Product structure of graphs with an excluded minor

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Abstract

This paper shows that K_t-minor-free (and $K_{s,t}$-minor-free) graphs G are subgraphs of products of a tree-like graph H (of bounded treewidth) and a complete graph K_m. Our results include optimal bounds on the treewidth of H and optimal bounds (to within a constant factor) on m in terms of the number of vertices of G and the treewidth of G. These results follow from a more general theorem whose corollaries include a strengthening of the celebrated separator theorem of Alon, Seymour, and Thomas [*J. Amer. Math. Soc.* 1990] and the Planar Graph Product Structure Theorem of Dujmović *et al.* [*J. ACM* 2020].

1 Introduction

Graph Product Structure Theory is a body of research which describes complicated graphs as subgraphs of products of simpler graphs. Typically, the simpler graphs are tree-like, in the sense that they have bounded treewidth, which is the standard measure of how similar a graph is to a tree. (We postpone the definition of treewidth and other standard graph-theoretic concepts until Section 2.) This area has recently received a lot of attention [2, 6, 7, 10, 15, 17, 19, 20, 25–27, 40] with highlights including the Planar Graph Product Structure Theorem of Dujmović et al. [15]; see Theorem 7 below.

Our main contribution is a powerful general result, Theorem 12, that essentially converts a tree-decomposition of a graph excluding a particular minor into a product that inherits some of the properties of the decomposition. Its applications include a strengthening of the celebrated Alon–Seymour–Thomas separator theorem as well as the Planar Graph Product Structure Theorem.

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Throughout the paper we work with strong products of graphs. The *strong product* $A \boxtimes B$ of graphs A and B has vertex-set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$. This paper focuses on products of the form $H \boxtimes K_m$ and $H \boxtimes P \boxtimes K_m$ where H is a graph of bounded treewidth, P is a path and m is some function of the original graph. An alternative view of the product $H \boxtimes K_m$ is as a 'blow-up' of the graph H, obtained by replacing each vertex of H be a copy of the complete graph K_m and each edge of H by a copy of the complete bipartite graph $K_{m,m}$.

In one of the cornerstone results of Graph Minor Theory, Alon, Seymour, and Thomas [1] proved that every K_t-minor-free graph has a balanced separator of size at most $t^{3/2}n^{1/2}$. In fact, they proved the following stronger result.[1]

Theorem 1 ([1])**.** *Every n-vertex K_t-minor-free graph G has treewidth* $\operatorname{tw}(G) < t^{3/2}n^{1/2}$.

Our first result is the following strengthening of Theorem 1 that describes K_t-minor-free graphs as blow-ups of simpler graphs, namely graphs with bounded treewidth.

Theorem 2. *For any integer $t \geqslant 4$, every n-vertex K_t-minor-free graph G is*

(a) *isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$, where $\operatorname{tw}(H) \leqslant t - 1$ and $m := \sqrt{(t-3)n}$;*

(b) *isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$, where $\operatorname{tw}(H) \leqslant t - 2$ and $m := 2\sqrt{(t-3)n}$.*

Theorem 2(a) immediately implies Theorem 1, since

$$\operatorname{tw}(G) \leqslant \operatorname{tw}(H \boxtimes K_{\lfloor m \rfloor}) \leqslant (\operatorname{tw}(H) + 1)m - 1 < t\sqrt{(t-3)n}.$$

The dependence on n in the blow-up factor m is best possible since the $n^{1/2} \times n^{1/2}$ planar grid graph G is K_5-minor-free and has treewidth $n^{1/2}$. If G is isomorphic to a subgraph of $H \boxtimes K_m$ where H has bounded treewidth, then $n^{1/2} \leqslant \operatorname{tw}(G) \leqslant (\operatorname{tw}(H) + 1)m - 1$ and so $m = \Omega(n^{1/2})$. The dependence on t is discussed in Section 6; see Q1 there.

While our proof of Theorem 2 uses some ideas from the proof of Theorem 1 (in particular, Lemma 10 below), it is in fact significantly simpler, avoiding the use of havens or any form of treewidth duality. Instead, the proof directly constructs an isomorphism from G to $H \boxtimes K_{\lfloor m \rfloor}$ where H is a graph obtained by repeated clique-sums (which implies the desired treewidth bound).

We also prove the following analogous theorem for excluded complete bipartite minors. Let $K^*_{s,t}$ be the graph whose vertex-set can be partitioned $A \cup B$, where $|A| = s$, $|B| = t$, A is a clique, B is an independent set, and every vertex in A is adjacent to every vertex in B, that is, $K^*_{s,t}$ is obtained from $K_{s,t}$ by adding all the edges inside the part of size s.

[1] The balanced separator result follows from Theorem 1 and the separator lemma of Robertson and Seymour [37, (2.6)].

Theorem 3. *For all integers $s, t \geqslant 2$, every n-vertex $K^*_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$, where $\text{tw}(H) \leqslant s$ and $m := 2\sqrt{(s-1)(t-1)n}$.*

Again the $n^{1/2} \times n^{1/2}$ planar grid (which is $K_{3,3}$-minor-free) shows the dependence on n in the blow-up factor is best possible—we must have $m = \Omega(n^{1/2})$.

In light of Theorem 1, it is natural to try to qualitatively strengthen Theorems 2 and 3 by bounding the blow-up factor by a function of the treewidth of G, and ideally by a linear function of $\text{tw}(G)$ since if $G \subseteq H \boxtimes K_m$ and $\text{tw}(H) = \mathcal{O}(1)$, then $m = \Omega(\text{tw}(G))$. In this direction, Campbell et al. [7, Thm. 18] proved that every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_m$ where $\text{tw}(H) \leqslant t-2$ and $m = \mathcal{O}_t(\text{tw}(G)^2)$. Similarly, they proved [7, Thm. 19] that every $K_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_m$ where $\text{tw}(H) \leqslant s$ and $m = \mathcal{O}_{s,t}(\text{tw}(G)^2)$. Here $\mathcal{O}_{s,t}(\cdot)$ and $\Omega_{s,t}(\cdot)$ hide dependence on s and t.

We achieve a blow-up factor that is linear in $\text{tw}(G)$, and is independent of t for K_t-minor-free graphs.

Theorem 4. *For any integer $t \geqslant 2$, every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_m$, where $\text{tw}(H) \leqslant t-2$ and $m := \text{tw}(G)+1$.*

The value of m in Theorem 4 is within a factor $t-1$ of best possible, since

$$\text{tw}(G) \leqslant \text{tw}(H \boxtimes K_m) \leqslant (\text{tw}(H)+1)m - 1 < (t-1)m.$$

Furthermore, the $t-2$ bound on the treewidth of H is best possible, since Campbell et al. [7, Thm. 18] proved that, for any function f and for all t, there is a K_t-minor-free graph G that is not a subgraph of $H \boxtimes K_{f(\text{tw}(G))}$ for any graph H with treewidth at most $t-3$.

For $K^*_{s,t}$-minor-free graphs we also obtain a blow-up factor that is linear in $\text{tw}(G)$.

Theorem 5. *For all integers $s, t \geqslant 2$, every $K^*_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_m$, where $\text{tw}(H) \leqslant s$ and $m := (t-1)(\text{tw}(G)+1)$.*

Here the value of m is within a factor $(s+1)(t-1)$ of best possible and the $\text{tw}(H) \leqslant s$ bound is best possible [7, Thm. 19].

An attraction of Theorems 3 and 5 is that $\text{tw}(H)$ depends on s and not on the size of the excluded minor. This is particularly relevant for graphs of Euler genus[2] g, since these contain no $K_{3,2g+3}$-minor. Thus the next result follow from Theorems 3 and 5.

[2] The *Euler genus* of a surface with h handles and c cross-caps is $2h+c$. The *Euler genus* of a graph G is the minimum integer $g \geqslant 0$ such that G embeds in a surface of Euler genus g; see [33] for more about graph embeddings in surfaces.

Corollary 6. *For any integer $g \geqslant 0$, every n-vertex graph G of Euler genus g is isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$, where $\operatorname{tw}(H) \leqslant 3$ and*

$$m := \min\{4\sqrt{(g+1)n},\ 2(g+1)(\operatorname{tw}(G)+1)\}.$$

Corollary 6 is a product strengthening of results about balanced separators (equivalently, about treewidth) in graphs embeddable on surfaces of genus g, independently due to Djidjev [11] and Gilbert, Hutchinson, and Tarjan [23]. In particular, Corollary 6 implies that $\operatorname{tw}(G) \leqslant (\operatorname{tw}(H)+1)m - 1 = 4m - 1 < 16\sqrt{(g+1)n}$ and that G has a balanced separator of size at most $4m \leqslant 16\sqrt{(g+1)n}$. Both these bounds are tight up to the multiplicative constant.

Theorems 4 and 5 are in fact special cases of a more general result, Theorem 12, that essentially converts any tree-decomposition of a graph excluding a particular minor into a strong product. The starting tree-decomposition may be chosen to suit one's needs. Making use of this flexibility, we deduce the Planar Graph Product Structure Theorem, Theorem 7(b).

Theorem 7 ([15])**.** *Every planar graph is isomorphic to a subgraph of:*

(a) *$H \boxtimes P$ for some graph H of treewidth 8 and for some path P.*

(b) *$H \boxtimes P \boxtimes K_3$ for some graph H of treewidth 3 and for some path P.*

Theorem 7 has been the key tool to resolve several open problems regarding queue layouts [15], nonrepetitive colouring [14], p-centred colouring [12], adjacency labelling [4, 13, 22], infinite graphs [28], twin-width [2, 5], and comparable box dimension [18].

The bound of 3 on the treewidth of H in (b) is tight [15] even if K_3 is replaced by any constant-sized complete graph. Note that $\operatorname{tw}(H \boxtimes K_3) \leqslant 3\operatorname{tw}(H) + 2$ for any graph H, so (b) implies (a) but with 8 replaced by 11. Our proof of Theorem 7(b) removes much of the topology from the original proof, avoiding the use of Sperner's planar triangulation lemma. This allows us to prove a more general $H \boxtimes P \boxtimes K_m$ structure theorem, Theorem 16, which we apply in the more general setting of apex-minor-free graphs, Theorem 20. This in turn has applications for p-centred colourings.

2 Preliminaries

We consider simple finite undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. For each vertex $v \in V(G)$, let $N_G(v) = \{w \in V(G) \colon vw \in E(G)\}$. For $S \subseteq V(G)$, let $N_G(S) = \bigcup\{N_G(v) \colon v \in S\} \setminus S$.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Say G is *H-minor-free* if H is not a minor of G. A *K_r-model* in a graph G consists of pairwise-disjoint vertex-sets $(U_1, \dots, U_r)$ such that, for each i, the induced subgraph $G[U_i]$ is connected and, for all distinct i, j, there is an edge between U_i and U_j. Clearly K_r is a minor of a graph G if and only if G contains a K_r-model.

2.1 Tree-decompositions and treewidth

A *tree-decomposition* $(T, \mathcal{W})$ of a graph G consists of a collection $\mathcal{W} = (W_x \colon x \in V(T))$ of subsets of $V(G)$, called *bags*, indexed by the nodes of a tree T, such that:

- for each vertex $v \in V(G)$, the set $\{x \in V(T) \colon v \in W_x\}$ induces a non-empty (connected) subtree of T; and
- for each edge $vw \in E(G)$, there is a node $x \in V(T)$ for which $v, w \in W_x$.

The *width* of such a tree-decomposition is $\max\{|W_x| \colon x \in V(T)\} - 1$. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [3, 24, 36] for surveys.

We use the following property to prove treewidth upper bounds. A graph G is a *clique-sum* of graphs G_1 and G_2, if for some clique $\{v_1, \dots, v_k\}$ in G_1 and for some clique $\{w_1, \dots, w_k\}$ in G_2, G is obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for each i.[3] In this case, it is well known and easily seen that $\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$.

2.2 Partitions

Instead of working with products, it is convenient to present our proofs using the following definition. A *partition* of a graph G is a graph H such that:

- each vertex of H is a set of vertices of G,
- each vertex of G is in exactly one vertex of H, and
- for each edge vw of G, if $v \in X \in V(H)$ and $w \in Y \in V(H)$ then $XY \in E(H)$ or $X = Y$.

[3] It is common in the literature for clique-sums to allow the deletion of edges after the identification. In this paper we do not allow this.

We call the vertices of H the *parts* of the partition. The *width* of a partition is the size of its largest part. The *treewidth* of a partition H is $\text{tw}(H)$. The next observation follows from the definitions and gives a useful characterisation of when a graph is isomorphic to a subgraph of a product of the form $H \boxtimes K_m$.

Observation 8. *A graph G has a partition H of width at most m if and only if G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$.*

In light of Observation 8, to prove our results it suffices to find a suitable partition. The following definition enables inductive proofs. A partition H of a graph G is *rooted* at a K_r-model $(U_1, \dots, U_r)$ in G if $U_1, \dots, U_r$ are vertices of H. Note that $U_1, \dots, U_r$ must be the vertices of an r-clique in H.

Finally, it will be useful to measure the 'complexity' of a vertex-set with respect to a tree-decomposition $(T, \mathcal{W})$ of G. For a vertex-set $S \subseteq V(G)$, the *$\mathcal{W}$-width* of S is the minimum number of bags of $\mathcal{W}$ whose union contains S. The *$\mathcal{W}$-width* of a collection of vertex-sets is the maximum $\mathcal{W}$-width of one of its sets. In a slight abuse of terminology, the *$\mathcal{W}$-width* of a partition H of G is the maximum $\mathcal{W}$-width of one of the vertices of H.

2.3 Hitting sets

Our proofs use results that say a collection of connected subgraphs of a graph (satisfying certain conditions) either has a small 'hitting set' (a small set of vertices that meets every subgraph in the collection) or contains some suitable graphs. The following folklore lemma (see [38, (8.7)]) essentially says that complements of chordal graphs are perfect. We include the proof for completeness.

Lemma 9. *For any integer $\ell \geqslant 0$ and any collection $\mathcal{F}$ of subtrees of a tree T, either*:

- (a) *there are $\ell + 1$ vertex-disjoint trees in $\mathcal{F}$, or*
- (b) *there is set S of at most ℓ vertices such that $S \cap V(T') \neq \varnothing$ for all $T' \in \mathcal{F}$.*

Proof. We proceed by induction on $|V(T)|$. The $|V(T)| = 1$ case is immediate. Let x be a leaf of T and y its unique neighbour. Let $T' \coloneqq T - x$.

First suppose that $T[\{x\}]$ is not in $\mathcal{F}$. Let $\mathcal{F}'$ be obtained by removing x from every tree in $\mathcal{F}$. By induction, either (a) or (b) occurs for $\mathcal{F}'$ and T'. If (a) occurs, then the corresponding trees in $\mathcal{F}$ are also vertex-disjoint (since if two trees of $\mathcal{F}$ contain x, then they also both contain y). If (b) occurs, then the set obtained also meets every tree in $\mathcal{F}$.

Second suppose that $T[\{x\}]$ is in $\mathcal{F}$. Let $\mathcal{F}''$ be the set of all trees in $\mathcal{F}$ that do not contain x. So $\mathcal{F}''$ is a collection of subtrees of T'. Now apply induction to $\mathcal{F}''$ and T' with $\ell - 1$ in place of ℓ. If (a) occurs, then these trees together with $T[\{x\}]$ are $\ell + 1$ vertex-disjoint trees in $\mathcal{F}$. If (b) occurs, then this set together with x meets every tree in $\mathcal{F}$. $\square$

In the setting of $\mathcal{O}(\sqrt{n})$ blow-ups we need the following hitting set lemma due to Alon, Seymour, and Thomas [1]. Let $\mathcal{F}$ be the collection of connected subgraphs of G that intersect all of $A_1, \dots, A_k$. Lemma 10 says that $\mathcal{F}$ either contains a small graph or has a small hitting set.

Lemma 10 ([1, (2.1)]). *Let G be a graph, $A_1, \dots, A_k$ be non-empty subsets of $V(G)$, and $x \geqslant 1$ be a real. Then either:*

- (a) *there is a subtree X of G with $|V(X)| \leqslant x$ such that $V(X) \cap A_i \neq \varnothing$ for each i, or*
- (b) *there is a set Y of at most $(k-1)|V(G)|/x$ vertices such that no component of $G - Y$ intersects all of $A_1, \dots, A_k$.*

The next result is a straightforward extension of Lemma 10.

Lemma 11. *Let G be a graph, $A_1, \dots, A_k$ be non-empty subsets of $V(G)$, $x \geqslant 1$ be a real, and $\ell \geqslant 1$ be an integer. Then either:*

- (a) *there are pairwise disjoint trees $X_1, \dots, X_\ell$ in G with $|V(X_j)| \leqslant x$ and such that $V(X_j) \cap A_i \neq \varnothing$ for each i and j, or*
- (b) *there is a set Y of at most $(\ell - 1)x + (k-1)|V(G)|/x$ vertices such that no component of $G - Y$ intersects all of $A_1, \dots, A_k$.*

Proof. We proceed by induction on ℓ. Lemma 10 proves the result if $\ell = 1$. Now assume that $\ell \geqslant 2$ and the result holds for $\ell - 1$. If outcome (b) holds for $\ell - 1$, then the same set Y satisfies outcome (b) for ℓ. So assume that (a) holds for $\ell - 1$. That is, there are pairwise disjoint trees $X_1, \dots, X_{\ell-1}$ in G with $|V(X_j)| \leqslant x$ and such that $V(X_j) \cap A_i \neq \varnothing$ for each i and j. Apply Lemma 10 to $G' \coloneqq G - V(X_1 \cup \dots \cup X_{\ell-1})$. If there is a tree X_ℓ in G' with $|V(X_\ell)| \leqslant x$ such that $V(X_\ell) \cap A_i \neq \varnothing$ for each i, then $X_1, \dots, X_\ell$ are the desired set of trees, and outcome (a) holds. Otherwise there exists $Y' \subseteq V(G')$ with $|Y'| \leqslant (k-1)|V(G)|/x$ such that no component of $G' - Y'$ intersects all of $A_1, \dots, A_k$. Let $Y \coloneqq V(X_1 \cup \dots \cup X_{\ell-1}) \cup Y'$. Thus $|Y| \leqslant (\ell-1)x + (k-1)|V(G)|/x$ and no component of $G - Y$ intersects all of $A_1, \dots, A_k$ (since $G' - Y' = G - Y$). That is, Y satisfies (b). $\square$

3 Main theorem and $\mathcal{O}(\text{tw}(G))$ blow-up

We now prove our main technical theorem and deduce Theorems 4 and 5 from it.

The following definition allows the K_t-minor-free and $K^*_{s,t}$-minor-free cases to be combined. Let $\mathcal{J}_{s,t}$ be the class of graphs G whose vertex-set has a partition $A \cup B$, where $|A| = s$ and $|B| = t$, A is a clique, every vertex in A is adjacent to every vertex in B, and $G[B]$ is connected. A graph is *$\mathcal{J}_{s,t}$-minor-free* if it contains no graph in $\mathcal{J}_{s,t}$ as a minor. The following is our main theorem.

Theorem 12. *Let $s, t \geqslant 2$ be integers, G be a $\mathcal{J}_{s,t}$-minor-free graph, and $(T, \mathcal{W})$ be a tree-decomposition of G. Then G has a partition of $\mathcal{W}$-width at most $t-1$ and treewidth at most s.*

This says that, given a $\mathcal{J}_{s,t}$-minor-free G and a tree-decomposition $(T, \mathcal{W})$ of G, there is a simple (low treewidth) partition that is also simple with respect to $\mathcal{W}$. Theorem 12 follows immediately from the next lemma (for example, by taking $r = 1$ and U_1 to consist of a single vertex).

Lemma 13. *Let $s, t \geqslant 2$ be integers, G be a $\mathcal{J}_{s,t}$-minor-free graph, and $(T, \mathcal{W})$ be a tree-decomposition of G. Suppose that $(U_1, \dots, U_r)$ is a K_r-model of $\mathcal{W}$-width at most $t-1$ where $r \leqslant s$. Then G has a partition of $\mathcal{W}$-width at most $t-1$ and treewidth at most s that is rooted at $(U_1, \dots, U_r)$.*

Proof. Let $U \coloneqq U_1 \cup \dots \cup U_r$. We proceed by induction on $|V(G)|$. If $V(G) = U$, then $(U_1, \dots, U_r)$ is the desired partition H where $H = K_r$ has treewidth $r - 1 \leqslant s$. Now assume that $V(G) \setminus U \neq \varnothing$. Let $A_i \coloneqq N_G(U_i) \setminus U$ for each i.

First suppose that some A_i is empty, say $A_1 = \varnothing$. By induction, $G - U_1$ has a partition H_1 of $\mathcal{W}$-width at most $t-1$ and treewidth at most s that is rooted at $(U_2, \dots, U_r)$. Add a new part U_1 adjacent to each of $U_2, \dots, U_r$ to obtain the desired H-partition of G. The neighbourhood of U_1 is a clique on $r - 1$ vertices, so $\text{tw}(H) = \max\{\text{tw}(H_1), r - 1\} \leqslant s$. Thus we may assume that A_i is non-empty for all i.

Next suppose that $G - U$ is disconnected. Then there is a partition U, V_1, V_2 of $V(G)$ into three non-empty sets such that there is no edge between V_1 and V_2. Let $G_1 \coloneqq G[U \cup V_1]$ and $G_2 \coloneqq G[U \cup V_2]$. For $j \in \{1, 2\}$, let $\mathcal{W}_j$ be the tree-decomposition of G_j obtained from $\mathcal{W}$ by deleting all the vertices of G not in G_j. By induction, each G_j has a partition H_j of $\mathcal{W}_j$-width at most $t-1$ and treewidth at most s that is rooted at $(U_1, \dots, U_r)$. Let H be the partition of G obtained from H_1 and H_2 by identifying the vertex U_i in H_1 with the vertex U_i in H_2 for each i. The graph H is a clique-sum of H_1 and H_2, so $\text{tw}(H) = \max\{\text{tw}(H_1), \text{tw}(H_2)\} \leqslant s$. Since every bag of $\mathcal{W}_1$ and $\mathcal{W}_2$ is a subset of a bag of $\mathcal{W}$, the partition H has $\mathcal{W}$-width at most $t-1$. Thus we may assume that $G-U$ is connected.

We now show there exists a set $Y \subseteq V(G) \setminus U$ of $\mathcal{W}$-width at most $t-1$ such that

$$\text{no component of } G-U-Y \text{ meets every } A_i. \tag{†}$$

Let $\mathcal{F}$ be the collection of all connected subgraphs F of $G-U$ such that $V(F) \cap A_i \neq \varnothing$ for all i. For each $F \in \mathcal{F}$, let $T_F \coloneqq T[\{x \in V(T) \colon W_x \cap V(F) \neq \varnothing\}]$. Since F is connected, T_F is a (connected) subtree of T.

First consider the case $r \leqslant s-1$.

First suppose there exists $F_1, F_2 \in \mathcal{F}$ such that T_{F_1} and T_{F_2} are disjoint. Let xy be any edge of T on the shortest path between T_{F_1} and T_{F_2}. Then $W_x \cap W_y$ separates[4] $V(F_1)$ and $V(F_2)$. Let S be a minimal subset of $W_x \cap W_y$ that separates $V(F_1)$ and $V(F_2)$. By construction, S has $\mathcal{W}$-width 1, $S \cap V(F_1) = \varnothing$, and $S \cap V(F_2) = \varnothing$. Then there is a partition $S \cup V_1 \cup V_2$ of $V(G) \setminus U$ such that $V(F_1) \subseteq V_1$, $V(F_2) \subseteq V_2$ and there is no edge between V_1 and V_2. We now show that $G[S \cup V_1]$ and $G[S \cup V_2]$ are connected. Consider some $s \in S$. Since S is minimal, there is a path from s to $V(F_1)$ internally disjoint from $S \cup V(F_2)$. Since there is no edge between V_1 and V_2, this path must lie entirely inside $S \cup V_1$. Since F_1 is connected, between any two vertices of S there is a path entirely inside $S \cup V_1$. Since $G-U$ is connected, there is a path from any vertex of V_1 to S inside $S \cup V_1$. Hence $G[S \cup V_1]$ is connected. Similarly for $G[S \cup V_2]$. For $j \in \{1,2\}$, let G_j be the graph obtained from G by contracting all of $S \cup V_j$ into a single vertex v_j. Each G_j is a minor of G and thus is $\mathcal{J}_{s,t}$-minor-free. Furthermore, since $V(F_j) \subseteq V_j$, $(U_1, \dots, U_r, \{v_j\})$ is a K_{r+1}-model in G_j. Let $\mathcal{W}_j$ be the tree-decomposition of G_j obtained from $\mathcal{W}$ by replacing every instance of a vertex in $S \cup V_j$ by v_j. By induction, each G_j has a partition H_j of $\mathcal{W}_j$-width at most $t-1$ and treewidth at most s that is rooted at $(U_1, \dots, U_r, \{v_j\})$. Let H be obtained from the disjoint union of H_1 and H_2 by identifying corresponding U_i, and identifying v_1 and v_2 into a single vertex S. If $X \subseteq V(G_j) \setminus \{v_j\}$ is a subset of a bag of $\mathcal{W}_j$, then X is a subset of a bag of $\mathcal{W}$. So if $X \subseteq V(G_j) \setminus \{v_j\}$ has $\mathcal{W}_j$-width at most $t-1$, then X has $\mathcal{W}$-width at most $t-1$. Since S also has $\mathcal{W}$-width $1 \leqslant t-1$, the partition H has $\mathcal{W}$-width at most $t-1$. The graph H is a clique-sum of H_1 and H_2, so $\operatorname{tw}(H) \leqslant \max\{\operatorname{tw}(H_1), \operatorname{tw}(H_2)\} \leqslant s$ and the partition has all the required properties.

Now assume that T_{F_1} and T_{F_2} intersect for all $F_1, F_2 \in \mathcal{F}$. By the Helly property, there is a node $x \in V(T)$ such that $x \in V(T_F)$ for all $F \in \mathcal{F}$. Let $Y \coloneqq W_x$. Then Y has $\mathcal{W}$-width 1 and intersects every $F \in \mathcal{F}$. Thus $G-U-Y$ contains no graph of $\mathcal{F}$ and

[4]Given a graph G and $V_1, V_2 \subseteq V(G)$, a set S *separates* V_1 and V_2 if no connected component of $G-S$ contains a vertex of both V_1 and V_2.

so every component of $G - U - Y$ avoids some A_i. This Y satisfies (†).

Now consider the case $r = s$.

Suppose that $\mathcal{F}$ contains t vertex-disjoint graphs $F_1, \dots, F_t$. Since $G - U$ is connected, there is a partition $Q_1, \dots, Q_t$ of $V(G) \setminus U$ such that $V(F_i) \subseteq Q_i$ and $G[Q_i]$ is connected, for all i. Contract each Q_i to a single vertex q_i and each U_i to a single vertex u_i to get a graph G' with vertex-set $\{u_1, \dots, u_s, q_1, \dots, q_t\}$. Since $G - U$ is connected, $G'[\{q_1, \dots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{s,t}$, a contradiction. Hence, there are no t vertex-disjoint graphs in $\mathcal{F}$. For any $F_1, F_2 \in \mathcal{F}$, if T_{F_1} and T_{F_2} are disjoint, then F_1 and F_2 are disjoint. So $\{T_F : F \in \mathcal{F}\}$ contains no t pairwise disjoint subtrees. Thus, by Lemma 9, there is a set $S \subseteq V(T)$ of size at most $t - 1$ that meets every T_F. Let $Y \coloneqq \bigcup_{x \in S} W_x$. Then Y has $\mathcal{W}$-width at most $t - 1$ and intersects every $F \in \mathcal{F}$. This Y satisfies (†).

We have shown in all cases that there exists $Y \subseteq V(G) \setminus U$ satisfying (†). Take a minimal such Y. Let $G_1, \dots, G_r$ be unions of components of $G - U - Y$ such that $V(G_1), \dots, V(G_r)$ is a vertex-partition of $V(G) - U - Y$ and $V(G_j) \cap A_j = \varnothing$ for each j. Some G_j may be empty; ignore such indices henceforth. Fix j and consider $w \in Y$. Since Y is minimal, there is a component of $G - U - (Y \setminus \{w\})$ that meets every A_i. Since Y satisfies (†), this component contains w. In particular, there is a path P_w from w to A_j in $G - U - (Y \setminus \{w\})$. P_w cannot meet G_j otherwise $G - U - Y$ has a component meeting A_j and G_j. Hence, for every $w \in Y$, there is a path P_w from w to A_j that avoids $V(G_j) \cup U$. Let Z_j be the subgraph induced by the union of U_j and all P_w (where $w \in Y$). By construction, Z_j is connected and disjoint from $V(G_j) \cup (U \setminus U_j)$.

Take the subgraph of G induced by $V(G_j) \cup Z_j \cup U$ and contract Z_j into a new vertex z_j. Call the graph obtained G'_j, which has vertex-set $V(G_j) \cup (U \setminus U_j) \cup \{z_j\}$. Now $(U_i : i \neq j, \{z_j\})$ is a K_r-model in G'_j. Let $\mathcal{W}_j$ be the tree-decomposition of G'_j obtained from $\mathcal{W}$ by deleting vertices of G not in $V(G_j) \cup Z_j \cup U$, and then replacing each vertex in Z_j by z_j. By induction, G'_j has a partition H_j of $\mathcal{W}_j$-width at most $t-1$ and treewidth at most s that is rooted at $(U_i : i \neq j, \{z_j\})$. Add to H_j the vertex U_j adjacent to all other U_i and to $\{z_j\}$. Since the neighbourhood of this added vertex is a clique of order $r \leqslant s$, H_j still has treewidth at most s. Let H be obtained from the disjoint union of $H_1, \dots, H_r$, by identifying corresponding U_i, and identifying $z_1, \dots, z_r$ into a single vertex Y. Note that if $X \subseteq V(G_j) \setminus \{z_j\}$ is a subset of a bag of $\mathcal{W}_j$, then X is a subset of a bag of $\mathcal{W}$. So if $X \subseteq V(G_j) \setminus \{z_j\}$ has $\mathcal{W}_j$-width at most $t-1$, then X has $\mathcal{W}$-width at most $t - 1$. Since Y has $\mathcal{W}$-width at most $t - 1$, the partition H has $\mathcal{W}$-width at most $t - 1$. The graph H is a clique-sum of $H_1, \dots, H_r$, so $\operatorname{tw}(H) \leqslant \max_j \operatorname{tw}(H_j) \leqslant s$.

We finally check that H is a partition of G. The vertices $U_1, \dots, U_r, Y$ form a clique in H, so all edges of G inside $Y \cup U$ appear in H. Every edge inside G_j appears in $G'_j - z_j$, thus appears in H_j and hence in H. Any edge between U and G_j is, by definition of G_j, an edge between G_j and $U \setminus U_j$ so appears in $G'_j - z_j$ and hence in H. Finally consider edges between Y and G_j. Let vw be an edge with $v \in V(G_j)$ and $w \in Y$. Note that $w \in Z_j$ and so the edge vz_j is present in G'_j and hence in H_j. Since z_j is replaced by Y, the edge vw is in H. □

Applying Theorem 12 to a tree-decomposition of minimum width gives the following.

Theorem 14. *For all integers $s, t \geqslant 2$, every $\mathcal{J}_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_m$, where $\operatorname{tw}(H) \leqslant s$ and $m := (\operatorname{tw}(G) + 1)(t - 1)$.*

Proof. Let G be a $\mathcal{J}_{s,t}$-minor-free graph. Fix a tree-decomposition $(T, \mathcal{W})$ of G in which every bag has size at most $\operatorname{tw}(G) + 1$. By Theorem 12, G has a partition H of $\mathcal{W}$-width at most $t - 1$ where $\operatorname{tw}(H) \leqslant s$. Since each bag of $\mathcal{W}$ has size at most $\operatorname{tw}(G) + 1$, the partition has width at most $(t - 1)(\operatorname{tw}(G) + 1) = m$. Hence, by Observation 8, G is isomorphic to a subgraph of $H \boxtimes K_m$. □

Observe that $\mathcal{J}_{t-2,2} = \{K_t\}$ so every K_t-minor-free graph is $\mathcal{J}_{t-2,2}$-minor-free. Hence Theorem 14 implies Theorem 4. Clearly, $K^*_{s,t}$ is a subgraph of every graph in $\mathcal{J}_{s,t}$ and so every $K^*_{s,t}$-minor-free graph is $\mathcal{J}_{s,t}$-minor-free. Hence, Theorem 14 implies Theorem 5.

4 Layered treewidth: planar and apex-minor-free graphs

A *layering* of a graph G is a partition $\mathcal{L} = (V_1, V_2, \dots)$ of $V(G)$ such that for each edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. A layering of G is equivalent to a partition P of G where P is a path. The next observation, first noted in [15], gives a useful characterisation of when a graph is isomorphic to a subgraph of a product of the form $H \boxtimes P \boxtimes K_m$.

Observation 15 ([15])**.** *A graph G has a layering $\mathcal{L}$ and a partition H such that each layer of $\mathcal{L}$ and each part of H intersect in at most m vertices if and only if G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$ for some path P.*

Proof. Suppose that G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$ where $V(H) = \{x_1, \dots, x_h\}$, $V(P) = \{y_1, y_2, \dots\}$, and $V(K_m) = \{z_1, \dots, z_m\}$. Then the isomorphism maps each vertex v of G to $(x_{a(v)}, y_{b(v)}, z_{c(v)})$ where $v \mapsto (a(v), b(v), c(v))$ is injective. Let $\mathcal{L}$ have layers $V_i = \{v \colon b(v) = i\}$ and the partition H have parts $\{v \colon a(v) = j\}$ for $j \in \{1, \dots, h\}$. Since $c(v)$ takes at most m values, each layer and part have at most m vertices in common.

Reversing this identification converts a suitable layering $\mathcal{L}$ and partition H into an isomorphism from G to a subgraph of $H \boxtimes P \boxtimes K_m$. $\square$

The *layered treewidth* $\operatorname{ltw}(G)$ of a graph G is the minimum integer k for which G has a layering $\mathcal{L}$ and tree-decomposition $(T, \mathcal{W})$ such that $|L \cap W| \leqslant k$ for each layer $L \in \mathcal{L}$ and each bag $W \in \mathcal{W}$. This notion was independently introduced by Dujmović, Morin, and Wood [16] and Shahrokhi [39]. Theorem 12 has the following corollary.

Theorem 16. *For all integers $s, t \geqslant 2$, every $\mathcal{J}_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where P is a path, $\operatorname{tw}(H) \leqslant s$, and $m := (t-1)\operatorname{ltw}(G)$.*

Proof. Let G be a $\mathcal{J}_{s,t}$-minor-free graph. Fix a layering $\mathcal{L}$ and tree-decomposition $(T, \mathcal{W})$ of G such that $|L \cap W| \leqslant \operatorname{ltw}(G)$ for every layer $L \in \mathcal{L}$ and each bag $W \in \mathcal{W}$. By Theorem 12, G has a partition H of $\mathcal{W}$-width at most $t-1$ where $\operatorname{tw}(H) \leqslant s$.

Let $X \in V(H)$ be a part and $L \in \mathcal{L}$ be a layer. Since the partition has $\mathcal{W}$-width at most $t-1$, there are bags $W_1, \dots, W_{t-1} \in \mathcal{W}$ such that $X \subseteq \bigcup_{i=1}^{t-1} W_i$. Since $|L \cap W_i| \leqslant \operatorname{ltw}(G)$ for each i, $|X \cap L| \leqslant (t-1)\operatorname{ltw}(G)$. The result now follows from Observation 15. $\square$

Again, since $\mathcal{J}_{t-2,2} = \{K_t\}$ and $K^*_{s,t}$ is a subgraph of every graph in $\mathcal{J}_{s,t}$, Theorem 16 has the following corollaries.

Theorem 17. *For any integer $t \geqslant 2$, every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where P is a path, $\operatorname{tw}(H) \leqslant t-2$, and $m := \operatorname{ltw}(G)$.*

Theorem 18. *For all integers $s, t \geqslant 2$, every $K^*_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where P is a path, $\operatorname{tw}(H) \leqslant s$, and $m := (t-1)\operatorname{ltw}(G)$.*

The Planar Graph Product Structure Theorem (Theorem 7(b)) follows from Theorem 17 (with $t = 5$) and the fact that every planar graph has layered treewidth at most 3, as proved by Dujmović et al. [16]. We sketch the proof for completeness.

Theorem 19 ([16, Thm. 12])**.** *Every planar graph has layered treewidth at most* 3.

Proof Sketch. We may assume that G is a planar triangulation. Let T be a breadth-first-search spanning tree rooted at an arbitrary vertex r. Let G^* be the dual of G and T^* be the spanning subgraph of G^* consisting of those edges not dual to edges in T. Von Staudt [41] showed that T^* is a spanning tree of G^*. For each vertex x of T^*, corresponding to face uvw of G, let W_x be the union of the ur-path in T, the vr-path in T, and the wr-path in T. Eppstein [21] showed that $(W_x \colon x \in V(T^*))$ is a tree-decomposition of G. Let $V_i := \{v \in V(G) \colon \operatorname{dist}_G(v, r) = i\}$ and so $(V_0, V_1, \dots)$ is a layering of G. Since T is a breadth-first-search spanning tree, each bag W_x has at most three vertices in each layer V_i. Hence $\operatorname{ltw}(G) \leqslant 3$. $\square$

We now show that the bound in Theorem 19 is tight. Suppose on the contrary that $\operatorname{ltw}(G) \leqslant 2$ for every planar graph G. Then each layer induces a subgraph with treewidth 1, which is thus a forest. Taking alternate layers, G has a vertex-partition into two induced forests (which would imply the 4-colour theorem). Chartrand and Kronk [8] constructed planar graphs G that have no vertex-partition into two induced forests, implying $\operatorname{ltw}(G) \geqslant 3$.

Theorem 7 is generalised as follows. The *vertex-cover number $\tau(G)$* of a graph G is the size of a smallest set $S \subseteq V(G)$ such that every edge of G has at least one end-vertex in S. By definition, G is a subgraph of every graph in $\mathcal{J}_{\tau(G),|V(G)|-\tau(G)}$. A graph X is *apex* if $X - v$ is planar for some vertex $v \in V(X)$. Dujmović et al. [16] showed that for any graph X, the class of X-minor-free graphs has bounded layered treewidth if and only if X is apex. Thus, the next result follows from Theorem 18.

Theorem 20. *For every apex graph X there exists $m \in \mathbb{N}$, such that every X-minor-free graph is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where P is a path and $\operatorname{tw}(H) \leqslant \tau(X)$.*

Dujmović et al. [15] proved a similar result to Theorem 20, but with a much larger bound on $\operatorname{tw}(H)$ (depending on constants from the Graph Minor Structure Theorem).

Theorem 20 has applications to p-centred colouring, as we now explain. For $p \in \mathbb{N}$, a vertex colouring of a graph G is *p-centred* if for every connected subgraph X of G, X receives more than p colours or some vertex in X receives a unique colour. The *p-centred chromatic number $\chi_p(G)$* is the minimum number of colours in a p-centred colouring of G. Centred colourings are important within graph sparsity theory as they characterise graph classes with bounded expansion [34]. A result of Dębski, Felsner, Micek, and Schröder [12, Lem. 8] implies that $\chi_p(H \boxtimes P \boxtimes K_m) \leqslant m(p+1)\chi_p(H)$ for every graph H. Pilipczuk and Siebertz [35, Lem. 15] proved that every graph of treewidth at most t has p-centred chromatic number at most $\binom{p+t}{t} \leqslant (p+1)^t$. In particular, Theorem 20 implies:

Theorem 21. *For every apex graph X with $\tau(X) \leqslant t$ there exists $m \in \mathbb{N}$ such that for every X-minor-free graph G,*

$$\chi_p(G) \leqslant m(p+1)^{t+1}.$$

Pilipczuk and Siebertz [35] proved that for every graph X there exists c such that every X-minor-free graph has p-centred chromatic number $\mathcal{O}(p^c)$. However, the known bounds on c are huge (depending on the Graph Minor Structure Theorem). Theorem 21 provides much improved bounds in the case of apex-minor-free graphs. As an example, since $K^*_{3,t}$ is apex with $\tau(K^*_{3,t}) \leqslant 3$, Theorem 21 implies there exists $m = m(t)$ such that $\chi_p(G) \leqslant m(p+1)^4$ for every $K^*_{3,t}$-minor-free graph G. This bound is only slightly greater than the best bound for planar graphs of $\mathcal{O}(p^3 \log p)$, and for graphs of Euler genus g (which are $K_{3,2g+3}$-minor-free) of $\mathcal{O}(gp + p^3 \log p)$, both due to Dębski et al. [12].

5 Blow-up $\mathcal{O}(\sqrt{n})$

In this section we employ a similar proof strategy but with a different hitting result (Lemma 11 in place of Lemma 9) to prove Theorems 2 and 3.

Theorem 22. *Let s, t, n be positive integers and define*

$$m := \begin{cases} \max\{t-1, 1\} & \text{if } s = 1 \text{ or } 2, \\ \sqrt{(s-2)n} & \text{if } s \geqslant 3 \text{ and } t = 1, \\ 2\sqrt{(s-1)(t-1)n} & \text{otherwise.} \end{cases}$$

Then every $\mathcal{J}_{s,t}$-minor-free graph G on n vertices is isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$ for some graph H of treewidth at most s.

Theorem 22 implies Theorems 2 and 3 since $\mathcal{J}_{t-1,1} = \mathcal{J}_{t-2,2} = \{K_t\}$ and $K^*_{s,t}$ is a subgraph of every graph in $\mathcal{J}_{s,t}$. Theorem 22 is implied by Observation 8 and the following lemma.

Lemma 23. *Let s, t, n be positive integers and define m as in Theorem 22. Suppose G is a $\mathcal{J}_{s,t}$-minor-free graph on n vertices and $(U_1, \dots, U_r)$ is a K_r-model in G where $r \leqslant s$ and $|U_i| \leqslant m$ for all i. Then G has a partition of width at most m and treewidth at most s that is rooted at $(U_1, \dots, U_r)$.*

Proof. Let $U := U_1 \cup \cdots \cup U_r$. We proceed by induction on n. If $n \leqslant r + m$, then the partition $(U_1, \dots, U_r, V(G) \setminus U)$ is the desired partition H where $H = K_{r+1}$ has treewidth $r \leqslant s$. Now assume that $n > r + m$. Note that if $n \leqslant t - 1$, then $n \leqslant m$ in all cases and so we may assume that $n > t - 1$. Let $A_i := N_G(U_i) \setminus U$ for each i.

By the same argument used in the proof of Lemma 13, we may assume that A_i is non-empty for all i and that $G - U$ is connected.

If $r \leqslant s - 1$ and there is some U_{r+1} of size at most m such that $(U_1, \dots, U_{r+1})$ is a K_{r+1}-model in G, then Lemma 23 for $U_1, \dots, U_{r+1}$ would imply it is also true for $U_1, \dots, U_r$ (with the same partition). In particular, if $r \leqslant s - 1$, then we may assume there is no U_{r+1} of size at most m such that $(U_1, \dots, U_{r+1})$ is a K_{r+1}-model in G. Call this property the 'maximality of r'.

We now show there exists a set $Y \subseteq V(G) \setminus U$ of size at most m such that

$$\text{no component of } G - U - Y \text{ meets every } A_i. \tag{‡}$$

First suppose that $s = 1$ and so $U = U_1$. Suppose that $|A_1| \geqslant t$. Let $v_1, \ldots, v_t$ be distinct vertices in A_1. Since $G - U$ is connected, it is possible to partition $V(G) \setminus U$ into vertex-sets $Q_1, \ldots, Q_t$ such that for all i, $v_i \in Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i into a single vertex q_i and U_1 into a single vertex u_1 to get a graph G' on vertex-set $\{u_1, q_1, \ldots, q_t\}$. Since $G - U$ is connected, $G'[\{q_1, \ldots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{1,t}$, a contradiction. Hence $|A_1| \leqslant t - 1 \leqslant m$. Then $Y = A_1$ satisfies (‡).

Next suppose that $s = 2$. If $r = 1$, then for any $x \in A_1$, the pair $(U_1, \{x\})$ is a K_2-model in G, which contradicts the maximality of r. Hence $r = 2$ and $U = U_1 \cup U_2$. Suppose $G - U$ contains t pairwise vertex-disjoint paths $P_1, \ldots, P_t$ from A_1 to A_2. Since $G - U$ is connected, there is a partition of $V(G) \setminus U$ into vertex-sets $Q_1, \ldots, Q_t$ such that, for all i, $V(P_i) \subseteq Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i to a single vertex q_i and each U_i to a single vertex u_i to get a graph G' on vertex-set $\{u_1, u_2, q_1, \ldots, q_t\}$. Since $G - U$ is connected, $G'[\{q_1, \ldots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{2,t}$, a contradiction. Thus, by Menger's theorem, there is a set $Y \subseteq V(G) \setminus U$ of size at most $t - 1 \leqslant m$ such that there is no path from A_1 to A_2 in $G - U - Y$. In particular, no component of $G - U - Y$ meets both A_1 and A_2 and so Y satisfies (‡). Thus we may assume that $s \geqslant 3$.

Suppose that $r \leqslant s - 1$. Apply Lemma 10 to $G - U$ with $x = \sqrt{(s-2)n} \geqslant 1$ and $k = r$. If (a) occurs, then there is a tree T on at most $x \leqslant m$ vertices intersecting each A_i. Then $(U_1, \ldots, U_r, T)$ is a K_{r+1}-model in G with all parts of size at most m, which contradicts the maximality of r. Hence, (b) occurs. That is, there is a vertex-set Y of size at most $(r-1)n/x \leqslant (s-2)n/x = x \leqslant m$ such that no component of $G - U - Y$ meets every A_i. This Y satisfies (‡).

Now assume that $r = s$. For $t = 1$ we are done: since $G - U$ is connected, contracting each of $U_1, \ldots, U_s, G - U$ to a single vertex gives a K_{s+1}-minor in G, which is a contradiction since $K_{s+1} \in \mathcal{J}_{s,1}$. Thus $t \geqslant 2$. Apply Lemma 11 to $G - U$ with $\ell = t$, $k = r = s$ and $x = \sqrt{\frac{s-1}{t-1}n} > 1$. Suppose (a) occurs. Then there are pairwise disjoint trees $T_1, \ldots, T_t$ in $G - U$ such that each T_j meets each A_i. Since $G - U$ is connected, it is possible to partition $V(G) \setminus U$ into vertex-sets $Q_1, \ldots, Q_t$ such that, for all i, $V(T_i) \subseteq Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i to a single vertex q_i and each U_i to a single vertex u_i to get a graph G' on vertex-set $\{u_1, \ldots, u_s, q_1, \ldots, q_t\}$. Since $G - U$ is connected, $G'[\{q_1, \ldots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{s,t}$, a contradiction. Hence, (b) occurs: there is a vertex-set Y of size at most $(t-1)x + (s-1)n/x = m$ such that no component of $G - U - Y$ meets every A_i. This Y satisfies (‡).

We have shown in all cases that there exists $Y \subseteq V(G) \setminus U$ satisfying (‡). We may now finish exactly as in the proof of Lemma 13 (with width instead of $\mathcal{W}$-width, so the argument is even simpler). $\square$

Since $K^*_{2,t}$ is planar and so $K^*_{2,t}$-minor-free graphs have bounded treewidth, one would expect a good bound (independent of n) on the blow-up factor. Campbell et al. [7] showed that every $K^*_{2,t}$-minor-free graph is isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(t^3)}$ where $\operatorname{tw}(H) \leqslant 2$. They state as an open problem whether this $\mathcal{O}(t^3)$ bound can be improved to $\mathcal{O}(t)$. Theorem 22 for $s = 2$ gives an affirmative answer to this question.

Theorem 24. *For every integer $t \geqslant 2$, every $K^*_{2,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{t-1}$, where $\operatorname{tw}(H) \leqslant 2$.*

Note that Theorem 24 implies $K^*_{2,t}$-minor-free graphs have treewidth $\mathcal{O}(t)$, which was first proved by Leaf and Seymour [31, (4.4)].

6 Concluding Remarks

In the arXiv version of this paper [29] we show that Theorem 2(a), Theorem 3, Theorem 5, Corollary 6, Theorem 18, Theorem 20, and Theorem 24 can be slightly strengthened by replacing the bound on the treewidth of H by the same bound on the simple treewidth of H. In particular, in Theorem 24, H is outerplanar and, in Corollary 6, H is planar with treewidth at most 3.

We conclude the paper by first discussing some possible ways in which Theorem 2 might be strengthened. Similar questions can be asked for $K_{s,t}$-minor-free graphs. Consider the following meta-theorem:

> Every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{m(G)}$ for some function m and some graph H of treewidth at most $f(t)$. $(\star)$

Note that Theorem 2 says that $(\star)$ holds for $m(G) = 2\sqrt{(t-3)n}$ where $n := |V(G)|$ and $f(t) = t - 2$ while Theorem 4 says it holds for $m(G) = \operatorname{tw}(G) + 1$ and $f(t) = t - 2$.

Q1. Is it possible to improve $f(t)$ in Theorem 2 (possibly sacrificing some dependence on t in $m(G)$)? In particular, can $(\star)$ be proved with $m(G) = \mathcal{O}_t(n^{1/2})$ and $f(t) = c$ for some constant c independent of t? It follows from a result of Linial, Matoušek, Sheffet, and Tardos [32] that, even for planar graphs, $c \geqslant 2$. On the other hand, $(\star)$ holds with H a star ($c = 1$) and $m(G) = \mathcal{O}_t(n^{2/3})$, and for any $\varepsilon > 0$ there exists c such that $(\star)$ holds with $f(t) \leqslant c$ and $m(G) = \mathcal{O}_t(n^{1/2+\varepsilon})$; see [20]. The real interest is when $m(G) = \mathcal{O}_t(n^{1/2})$.

As noted in Section 1, there is no corresponding improvement to Theorem 4 since $f(t) = t - 2$ is best possible when $m(G)$ is a function of $\mathrm{tw}(G)$.

Q2. We highlight the $t = 5$ case of Q1: is every K_5-minor-free n-vertex graph G isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(\sqrt{n})}$ for some graph H of treewidth at most 2? Having treewidth at most 2 is equivalent to being K_4-minor-free, so this problem is particularly appealing. It is open even when G is planar.

Q3. Optimising the dependence on t in Theorem 2 is an interesting question. Note that Kawarabayashi and Reed [30] proved that K_t-minor-free graphs have balanced separators of order $\mathcal{O}(t\sqrt{n})$, which is best possible. Does ($\star$) hold with $f(t) \cdot m(G) = \mathcal{O}(t\sqrt{n})$?

Finally we mention a connection to row treewidth. Bose et al. [6] defined the *row treewidth* of a graph G to be the minimum treewidth of a graph H such that G is isomorphic to a subgraph of $H \boxtimes P$ for some path P. For example, Theorem 7(a) says that planar graphs have row treewidth at most 8, which was improved to 6 by Ueckerdt, Wood, and Yi [40]. It is easily seen that $\mathrm{ltw}(G) \leqslant \mathrm{rtw}(G) + 1$ for every graph G. The next result, which provides a partial converse, follows from Theorem 17 since $\mathrm{tw}(H \boxtimes K_m) \leqslant (\mathrm{tw}(H) + 1)m - 1$.

Corollary 25. *For every K_t-minor-free graph G,*

$$\mathrm{rtw}(G) \leqslant (t-1)\,\mathrm{ltw}(G) - 1.$$

Corollary 25 is in marked contrast to a result of Bose et al. [6] who constructed graphs with layered treewidth 1 and arbitrarily large row-treewidth. Thus the K_t-minor-free (or some other sparsity) assumption in Corollary 25 is necessary.

Q4. For what other graph classes $\mathcal{G}$ (beyond those defined by an excluded minor) is row treewidth bounded by a function of layered treewidth for graphs in $\mathcal{G}$?

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Note Added in Proof

Following the initial release of this paper, there has been significant progress on some of the above questions. Distel, Dujmović, Eppstein, Hickingbotham, Joret, Micek, Morin, Seweryn, and Wood [9] answered Q1 in the affirmative by proving ($\star$) with

$f(t) = 4$ and $m(G) = \mathcal{O}_t(n^{1/2})$. They also solved Q2 for planar graphs, and indeed for $K_{3,t}$-minor-free graphs. In particular, they showed that every n-vertex $K_{3,t}$-minor-free graph is isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(t\sqrt{n})}$ for some graph H of treewidth 2.

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