Number Theory — Examples Sheet 3

Michaelmas Term 2013 *V. Neale*

Throughout this sheet, ϕ denotes the Euler totient function, μ the Möbius function, $d(n)$ the number of positive divisors of n, and $\sigma(n)$ the sum of the positive divisors of n.

1. Prove that for $\Re(s) > 1$, we have
$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$
Can you find Dirichlet series for $1/\zeta(s)$ and $\zeta(s-1)/\zeta(s)$ (for suitable values of s)?

2. Find all natural numbers n for which $\sigma(n) + \phi(n) = nd(n)$.

3. (i) Compute $\sum_{d|n} \mu(d)$ for natural numbers n.
   (ii) Let f be a function defined on the natural numbers, and define g by $g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$. Find an expression for f in terms of g.
   (iii) Find a relationship between μ and ϕ.

4. Compute $\sum_{d|n} \Lambda(d)$ for natural numbers n. (Here Λ is the von Mangoldt function.)

5. Use Legendre's formula to compute $\pi(207)$.

6. Let N be a positive integer greater than 1.
   (i) Show that the exact power of a prime p dividing $N!$ is $\sum_{k=1}^{\infty} \lfloor \frac{N}{p^k} \rfloor$.
   (ii) Prove the inequality $N! > (\frac{N}{e})^N$.
   (iii) Deduce that
$$\sum_{p \leqslant N} \frac{\log p}{p-1} > \log N - 1.$$

7. Prove that every non-constant polynomial with integer coefficients assumes infinitely many composite values.

8. Prove that every integer $N > 6$ can be expressed as a sum of distinct primes.

9. Prove that for every $n \geqslant 1$, the set of numbers $\{1, 2, \ldots, 2n\}$ can be partitioned into pairs $\{a_1, b_1\}$, $\{a_2, b_2\}$, ..., $\{a_n, b_n\}$ so that the sum $a_i + b_i$ of each pair is prime.

10. Calculate a_0, ..., a_4 in the continued fraction expansions of e and π.

11. Let a be a positive integer. Determine explicitly the real number whose continued fraction is $[a, a, a, \ldots]$.

12. Determine the continued fraction expansions of $\sqrt{3}$, $\sqrt{7}$, $\sqrt{13}$, $\sqrt{19}$, $\sqrt{46}$.

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