Coupled flows and calibrated geometry

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Motivation

Question

What is the "best" way to deform a geometric structure?

- Riemannian metric $g \rightsquigarrow$ Ricci flow $\partial_t g_t = -2\,\mathrm{Ric}(g_t)$
  Critical g: Ricci-flat $\supseteq$ (most) special holonomy
- $\iota : N \hookrightarrow (M, g) \rightsquigarrow$ mean curvature flow $\partial_t \iota_t = H(\iota_t)$
  Critical ι: minimal $\supseteq$ calibrated
- A connection $\rightsquigarrow$ Yang–Mills flow $\partial_t A_t = -\mathrm{d}^*_{A_t} F(A_t)$
  Critical A: Yang–Mills $\supseteq$ instantons

Idea: try to find symplectic and G_2 analogues via "coupling"

Lagrangians

(M, ω) symplectic, $\iota : L^n \hookrightarrow M^{2n}$ Lagrangian $\iota^*\omega = 0$

- compatible almost complex structure $J \rightsquigarrow g(.,.) = \omega(., J.)$
- mean curvature flow does *not* usually preserve Lagrangians

Theorem (Smoczyk 1996)

M Kähler, L compact

- *Kähler–Ricci flow* $\partial_t \omega_t = -\operatorname{Ric}(\omega_t)$
- *"coupled" mean curvature flow* $\partial_t \iota_t = H(\iota_t)$

$\Rightarrow \iota_t : L \hookrightarrow (M, \omega_t)$ *Lagrangian for all t*

M Calabi–Yau $\rightsquigarrow$ Lagrangian mean curvature flow

- Critical points $\leftrightarrow$ special Lagrangians

Canonical geometry

(M, ω) symplectic, J compatible $\leftrightarrow$ almost Kähler

- $\rightsquigarrow$ canonical (Chern) connection ∇
- $\iota : L^n \hookrightarrow M^{2n}$ totally real $\leftrightarrow J(TL) \pitchfork TL$
- K_M canonical bundle $\rightsquigarrow K_M|_L$ trivial
- $\rightsquigarrow$ canonical unit section Ω_L of $K_M|_L$
- $\rightsquigarrow$ connection 1-form $\nabla \Omega_L = i\mu_L \otimes \Omega_L$

Definition

- μ_L *is the Maslov form of* L
- *Maslov flow* $\partial_t \iota_t = -J\mu_L(\iota_t)$

M Kähler, L Lagrangian $\Rightarrow -J\mu_L = H$

Symplectic coupled flow

Key facts

- Chern–Weil: $P(X,Y) = \mathrm{tr}_\omega R_\nabla(X,Y)$ represents $4\pi c_1(M)$
- M Kähler $\Rightarrow$ $P = 2\,\mathrm{Ric}(\omega)$
- $\mathcal{L}_{-J\mu_L}\omega = \mathrm{d}\mu_L = \frac{1}{2}\iota^* P$

Theorem (L.-Pacini 2013)

(M,ω) symplectic, J compatible, $\iota : L^n \hookrightarrow M^{2n}$ totally real

- *(Streets–Tian 2011) symplectic curvature flow $\partial_t\omega_t = -\frac{1}{2}P_t$*
- *coupled Maslov flow $\partial_t\iota_t = -J\mu_L(\iota_t)$*

$\Rightarrow \iota_t : L \hookrightarrow (M,\omega_t,J_t)$ satisfies $\iota_t^*\omega_t = \iota^*\omega$ for all t

Corollary

Maslov flow plus symplectic curvature flow preserves Lagrangians

Kodaira–Thurston manifold

$$G = \left\{ \begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{pmatrix} : x, y, z \in \mathbb{R}, s > 0 \right\}, \Gamma = G \cap \mathrm{GL}(4, \mathbb{Z})$$

- $M = \Gamma \backslash G$, $\omega = \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}s \wedge \mathrm{d}y \rightsquigarrow (M^4, \omega)$ symplectic
- $b_1(M) = 3 \Rightarrow M$ not Kähler
- (Pook 2012) symplectic curvature flow exists for all t with $\omega_t = \omega$ but g_t non-constant and does not converge
- $\rightsquigarrow$ (after rescaling) collapses to flat T^2

$L = T^2$

- $\{\gamma \in G : x = y = 0\} \rightsquigarrow \iota : L \hookrightarrow (M, \omega)$ Lagrangian
- Maslov form $\mu_L = 0 \rightsquigarrow \iota_t = \iota$ Lagrangian for all t
- L_t collapses to a point as $t \to \infty$

J-volume

$\iota : L^n \hookrightarrow (M^{2n}, \omega, J)$ totally real $\rightsquigarrow \Omega_L$ canonical section of $K_M|_L$

Definition

- *J-volume form:* $\mathrm{vol}_L^J = \iota^* \Omega_L$
- *J-volume functional:* $\mathrm{Vol}^J = \int_L \mathrm{vol}_L^J$

Theorem (L.–Pacini 2013)

(M, ω, J) *Kähler* $\Rightarrow$ *Maslov flow is negative gradient flow of* Vol^J

- *Critical points:* $\mu_L = 0 \leftrightarrow$ *stationary for* Vol^J

M Calabi–Yau, Ω holomorphic volume form $\Rightarrow \iota^* \operatorname{Re} \Omega \leq \mathrm{vol}_L^J$

- "calibrated" $\iota^* \operatorname{Re} \Omega = \mathrm{vol}_L^J$: "special totally real"
- special totally real $\Rightarrow$ homologically J-volume-minimizing

Convexity

$\mathcal{T}$ totally real immersions of L modulo $\mathrm{Diff}(L)$

- $\exists$ canonical connection on $\mathcal{T} \rightsquigarrow$ geodesics
- $JX \in T_{\iota(L)}\mathcal{T}$: geodesic given by "$J$-holomorphic thickening" of integral curves of X

Theorem (L.–Pacini 2014)

M Kähler $\mathrm{Ric} \leq 0 \Rightarrow \mathrm{Vol}^J$ *convex along geodesics*

M Kähler–Einstein $\mathrm{Ric} < 0$

- Maslov flow $=$ Vol^J gradient flow
- Critical points $=$ minimal Lagrangians
- Convexity $\Rightarrow$ minimal Lagrangians stable (Oh 1990)
- (Joyce 2014) Are minimal Lagrangians in $[L]$ unique?

Coassociatives

(M^7, φ) 7-manifold with G_2 structure, C^4 oriented

Definition

$\iota : C \hookrightarrow M$ *coassociative* $\Leftrightarrow \iota^* *_\varphi \varphi = \mathrm{vol}_C \Leftrightarrow \iota^* \varphi = 0$

- $\mathrm{d}\varphi = 0 \Rightarrow$ no obstructions to local existence of coassociatives
- $\rightsquigarrow$ is there a "best" way to deform φ with $\mathrm{d}\varphi = 0$?

Lemma

$\iota : C \hookrightarrow M$ *coassociative*, $\mathrm{d}\varphi = 0$

- $H = -\mathrm{d} *_\varphi \varphi(e_1, e_2, e_3, e_4, .)$ *where* $\{e_1, e_2, e_3, e_4\}$ *orthonormal frame on* C
- $H \lrcorner \varphi = -\pi_+ \iota^* \mathrm{d}^*_\varphi \varphi$ *where* $\pi_+ : \Omega^2(C) \to \Omega^2_+(C)$

G_2 coupled flow

Key facts

- $\mathcal{L}_H\varphi = \mathrm{d}(H \lrcorner \varphi) = -\iota^*\mathrm{d}\mathrm{d}^*_\varphi\varphi = -\iota^*\Delta_\varphi\varphi$
- compact coassociative $\Rightarrow$ moduli space of dimension b^2_+

Theorem (L.–Pacini 2014)

(M^7, φ) $\mathrm{d}\varphi = 0$, $\iota : C \hookrightarrow M$ compact coassociative

- *(Bryant 1992) Laplacian flow $\partial_t\varphi_t = \Delta_{\varphi_t}\varphi_t$*
- *coupled mean curvature flow $\partial_t\iota_t = H(\iota_t)$*

$\Rightarrow \iota_t : C \hookrightarrow (M, \varphi_t)$ coassociative for all t

Work in progress

- Can we extend this flow to "weak coassociatives"?
- Is this extension still the gradient flow of a functional?

Fernández closed G_2-solvmanifold

$$G = \left\{ \begin{pmatrix} 1 & 0 & x_2 & x_4 & x_6 \\ 0 & 1 & x_3 & x_5 & x_7 \\ 0 & 0 & 1 & 0 & x_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_i \in \mathbb{R} \right\}, \ \Gamma = G \cap \mathrm{GL}(5, \mathbb{Z})$$

- $M = \Gamma \backslash G \rightsquigarrow (M^7, \varphi)$ with $\mathrm{d}\varphi = 0$
- $b_1(M) = 5 \Rightarrow$ no torsion-free G_2 structure
- (Bryant 1992) Laplacian flow exists for all t but does not converge
- $\rightsquigarrow$ (after rescaling) collapses to flat T^3

$C = T^4$

- $\{\gamma \in G : x_1 = x_2 = x_3 = 0\} \rightsquigarrow \iota : C \hookrightarrow (M, \varphi)$ coassociative
- $H = 0 \rightsquigarrow \iota_t = \iota$ coassociative for all t
- C_t collapses to a point as $t \to \infty$

Future directions

(M, α) contact, $\iota : L^n \hookrightarrow M^{2n+1}$ Legendrian $\iota^*\alpha = 0$

- (cf. Smoczyk 2003) M Sasakian $\Rightarrow$ Sasaki–Ricci flow plus "projected" mean curvature flow preserves Legendrians
- $n = 1$: "projected" analogous to total length-preserving
- Is there a contact analogue of the symplectic coupled flow?
- $\rightsquigarrow$ "best" flow of contact structures?

(M^7, φ) 7-manifold with G_2 structure, A connection

- A G_2 instanton $\Leftrightarrow F(A) \wedge \varphi = -*F(A) \Leftrightarrow F(A) \wedge *\varphi = 0$
- $\mathrm{d}^*\varphi = 0 \Rightarrow$ no obstructions to local existence of G_2 instantons
- Is there an instanton analogue of coassociative coupled flow?
- $\rightsquigarrow$ "best" flow of φ with $\mathrm{d}^*\varphi = 0$?