Graphs without a 3-connected subgraph are 4-colourable

Édouard Bonnet[a] Carl Feghali[a] Tung Nguyen[b] Alex Scott[c]
Paul Seymour[b] Stéphan Thomassé[a] Nicolas Trotignon[a]

Abstract

In 1972, Mader showed that every graph without a 3-connected subgraph is 4-degenerate and thus 5-colourable. We show that the number 5 of colours can be replaced by 4, which is best possible.

Mathematics Subject Classifications: 05C15, 05C40

1 Introduction

Throughout the paper all graphs are finite and simple, and we only use standard notions and notation. We recall that a graph is *k-connected* if it has at least $k+1$ vertices and no vertex cutset with at most $k-1$ vertices. In 1972, Mader [2] proved the following theorem.

Theorem 1. *For every integer $k \geqslant 1$, every graph with average degree at least $4k$ contains a $(k+1)$-connected subgraph.*

Focusing on the case $k = 2$ of Theorem 1, we call a graph *fragile* if it has no 3-connected subgraph. From Theorem 1, every non-null fragile graph has a vertex of degree at most 7. By restricting the proof of Mader to the case $k = 2$, it is easy to show that all fragile graphs G on at least four vertices satisfy $|E(G)| \leqslant 2.5|V(G)| - 5$ (we supply the proof in Section 3 for the sake of completeness). So the average degree of G is smaller than 5. Thus every fragile graph contains a vertex of degree at most 4, and this is best possible as shown by the graph in Figure 1. Every fragile graph is therefore 5-colourable.

Despite recent progress on related questions, there is no available proof that the number 5 of colours can be improved. The objective of this paper is to prove the following

[a]CNRS, ENS de Lyon, Université Claude Bernard Lyon 1, LIP UMR 5668, 69342 Lyon Cedex 07, France (edouard.bonnet@ens-lyon.fr, carl.feghali@ens-lyon.fr, stephan.thomasse@ens-lyon.fr, nicolas.trotignon@ens-lyon.fr).
[b]Princeton University, Princeton, NJ 08544, USA (pds@math.princeton.edu, huytung.nht@gmail.com).
[c]Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK (scott@maths.ox.ac.uk).

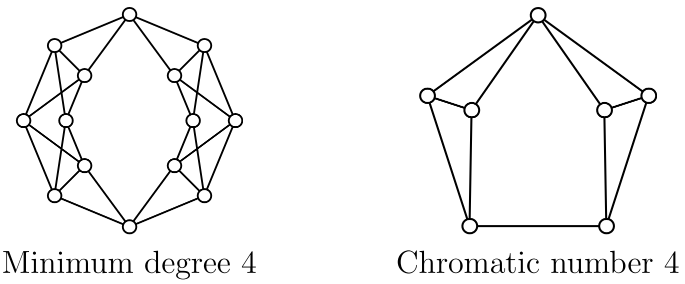

Figure 1: Graphs with no 3-connected subgraph.

theorem that implies that every fragile graph is 4-colourable. It was announced without proof in [3] (which also contains a thorough literature review) and was independently rediscovered by the first two and last two authors of this article.

Theorem 2. *For all $m \geqslant 4$, every graph with chromatic number at least $m+1$ has a 3-connected subgraph with chromatic number at least m.*

Theorem 2 is best possible as shown by the graph in Figure 1. The proof of Theorem 2 is given in Section 2. Several remarks and open questions are presented in Section 3.

2 Proof of Theorem 2

For every integer $m \geqslant 4$, a graph G is *m-fragile* if all 3-connected subgraphs of G are $(m-1)$-colourable. Observe that a fragile graph is m-fragile for all $m \geqslant 4$. Theorem 2 can be rephrased as: for all $m \geqslant 4$, every m-fragile graph is m-colourable. To prove Theorem 2, we shall establish the following stronger statement. By a *k-colouring of a graph G*, we mean a function c that associates to each vertex of G an integer in $\{1, \dots, k\}$ and such that for all edges xy of G, $c(x) \neq c(y)$.

Theorem 3. *For every integer $m \geqslant 4$, every m-fragile graph G satisfies the following four conditions.*

1. *For all non-adjacent $x, y \in V(G)$, G admits an m-colouring c such that $c(x) = c(y)$.*
2. *For all distinct $x, y \in V(G)$, G admits an m-colouring c such that $c(x) \neq c(y)$.*
3. *For all distinct $x, y, z \in V(G)$, G admits an m-colouring c such that*
$$c(x) \notin \{c(y), c(z)\}.$$
4. *For all distinct $x, y, z \in V(G)$ that are not all pairwise adjacent, G admits an m-colouring c such that $|\{c(x), c(y), c(z)\}| = 2$.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leqslant 3$, then G obviously satisfies conditions (1)–(4). For the induction step, suppose $|V(G)| \geqslant 4$ and that the statement holds for every graph with fewer vertices than G.

If G is 3-connected, then G satisfies conditions (1)–(4) because by assumption G is $(m-1)$-colourable. So G can be coloured with colours 1 to $m-1$, and colour m is available to satisfy any of the conditions (1)–(4) (for instance x and y can be recoloured with colour m to satisfy (1)). Hence we may assume from here on that G is not 3-connected.

Since G is not 3-connected, there exist two induced subgraphs G_1, G_2 of G such that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $V(G_1) \setminus V(G_2) \neq \emptyset$, $V(G_2) \setminus V(G_1) \neq \emptyset$, and $S = V(G_1) \cap V(G_2)$ has size at most 2. Moreover, since G is m-fragile, G_1 and G_2 are also m-fragile and, as $|V(G_1)|, |V(G_2)| < |V(G)|$, we may apply the induction hypothesis to both G_1 and G_2.

If $S = \emptyset$, the induction step is obvious and we omit the details. So we may set $S = \{u, v\}$ (possibly $u = v$). We have to prove that for each of the precolouring conditions C among (1)–(4) on any given set $X \subseteq V(G)$ (namely, $X = \{x, y\}$ for conditions (1) and (2) and $X = \{x, y, z\}$ for conditions (3) and (4)) some appropriate 4-colouring exists. Suppose first that $X \subseteq V(G_1)$. Then, by the induction hypothesis, G_1 admits a colouring c_1 that satisfies C. By applying (1) or (2) to the vertices u and v of G_2 (or trivially if $u = v$), and up to a relabeling of the colours, we can force a colouring c_2 of G_2 such that $c_2(u) = c_1(u)$ and $c_2(v) = c_1(v)$. Note that the case when uv is an edge corresponds to the usual amalgamation of two colourings on a clique cutset. Hence, $c_1 \cup c_2$ is a colouring of G that satisfies C. The proof is similar when $X \subseteq V(G_2)$. Hence, from here on, we may assume that

$$X \text{ intersects both } V(G_1) \setminus V(G_2) \text{ and } V(G_2) \setminus V(G_1). \qquad (\star)$$

We now prove four claims, from which Theorem 3 trivially follows. Note that, unless specified otherwise, we shall make no assumption on whether $u \neq v$ or $uv \in E(G)$.

Claim 1. *The graph G satisfies* (1).

Proof. By $(\star)$, we may assume that $x \in V(G_1) \setminus V(G_2)$ and $y \in V(G_2) \setminus V(G_1)$. Suppose first that $u = v$. By (2) applied to G_1, let c_1 be a colouring of G_1 such that $c_1(x) \neq c_1(u)$, and similarly, let c_2 be a colouring of G_2 such that $c_2(y) \neq c_2(u)$. Up to a relabelling of the colours, we may assume that $c_1(u) = c_2(u)$ and $c_1(x) = c_2(y)$. Hence, $c_1 \cup c_2$ is a colouring of G satisfying (1). So, we may suppose from here on that $u \neq v$.

We build three colourings a_1, b_1 and c_1 of G_1 and three colourings a_2, b_2 and c_2 of G_2 that are represented in Figure 2 for the reader's convenience.

By (3) applied to x, u, v (in this order) in G_1, we obtain a colouring a_1 of G_1 such that $a_1(x) \notin \{a_1(u), a_1(v)\}$. Similarly, we obtain a colouring a_2 of G_2 such that $a_2(y) \notin \{a_2(u), a_2(v)\}$. Up to a relabeling, we may assume that $a_1(x) = a_2(y) = 1$, $a_1(u) = a_2(u) = 2$ and $a_1(v), a_2(v) \in \{2, 3\}$. If $a_1(v) = a_2(v)$, then $a_1 \cup a_2$ is a colouring of G that satisfies (1). Hence, up to symmetry, we may assume that $a_1(v) = 3$ and $a_2(v) = 2$.

By (3) applied to u, x, v in G_1, we obtain a colouring b_1 of G_1 such that $b_1(u) \notin \{b_1(x), b_1(v)\}$. Similarly, we obtain a colouring b_2 of G_2 such that $b_2(u) \notin \{b_2(y), b_2(v)\}$.

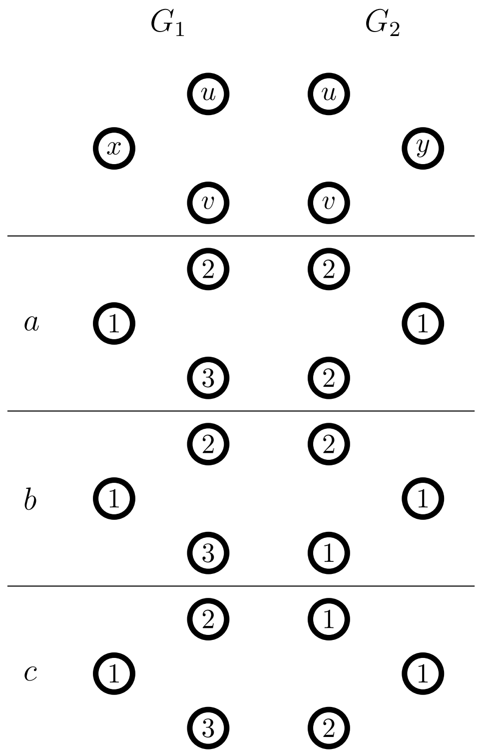

Figure 2: Colourings obtained in the proof of Claim 1.

Up to a relabeling, we may assume that $b_1(x) = b_2(y) = 1$, $b_1(u) = b_2(u) = 2$ and $b_1(v), b_2(v) \in \{1, 3\}$. If $b_1(v) = b_2(v)$, then $b_1 \cup b_2$ is a colouring of G that satisfies (1). Hence, we may assume that $b_1(v) \neq b_2(v)$. If $b_2(v) = 3$, then $a_1 \cup b_2$ is a colouring of G that satisfies (1). Hence, we may assume that $b_1(v) = 3$ and $b_2(v) = 1$.

By (3) applied to v, x, u in G_1, we obtain a colouring c_1 of G_1 such that $c_1(v) \notin \{c_1(x), c_1(u)\}$. Similarly, we obtain a colouring c_2 of G_2 such that $c_2(v) \notin \{c_2(y), c_2(u)\}$. Up to a relabeling, we may assume that $c_1(x) = 1$ and either $c_1(u) = 1$ and $c_1(v) = 2$ or $c_1(u) = 2$ and $c_1(v) = 3$. Up to a relabeling, we may also assume that $c_2(y) = 1$ and either $c_2(u) = 1$ and $c_2(v) = 2$ or $c_2(u) = 2$ and $c_2(v) = 3$. If $c_1(u) = c_2(u)$, then $c_1 \cup c_2$ is a colouring of G that satisfies (1). Hence, we may assume that $c_1(u) \neq c_2(u)$. If $c_2(u) = 2$ (and so $c_2(v) = 3$), then $a_1 \cup c_2$ is a colouring of G that satisfies (1). Hence, we may assume that $c_1(u) = 2$, $c_1(v) = 3$, $c_2(u) = 1$ and $c_2(v) = 2$.

By (4) applied to x, u, v in G_1, we obtain a colouring d_1 of G_1 such that $|\{d_1(x), d_1(u), d_1(v)\}| = 2$ (note that x, u and v are not pairwise adjacent because $a_2(u) = a_2(v)$ implies $uv \notin E(G)$). Up to a relabeling, we may assume that $d_1(x) = 1$ and $\{d_1(x), d_1(u), d_1(v)\} = \{1, 2\}$. If $d_1(u) = 1$ and $d_1(v) = 2$, then $d_1 \cup c_2$ satisfies (1). And if $d_1(u) = 2$ and $d_1(v) = 1$, then $d_1 \cup b_2$ satisfies (1). Finally, if $d_1(u) = 2$ and $d_1(v) = 2$, then $d_1 \cup a_2$ satisfies (1). The claim is proved. □

Claim 2. *The graph G satisfies* (3).

Proof. If $x \in \{u, v\}$ (say $x = u$ up to symmetry), then by ($\star$) we may assume that $y \in V(G_1) \setminus V(G_2)$ and $z \in V(G_2) \setminus V(G_1)$. If $u \neq v$, then by (3) applied separately to x, v and y in G_1 and to x, v and z in G_2, we obtain up to a relabeling a colouring a_1 of G_1 and a colouring a_2 of G_2 such that $a_1(x) = a_2(x) = 1$, $a_1(v) = a_2(v) = 2$, $a_1(y) \neq 1$ and $a_2(z) \neq 1$. Hence, $a_1 \cup a_2$ is a colouring of G that satisfies (3). If $u = v$, then by (2) applied separately to x and u in G_1 and to y and u in G_2, we obtain up to a relabeling a colouring a_1 of G_1 and a colouring a_2 of G_2 such that $a_1(x) = a_2(x) = 1$, $a_1(y) \neq 1$ and $a_2(z) \neq 1$. Hence, $a_1 \cup a_2$ is a colouring of G that satisfies (3). We may therefore assume that $x \notin \{u, v\}$, and so up to symmetry that $x \in V(G_1) \setminus V(G_2)$.

Hence, by ($\star$) and up to symmetry, we may restrict our attention to the following two cases.

Case 1: $x \in V(G_1) \setminus V(G_2)$ and $y, z \in V(G_2)$.

If $uv \in E(G)$ (so in particular $u \neq v$), then by (3) applied to x, u and v and up to a relabeling, there exists a colouring a_1 of G_1 such that $a_1(x) = 1$, $a_1(u) = 2$, and $a_1(v) = 3$. We claim that there exists a colouring a_2 of G_2 that requires at most $m - 1$ colours for u, v, y, z. If $m \geqslant 5$, this is trivial, so suppose $m = 4$. Then the graph induced by u, v, y and z is not a complete graph on four vertices, because such a graph is 3-connected with chromatic number 4 and would imply that G is not m-fragile. Hence, either $|\{u, v, y, z\}| \leqslant 3$ or there are non-adjacent vertices among u, v, y and z. In either case, there exists a colouring a_2 of G_2 that requires at most $m - 1 = 3$ colours for u, v, y, z (trivially if $|\{u, v, y, z\}| \leqslant 3$ or by applying (1) to a non-edge otherwise). This proves our claim. Up to a relabeling, we may assume that $a_2(u) = 2$, $a_2(v) = 3$ and

$\{a_2(y), a_2(z)\} \subseteq \{2, \dots, m\}$. Hence, $a_1 \cup a_2$ is a colouring of G satisfying (3). We may therefore assume from here on that $uv \notin E(G)$.

Suppose that there exists a colouring a_1 of G_1 such that $a_1(x) \neq a_1(u) = a_1(v)$ (this is the case when $u = v$ because of (2) applied to x and u in G_1). So, up to a relabeling, we may assume $a_1(x) = 1$ and $a_1(u) = a_1(v) = 2$. Then by (1) applied to u and v in G_2 (or trivially if $u = v$), there exists a colouring a_2 of G_2 such that $a_2(u) = a_2(v)$. Hence, $|\{a_2(u), a_2(v), a_2(y), a_2(z)\}| \leqslant 3$. So, up to a relabeling, we may assume that $a_2(u) = a_2(v) = 2$ and $\{a_2(y), a_2(z)\} \subseteq \{2, 3, 4\}$. So $a_1 \cup a_2$ is a colouring of G that satisfies (3). We may therefore assume that no colouring as a_1 exists. In particular, $u \neq v$.

Hence, when applying (3) to x, u and v, up to a relabeling, we obtain a colouring b_1 of G_1 such that $b_1(x) = 1$, $b_1(u) = 2$ and $b_1(v) = 3$. And when applying (4) to x, u and v (which is allowed since $uv \notin E(G)$), up to a relabeling and to the symmetry between u and v, we obtain a colouring c_1 of G_1 such that $c_1(x) = 1$, $c_1(u) = 1$ and $c_1(v) = 2$.

By (2) applied to u and v, there exists a colouring d_2 of G_2 such that $d_2(u) \neq d_2(v)$. If $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| \leqslant 3$, then up to a relabeling, we may assume that $d_2(u) = 2$, $d_2(v) = 3$ and $\{d_2(y), d_2(z)\} \subseteq \{2, 3, 4\}$, So $b_1 \cup d_2$ is a colouring of G that satisfies (3). And if $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| = 4$, then we may assume up to a relabeling that $d_2(u) = 1$, $d_2(v) = 2$, $d_2(y) = 3$ and $d_2(z) = 4$, so $c_1 \cup d_2$ is a colouring that satisfies (3).

Case 2: $x, y \in V(G_1) \setminus V(G_2)$ and $z \in V(G_2) \setminus V(G_1)$.

By (3) applied to x, y and u, up to a relabeling, we obtain a colouring a_1 of G_1 such that $a_1(x) = 1$, $a_1(y) = 2$ and $a_1(u) \in \{2, 3\}$. If $a_1(v) \neq 1$, then colour 1 is not used on u or v under a_1. By (1) or (2) applied to u and v (or just trivially if $u = v$), we obtain up to a relabeling a colouring a_2 of G_2 such that $a_2(u) = a_1(u)$ and $a_2(v) = a_1(v)$. Thus, colour 1 is not used on u or v under a_2 either and so, up to a relabeling, we may assume that $a_2(z) \neq 1$. Hence $a_1 \cup a_2$ is a colouring of G that satisfies (3). We may therefore assume that $a_1(v) = 1$ (so $u \neq v$).

By (3) applied to v, u and z, up to a relabeling, we obtain a colouring b_2 of G_2 such that $b_2(v) = 1$, $b_2(u) = a_1(u)$ and $b_2(z) \neq 1$. Hence $a_1 \cup b_2$ is a colouring of G that satisfies (3). □

Claim 3. *The graph G satisfies* (2).

Proof. By Claim 2, we may apply (3) to x, y and any vertex of G. We obtain a colouring of G that satisfies (2). □

Claim 4. *The graph G satisfies* (4).

Proof. By ($\star$), we may assume that $x \in V(G_1) \setminus V(G_2)$, $y \in V(G_2)$ and $z \in V(G_2) \setminus V(G_1)$. Moreover, we suppose that if $y \in \{u, v\}$, then $y = v$.

Suppose that $uv \in E(G)$ (in particular, $u \neq v$). Then by (3) applied to x, u and v and up to a relabeling, there exists a colouring a_1 of G_1 such that $a_1(x) = 1$, $a_1(u) = 2$ and $a_1(v) = 3$. By (3) applied to u, y and z (that are distinct since $y \neq u$ and $z \in V(G_2) \setminus V(G_1)$) and up to a relabeling, we obtain a colouring a_2 of G_2 such that

$a_2(u) = 2$, $a_2(v) = 3$ and $\{a_2(y), a_2(z)\}$ is either $\{3, 1\}$, $\{3\}$ or $\{4\}$. In either case, $a_1 \cup a_2$ is a colouring of G satisfying (4). We may therefore assume from here on that $uv \notin E(G)$.

Suppose that there exists a colouring a_1 of G_1 such that $a_1(x) \neq a_1(u) = a_1(v)$ (this is the case when $u = v$ because of (2) applied to x and u in G_1). Then up to a relabeling we may assume that $a_1(x) = 1$ and $a_1(u) = a_1(v) = 2$. By (1) applied to u and v in G_2 (or trivially if $u = v$), there exists up to a relabeling a colouring a_2 of G_2 such that $a_2(u) = a_2(v) = 2$. If $a_2(y) = a_2(z)$, then up to relabeling, we may assume that $a_2(y) = a_2(z) \neq 1$, so (4) is satisfied by $a_1 \cup a_2$. And if $a_2(y) \neq a_2(z)$, then up to a relabeling, we may assume $a_2(y) = 1$ or $a_2(z) = 1$, and (4) is again satisfied by $a_1 \cup a_2$. We may therefore assume that no colouring as a_1 exists. In particular, $u \neq v$.

Hence, when applying (3) to x, u and v, up to a relabeling, we obtain a colouring b_1 of G_1 such that $b_1(x) = 1$, $b_1(u) = 2$ and $b_1(v) = 3$. And when applying (4) to x, u and v (which is allowed since $uv \notin E(G)$), up to a relabeling and to the symmetry between u and v, we obtain a colouring c_1 of G_1 such that $c_1(x) = 1$, $c_1(u) = 1$ and $c_1(v) = 2$.

On the other hand, by (2) applied to u and v, there exists a colouring d_2 of G_2 such that $d_2(u) \neq d_2(v)$. If $d_2(y) = d_2(z)$, then up to a relabeling, we may assume that $d_2(u) = 2$, $d_2(v) = 3$ and $d_2(y) \neq 1$. Thus, $b_1 \cup d_2$ is a colouring that satisfies (4). Hence, from here on, we may assume that $d_2(y) \neq d_2(z)$.

If $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| \geqslant 3$, then we may assume up to a relabeling that $d_2(u) = 2$, $d_2(v) = 3$ and $1 \in \{d_2(y), d_2(z)\}$, so $b_1 \cup d_2$ is a colouring that satisfies (4). If $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| = 2$, then up to a relabeling, we may assume that $d_2(u) = 1$, $d_2(v) = 2$, so that $\{d_2(y), d_2(z)\} = \{1, 2\}$. So $c_1 \cup d_2$ is a colouring of G that satisfies (4). □

Theorem 3 immediately follows from Claims 1–4. □

3 Conclusion and open questions

We collect here several remarks and open questions.

3.1 Fragile graphs have average degree less than 5

As announced in the introduction, we recall the proof that every fragile graph G on at least four vertices satisfies $|E(G)| \leqslant 2.5|V(G)| - 5$. When G has 4 vertices, the inequality holds since the graph on 4 vertices and 6 edges is a complete graph and is 3-connected. For the induction step, we decompose G into G_1 and G_2 as in the previous section. If $|V(G_1)| \leqslant 3$, then G contains a vertex x of degree at most 2. Hence,

$$|E(G)| \leqslant |E(G \setminus x)| + 2 \leqslant 2.5|V(G \setminus x)| - 5 + 2 = 2.5(|V(G)| - 1) - 3 \leqslant 2.5|V(G)| - 5.$$

We may therefore assume that $|V(G_1)| \geqslant 4$ and symmetrically $|V(G_2)| \geqslant 4$. Hence the induction hypothesis can be applied to both G_1 and G_2 so that the result follows from these inequalities:

$$
\begin{aligned}
|E(G)| &\leqslant |E(G_1)| + |E(G_2)| \\
&\leqslant 2.5|V(G_1)| - 5 + 2.5|V(G_2)| - 5 \\
&= 2.5(|V(G_1)| + |V(G_2)|) - 10 \\
&\leqslant 2.5(|V(G)| + 2) - 10 \\
&= 2.5|V(G)| - 5.
\end{aligned}
$$

We do not know whether a fragile graph with minimum degree 4 and chromatic number 4 exists.

3.2 Girth conditions

It is easy to prove by induction that every fragile graph of girth at least 4 on at least 3 vertices satisfies $|E(G)| \leqslant 2|V(G)| - 4$ (the proof is as in Section 3.1). This implies that every fragile graph with girth at least 4 contains a vertex of degree at most 3, so is 4-colourable. We tried to improve this bound, but we instead found a fragile graph with girth 4 and chromatic number 4, as we now present.

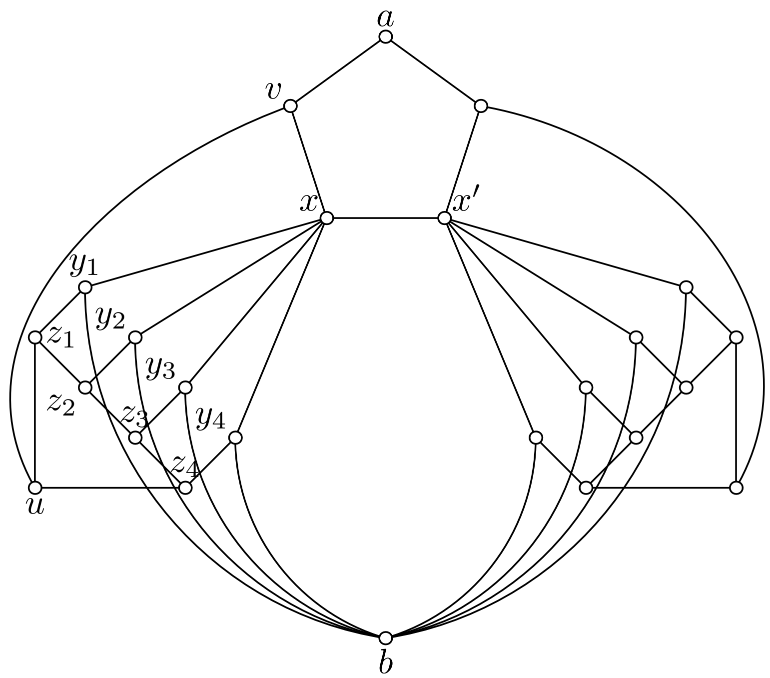

Figure 3: The graph G_1.

Let G_1 be the graph represented in Figure 3. It has girth 4 and is 2-degenerate; so in particular it is fragile and has chromatic number at most 3. For all 3-colourings of G_1, vertices a and b receive different colours. Indeed, suppose for a contradiction that for some 3-colouring of G_1, a and b receive the same colour, say colour 1. Then, one of x

and x', say x up to symmetry, must receive a colour different from 1, say colour 2. So, the vertices $y_1, \ldots, y_4$ must all receive the same colour, say colour 3. It follows that the vertices $z_1, \ldots, z_4$ are coloured with colour 1 and 2 alternately. Hence, u receives colour 3. Now, v has three neighbors, namely a, x and u that are coloured with colours 1, 2 and 3 respectively, a contradiction.

It follows that the triangle-free graph G_2 represented in Figure 4 is not 3-colourable, but it is fragile since $\{a', b'\}$ is a cutset, and G_1 is 2-degenerate even if two vertices adjacent to a and b are added.

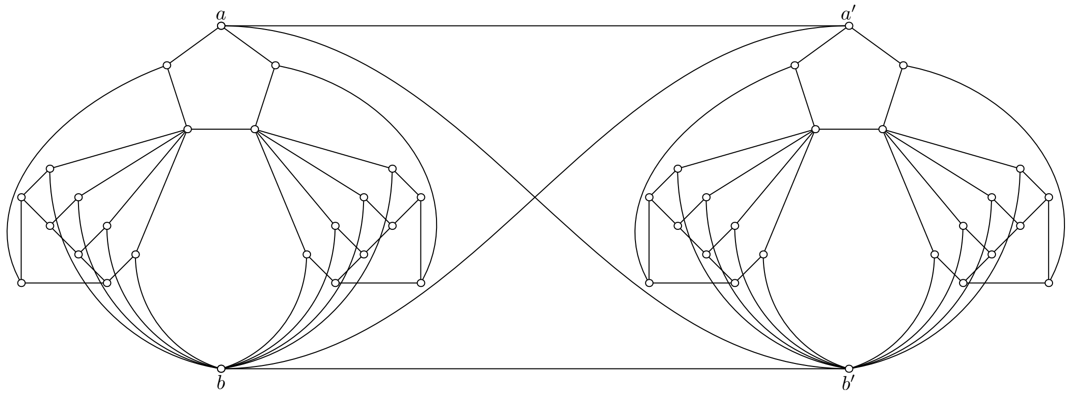

Figure 4: The graph G_2.

We could also obtain a fragile graph with no cycle of length 4 and chromatic number 4, see Figure 5.

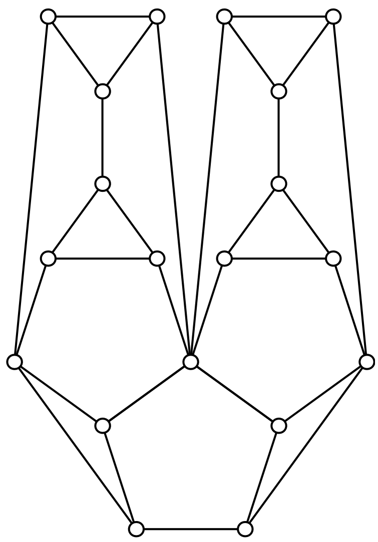

Figure 5: A fragile graph with no cycle of length 4 and chromatic number 4.

This raises the following question: Is there a finite girth that makes fragile graphs

3-colourable? A possible approach could be to prove that if the girth of a fragile graph is large enough, then the graph is 2-degenerate. But this approach fails because of the following construction. Consider an integer $g \geqslant 3$ and a connected cubic graph G of girth g (this exists, see for instance [1]). Remove an edge uv of G. This yields a 2-degenerate, and therefore fragile graph. Consider a copy G' of $G \setminus uv$, with the vertices u' and v' corresponding to u and v respectively. Now add an edge uu' and an edge vv'. The obtained graph is fragile, cubic and has girth g.

Trivially, a graph G is fragile if and only if every subgraph H of G is either on at most 3 vertices or admits a cutset of size at most 2. In fragile graphs of girth at least 4, one can further impose the cutset to be an independent set.

Lemma 4. *A graph G with girth at least 4 is fragile if and only if every subgraph H of G is either on at most 2 vertices or admits an independent cutset of size at most 2.*

Proof. We prove the statement by induction on $|V(G)|$. The equivalence can be checked to hold on graphs of up to 3 vertices. If $|V(G)| \geqslant 4$, then since G is not 3-connected, it admits a cutset S of size at most 2. Suppose that S is not independent, so $S = \{u, v\}$ and $uv \in E(G)$. Let C be a connected component of $G \setminus S$. Since G has girth at least 4, no vertex of C is adjacent to both u and v. Hence, if $|C| = 1$, G admits a cutset of size 1 (and therefore independent). So we may assume that $|C| \geqslant 2$. So, by the induction hypothesis, $G[S \cup C]$ admits an independent cutset S'. It is easy to check that S' is also a cutset of G. □

3.3 Algorithms

By subdividing twice every edge of any graph G, a fragile graph G' is obtained. Poljak [4] proved that $\alpha(G') = \alpha(G) + |E(G)|$. It follows that a polynomial-time algorithm that computes a maximum independent set for any fragile graph would yield a similar algorithm for all graphs. This proves that computing a maximum independent set in a fragile graph is NP-hard.

We also observe that, in G', every edge uv becomes a path $ux_{uv}y_{uv}v$. Consider the graph G'' obtained from G' by adding, for every vertex x_{uv}, a new vertex x'_{uv} adjacent to u, x_{uv} and y_{uv}. It is easy to check that G'' is fragile and for all 3-colourings of G'' and all edges uv of G, u and v have different colours (in G''). It follows that if G'' is 3-colourable, then so is G. Conversely it is easy to check that if G is 3-colourable, so is G''. This proves that deciding whether a fragile graph is 3-colourable is NP-complete. By the same kind of argument, we can prove that deciding whether a graph is 3-colourable stays NP-complete even when we restrict ourselves to fragile triangle-free graphs. To see this, pick any graph G, remove all edges uv, and replace them by a copy of the graph G_1 from Figure 3 with a identified to u and b identified to v. This yields a triangle-free fragile graph that is 3-colourable if and only if G is 3-colourable.

Our proof that every fragile graph is 4-colourable yields an algorithm that actually computes a 4-colouring. A crude implementation of this algorithm would run in exponential time, but it is easy to turn it into a polynomial time algorithm by maintaining for each 2-tuples and 3-tuples X of vertices of the input graph, a colouring satisfying the constraints (1)–(4) when applicable to X.

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