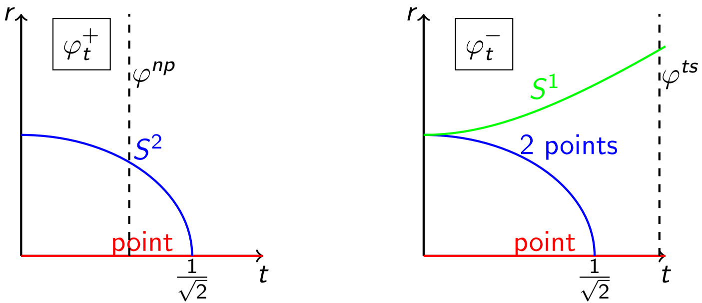
Theorem (L.–Oliveira)

A dG_2*-instanton on* $(X^7, \varphi_t^{\pm}) \Leftrightarrow$



Observations

Proof: explicit calculation $\rightsquigarrow$ solve quadratic equations



- distinct solution spaces for isometric φ_t^+ and φ_t^-
- distinct solution spaces for nearly parallel φ^{ts} and φ^{np}

Can construct examples on non-trivial line bundles on Aloff–Wallach space with similar behaviour

Deformation theory: obstructions

Key question: is deformation theory unobstructed or obstructed?

Unobstructed: moduli space locally smooth manifold of expected dimension, i.e. linearised $\mathrm{d}G_2$-instanton operator $\mathcal{L}$ surjective

$$\mathcal{L} = (\tfrac{1}{2}F_A^2 + *\varphi) \wedge \mathrm{d} : \Omega^1 \to \mathrm{d}\Omega^5$$

$\rightsquigarrow$ infinitesimal deformations guaranteed to extend to deformations

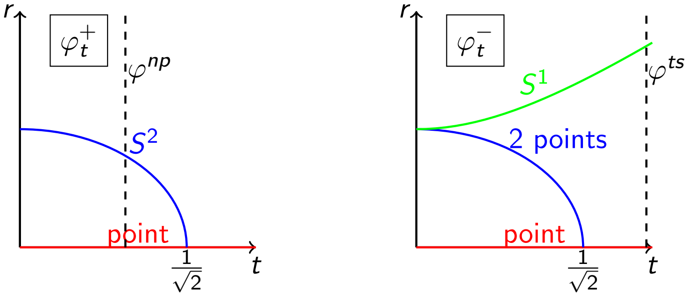
Obstructed: $\mathcal{L}$ not surjective $\rightsquigarrow$ some infinitesimal deformations may not extend

Theorem (L.–Oliveira)

- *Non-trivial* $\mathrm{d}G_2$*-instantons constructed for* φ^{ts} *and* φ^{np} *are obstructed*
- *Trivial* $\mathrm{d}G_2$*-instanton unobstructed for* φ^{ts} *and* φ^{np} *but obstructed for* $\varphi^{\pm}_{1/\sqrt{2}}$

Deformation theory: moduli space

Proof: (Kawai–Yamamoto) $\rightsquigarrow$ unobstructed $\Leftrightarrow$ rigid and isolated



At $t = \frac{1}{\sqrt{2}}$: $\exists$ infinitesimal deformation of trivial dG_2-instanton $\Rightarrow$ obstructed

Corollary

Moduli space of dG_2-instantons on trivial line bundle for φ^{ts} and φ^{np} contains at least two components of different dimensions

Chern–Simons-type functional

- A_0 reference connection on line bundle L on (X^7, φ)
- A connection on L
- $\rightsquigarrow \mathbb{A} = A_0 + s(A - A_0)$ connection on L over $X \times [0,1]$
- $\rightsquigarrow$ curvature $\mathbb{F}$

Proposition (Karigiannis–N.C. Leung)

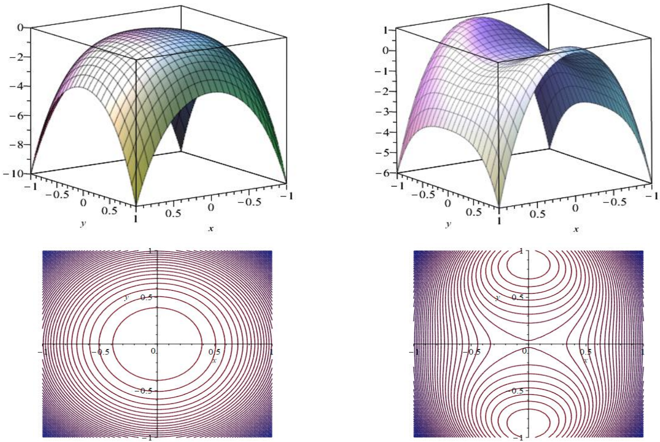
A dG_2*-instanton* $\Leftrightarrow$ *A critical point of functional*

$$\mathcal{F}(A) = \int_{X\times[0,1]} e^{\mathbb{F} + *\varphi}$$

Recall: our examples $A = i(a_1\eta_1 + a_2\eta_2 + a_3\eta_3)$
$\rightsquigarrow$ restriction of $\mathcal{F}$ is function of two variables x and y:

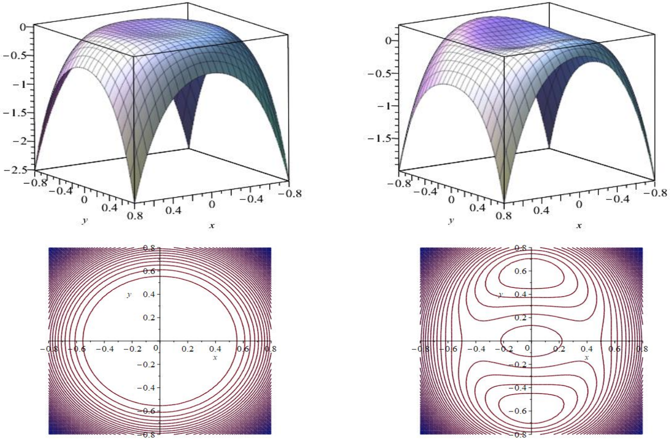
$$x = a_3 \quad \text{and} \quad y^2 = a_1^2 + a_2^2$$

$\mathcal{F}$ for φ_1^+ and $\varphi^{ts} = \varphi_1^-$ with level sets



- trivial connection only critical point $\Rightarrow$ local maximum
- otherwise trivial connection saddle point
- non-trivial dG_2-instantons are local maxima

$\mathcal{F}$ for $\varphi^{np} = \varphi^{+}_{1/\sqrt{5}}$ and $\varphi^{-}_{1/\sqrt{5}}$ with level sets



- trivial connection is local minimum
- continuous families of dG_2-instantons are local maxima
- two isolated examples are saddle points

Questions

- non-trivial dG_2-instantons for holonomy G_2-manifolds?
- (adiabatic) limits of dG_2-instantons?
- dependence of moduli space on G_2-structure?
- "mirror count"?
- applications of $\mathcal{F}$ to compactness or deformation theory?
- Spin(7) version?