A LOGARITHMIC BOUND FOR THE CHROMATIC NUMBER OF THE ASSOCIAHEDRON

LOUIGI ADDARIO-BERRY, BRUCE REED, ALEX SCOTT, AND DAVID R. WOOD

Abstract. We show that the chromatic number of the n-dimensional associahedron grows at most logarithmically with n, improving a bound from and proving a conjecture of Fabila-Monroy et al. (2009).

1. Introduction

The associahedron $\mathcal{A}_n$ is an $(n-3)$-dimensional convex polytope that arises in numerous branches of mathematics, including algebraic combinatorics [6, 11, 14, 26] and discrete geometry [2, 19, 20]. Associahedra are also called Stasheff polytopes after the work of Stasheff [26], following earlier work by Tamari [27]. We are only interested in the 1-skeleton of the associahedron, so we consider it as a graph, defined as follows.

The elements of the associahedron $\mathcal{A}_n$ are triangulations T of the convex n-gon with vertices labeled by $\{0, \dots, n-1\}$ in clockwise order. For any such triangulation T, we always denote triangles of T by the sequence ABC of their vertices, ordered so that $A < B < C$. We write $E(T)$ for the set of edges contained in T. Every triangulation T of $\mathcal{A}_n$ contains the edges $01, 12, \dots, (n-1)0$; we refer to these as *boundary edges*. For T in $\mathcal{A}_n$, each non-boundary edge $e \in E(T)$ is contained in a unique quadrilateral $Q = Q_T(e) = ABCD$ with $A < B < C < D$; we always list the vertices of quadrilaterals in increasing order. *Flipping* the edge e means replacing e by the other diagonal of Q; see Figure 1. Two triangulations T, T' in $\mathcal{A}_n$ are adjacent in $\mathcal{A}_n$ if they may be obtained from one another by a single flip.

Graph-theoretic properties of associahedra have been well-studied. For example, it is easily seen that $\mathcal{A}_n$ is $(n-3)$-regular. Lucas [15] and Hurtado and Noy [12] both proved that $\mathcal{A}_n$ is Hamiltonian. Hurtado and Noy [12] also showed that $\mathcal{A}_n$ has connectivity $n-3$, as well as determining its automorphism group. Parlier and Petri proved bounds on the genus of $\mathcal{A}_n$. The diameter of $\mathcal{A}_n$ and several related questions have been studied extensively [1, 3–5, 7, 8, 16, 21–24]. Sleator et al. [24] proved that the diameter equals $2n-10$ for sufficiently large n, and recently Pournin [21] showed that $2n-10$ is the answer for $n > 12$. Several authors [9, 17, 18] studied random walks in $\mathcal{A}_n$.

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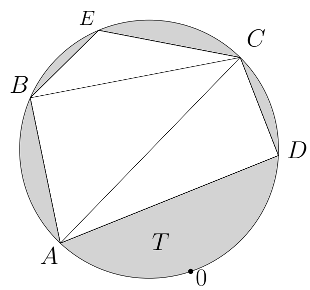
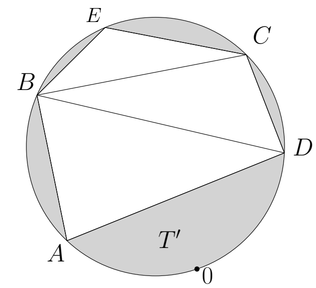


(A) A portion of a triangulation T.



(B) A portion of the triangulation T' formed from T by flipping AC.

FIGURE 1. Portions of two adjacent triangulations of an n-gon.

This paper studies the chromatic number of $\mathcal{A}_n$, a quantity which was first considered by Fabila-Monroy et al. [10]. That work gave an explicit $\lceil \frac{n}{2} \rceil$-colouring of $\mathcal{A}_n$, and observed that $\chi(\mathcal{A}_n) \in O(n/\log n)$, based on the result of Johansson [13] which says that every triangle-free graph with maximum degree Δ is $O(\Delta/\log\Delta)$-colourable. No non-constant lower bound on $\chi(\mathcal{A}_n)$ is known. Indeed, the best known lower bound is $\chi(\mathcal{A}_{10}) \geqslant 4$ [private communication, Ruy Fabila-Monroy]. Fabila-Monroy et al. [10] conjectured a $O(\log n)$ upper bound. We prove this conjecture.

Theorem 1. $\chi(\mathcal{A}_n) \in O(\log n)$.

2. The Proof

We prove Theorem 1 by tracking how several carefully chosen properties of triangulations change when an edge is flipped. To see how this yields a route to bounding the chromatic number of $\mathcal{A}_n$, first recall that if f is a graph homomorphism from $\mathcal{A}_n$ to some graph G, which is to say that $f : V(\mathcal{A}_n) \to V(G)$ is adjacency-preserving, then $\chi(\mathcal{A}_n) \leqslant \chi(G)$. This fact may be generalized as follows. Suppose that $(G_i)_{i\in I}$ is a finite set of graphs and $(f_i : V(\mathcal{A}_n) \to V(G_i))_{i\in I}$ are functions such that for all adjacent triangulations T, T' in $\mathcal{A}_n$, there exists $i \in I$ for which $f_i(T)$ and $f_i(T')$ are adjacent in G_i. For each $i \in I$, let κ_i be a proper colouring of G_i with $\chi(G_i)$ colours, and colour each T in $\mathcal{A}_n$ with the vector $(\kappa_i(f_i(T)))_{i\in I}$. If T and T' are adjacent in $\mathcal{A}_n$, then $(\kappa_i(f_i(T)))_{i\in I}$ and $(\kappa_i(f_i(T')))_{i\in I}$ differ in at least one coordinate. Thus this is a proper colouring of $\mathcal{A}_n$, and $\chi(\mathcal{A}_n) \leqslant \prod_{i\in I}\chi(G_i)$. The remainder of the paper is devoted to defining the five functions that we use (see (3)), and showing they have the requisite properties.

Two fundamental notions that we use are the *type* of a quadrilateral and the *scale* of an edge. For a quadrilateral $Q = Q_T(e) = ABCD$ contained within triangulation T, we say Q is *type-1* if $e = AC$, and otherwise say Q is *type-2*; we say that an edge e is *type-1* or *type-2* according to the type of the quadrilateral $Q_T(e)$. For example, in Figure 1(A), $Q_T = ABCD$ is typ]e-1 and $Q_T(BC) = ABEC$ is type-2, and in Figure 1(B), $Q_{T'}(BC) = BECD$ is type-1 and $Q_{T'}(BD) = ABCD$ is type-2.

Fix an integer $\alpha \geqslant 3$ to be chosen later (in fact we end up taking $\alpha = 3$). For an edge $e = UV$, define the *scale* of e to be

$$\sigma_e := \lceil \log_\alpha |U - V| \rceil \in \{0, 1, \dots, \lceil \log_\alpha (n-1) \rceil\}.$$

Note that $\sigma_e = 0$ if and only if e is a boundary edge. The scales of the edges incident to triangles within a fixed quadrilateral Q are a key input to the functions we define.

We first consider the effect of edge flips on triangles ABC, where two of the three incident edges have the same scale. If AB (resp. BC, AC) is the unique edge whose scale is different from the others, then we say ABC is a *type-ℓ* (resp. *type-m*, *type-r*) triangle. If all three edges have the same scale, then we say ABC is a *type-z* triangle. Let (ℓ_T, m_T, r_T, z_T) be the vector counting the number of type-ℓ, type-m, type-r and type-z triangles in T.

For the remainder of the paper, we fix a triangulation T, and consider the effect of flipping an edge $e = AC$ within a type-1 quadrilateral $Q_T(e) = ABCD$, to form another triangulation T'.

Proposition 2. *If σ_{AC}, σ_{BD} and σ_{BC} are all different, then $(\ell_{T'}, m_{T'}, r_{T'}, z_{T'}) \neq (\ell_T, m_T, r_T, z_T)$.*

Proof. By assumption, $\sigma_{BD} \neq \sigma_{AC}$ and $\sigma_{BC} < \min(\sigma_{AC}, \sigma_{BD})$. We argue by contradiction. To this end, suppose that $(\ell_{T'}, m_{T'}, r_{T'}, z_{T'}) = (\ell_T, m_T, r_T, z_T)$. Since $\alpha \geqslant 3$,

$$\log_\alpha(D-A) \leqslant \log_\alpha(3\max(B-A, C-B, D-C)) \leqslant 1 + \log_\alpha \max(B-A, C-B, D-C).$$

Taking ceilings, it follows that

$$\sigma_{AD} \leqslant 1 + \max(\sigma_{AB}, \sigma_{BC}, \sigma_{CD}). \tag{1}$$

The preceding equation requires one of three inequalities to hold; the next three paragraphs treat the possibilities one at a time.

Suppose that $\sigma_{AB} = \max(\sigma_{AB}, \sigma_{BC}, \sigma_{CD})$. Using (1) and the fact that $\sigma_{AB} \leqslant \sigma_{AC} \leqslant \sigma_{AD}$, we find that either $\sigma_{AC} = \sigma_{AB}$ or $\sigma_{AC} = \sigma_{AB} + 1 = \sigma_{AD}$.

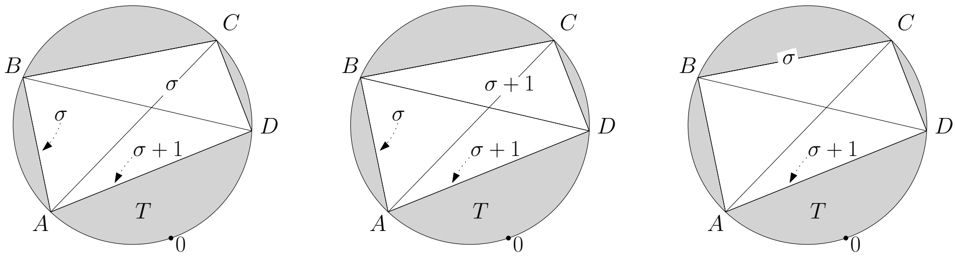
If $\sigma_{AC} = \sigma_{AB}$, as in Figure 2(A), then ABC is a type-m triangle so, since we assume the triangle type vector is unchanged by flipping edge AC, either ABD or BCD is also type-m. BCD is not type-m, as $\sigma_{BC} < \sigma_{BD}$ by assumption, so ABD is type-m and hence $\sigma_{AB} = \sigma_{AD}$. But then ABC and ACD are both type-m, which gives a contradiction as BCD is not.

If $\sigma_{AC} = \sigma_{AB} + 1$, as in Figure 2(B), then ACD is type-m or type-z, so either ABD or BCD is type-m or type-z. But BCD is neither, as $\sigma_{BC} < \sigma_{BD}$, and ABD is neither as $\sigma_{AB} \neq \sigma_{AD}$.

Next suppose that $\sigma_{BC} = \max(\sigma_{AB}, \sigma_{BC}, \sigma_{CD})$, as in Figure 2(C). Since $\sigma_{BC} \neq \sigma_{AC}$ and $\sigma_{BC} \neq \sigma_{BD}$ by assumption, and all scales are at most σ_{AD}, it must be that $\sigma_{AC} = \sigma_{BD} = \sigma_{AD}$; but this is ruled out by assumption.

Finally, suppose that $\sigma_{CD} = \max(\sigma_{AB}, \sigma_{BC}, \sigma_{CD})$. This case is the same as the first case, as we can apply the argument to a reversed copy of the associahedron (which exchanges type-ℓ and type-m triangles, while leaving the other two types invariant). $\square$

Proposition 3. *If $\sigma_{AC} = \sigma_{BD} = \sigma_{BC}$ and $(\ell_{T'}, m_{T'}, r_{T'}, z_{T'}) = (\ell_T, m_T, r_T, z_T)$, then either $\sigma_{AD} = \sigma_{BC}$ or $\sigma_{AB} = \sigma_{BC} = \sigma_{CD}$.*



(A) $\sigma_{AB} = \sigma = \sigma_{AC}$. (B) $\sigma_{AB} = \sigma, \sigma_{AC} = \sigma{+}1$. (C) $\sigma_{BC} = \sigma$.

FIGURE 2. Writing $\sigma = \max(\sigma_{AB}, \sigma_{BC}, \sigma_{CD})$, the subfigures correspond to possible configurations arising in the proof of Proposition 2.

Proof. In this case ABC is type-ℓ or type-z, and BCD is type-m or type-z. If ABC is type-ℓ then since the triangle type vector does not change when flipping, it must be that ABD is type-ℓ, which implies that $\sigma_{AD} = \sigma_{BD}$, yielding the result in this case since $\sigma_{BD} = \sigma_{BC}$. Similarly, if BCD is type-m then it must be that ACD is type-m, which implies that $\sigma_{AD} = \sigma_{AC}$. Otherwise, both ABC and BCD are type-z, in which case we indeed have $\sigma_{AB} = \sigma_{BC} = \sigma_{CD}$. □

For each triangulation T in $\mathcal{A}_n$ and $k \in \{1, 2\}$ and $i \in \{0, 1, \dots, \lceil \log_\alpha(n-1) \rceil\}$, let

$$s_i^k(T) := \#\{e \in E(T) : Q(e) \text{ is type-}k,\ \sigma_e = i\}\,.$$

Assign an integer label $c(T)$ to T given by

$$c(T) \;:=\; \left(\sum_{i=0}^{\lceil \log_\alpha(n-1) \rceil} 2is_i^1(T) \quad + \sum_{i=0}^{\lceil \log_\alpha(n-1) \rceil} 3is_i^2(T) \right) \quad \bmod\ (3\lceil \log_\alpha n \rceil)\,.$$

The utility of such a labelling rule is explained by the following fact. We continue to work with triangulations T and T' related by an edge flip within quadrilateral $ABCD$ with $AC \in E(T)$ and $BD \in E(T')$, as above.

Proposition 4. *If exactly two of σ_{AC}, σ_{BD} and σ_{BC} are equal, then $c(T') \neq c(T)$.*

Proof. First suppose that BC is not a boundary edge, and let V be the unique vertex of T with $B < V < C$ adjacent to both B and C. Note that $ABCD = Q_T(AC)$ is type-1 in T and $ABCD = Q_{T'}(BD)$ is type-2 in T'. Also, $Q_T(BC) = ABVC$ is type-2 in T and $Q_{T'}(BC) = BVCD$ is type-1 in T'. It is not hard to check that no other quadrilaterals change type when moving from T to T'. Thus

$$\begin{aligned} c(T') - c(T) &= 3\sigma_{BD} - 2\sigma_{AC} + 2\sigma_{BC} - 3\sigma_{BC} && \bmod\ (3\lceil \log_\alpha n \rceil) \\ &= 3\sigma_{BD} - 2\sigma_{AC} - \sigma_{BC} && \bmod\ (3\lceil \log_\alpha n \rceil)\,. \end{aligned}$$

It follows that if $\sigma_{BC} < \sigma_{BD} = \sigma_{AC}$, then

$$c(T') - c(T) = \sigma_{BD} - \sigma_{BC} \quad \bmod\ (3\lceil \log_\alpha n \rceil) \neq 0\,;$$

the difference is non-zero modulo $(3\lceil \log_\alpha n \rceil)$ since all scales are at most $\lceil \log_\alpha n \rceil$. Similarly, if $\sigma_{BC} = \sigma_{BD} < \sigma_{AC}$ then

$$c(T') - c(T) = 2(\sigma_{AC} - \sigma_{BD}) \quad \bmod\ (3\lceil \log_\alpha n \rceil) \neq 0\,.$$

Finally, if $\sigma_{AC} = \sigma_{BC} < \sigma_{BD}$ then (since BD and AC are non-boundary edges)

$$c(T') - c(T) = 3(\sigma_{BD} - \sigma_{AC}) \mod (3\lceil \log_\alpha n \rceil) \neq 0$$

Since $\sigma_{BC} \leqslant \min(\sigma_{AC}, \sigma_{BD})$, these are the only possibilities. Here we use that $\sigma_{BD} \geqslant 1$ and $\sigma_{AC} \geqslant 1$ (since $D - B \geqslant 2$ and $C - A \geqslant 2$).

The case when BC is a boundary edge is very similar, but easier. In this case,

$$c(T') - c(T) = 3\sigma_{BD} - 2\sigma_{AC} \mod (3\lceil \log_\alpha n \rceil).$$

Since BC is a boundary edge, AC and BD are not, so $\sigma_{BC} = 0$ and $\sigma_{AC} \neq 0$ and $\sigma_{BD} \neq 0$. It follows by assumption that $\sigma_{AC} = \sigma_{BD}$, so

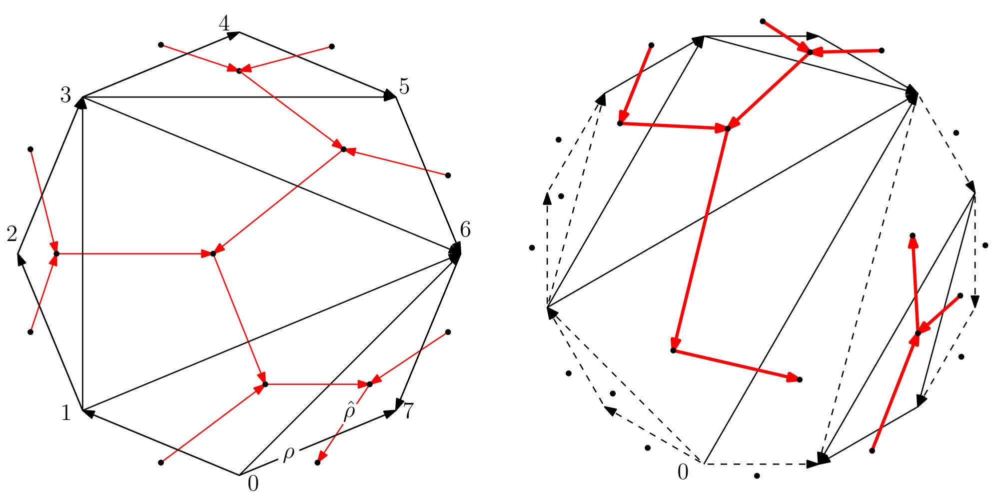
$$c(T') - c(T) = \sigma_{AC} \mod (3\lceil \log_\alpha n \rceil) \neq 0. \qquad \square$$

Propositions 2, 3 and 4 imply that the label $c(T)$ and the type vector (ℓ_T, m_T, r_T, z_T) together distinguish T from T' except in the following cases.

(a) AC, BD, BC, and AD have the same scale and AB, CD have smaller scales.
(b) AC, BD, BC, AD and AB have the same scale and CD has a smaller scale.
(c) AC, BD, BC, AD and CD have the same scale and AB has a smaller scale.
(d) AC, BD, BC, AB and CD have the same scale and AD has a larger scale.
(e) All six edges AB, AC, AD, BC, BD and CD have the same scale.

To handle cases (a), (b) and (c) we track two additional parameters, and show that the parity of one or both parameters is different for T and T'. In case (d) we again prove there is a change of parity, but of a third, more complicated parameter. For case (e) we use induction.

Figure 3(A) should make the following definitions clear. Orient the edges of the triangulation T so that the head of each edge has larger label. The *root edge* of T is the edge $\rho = (0, n-1)$. Now construct the following oriented tree $\widehat{T}$. First, augment T by adding a vertex v to the unbounded face, and join it to all vertices of the polygon. Let $\widehat{T}_0$ be the planar dual of the augmented graph; then $\widehat{T}$ is



(A) The dual tree of a triangulation of an 8-gon.

(B) The subgraph $\widehat{S}$ of $\widehat{T}$ corresponding to a subgraph S of a triangulation T.

FIGURE 3. The dual trees of an 8-gon and of a sub-triangulation of a 12-gon

formed from $\widehat{T}_0$ by removing all edges of $\widehat{T}_0$ lying entirely within the unbounded face of T. For each edge e of T there is a unique edge $\hat{e}$ of $\widehat{T}$ crossing e. Orient $\hat{e}$ from the left to the right of e (when following e from tail to head). Root $\widehat{T}$ at the edge $\hat{\rho}$, whose head is the unique node of $\widehat{T}$ with out-degree 0. Note that $\widehat{T}$ is a tree, which we call the *dual tree* of T.

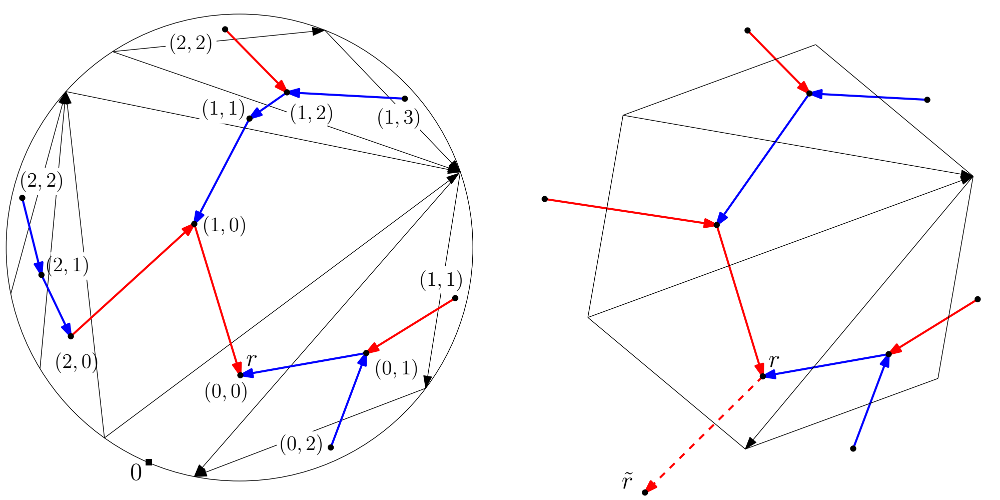
Given an edge $e = UV$ of T with $e \neq \rho$, the triangle containing the head of $\hat{e}$ is incident to both U and V; let W be its third node. By the above choice of orientation for $\hat{e}$, either $W < \min(U, V)$ or $W > \max(U, V)$. In the first case, $\hat{e}$ is a *left turn*, and in the second case it is a *right turn*.

Given a subgraph S of triangulation T, as illustrated in Figure 3(B), let $\widehat{S}$ be the "dual" subgraph of $\widehat{T}$ with the same vertex set as $\widehat{T}$ and with edge set

$$E(\widehat{S}) := \{\hat{e} : e \in E(S)\}.$$

A node of $\widehat{S}$ is a *leaf* if it has degree 1. For each node v of $\widehat{S}$, let $g_S(v)$ and $d_S(v)$ be defined as follows. Write r for the root (the unique node of out-degree 0) of the tree component of $\widehat{S}$ containing v. Then $g_S(v)$ and $d_S(v)$ are the number of left- and right-turns on the path from v to r, respectively; see Figure 4(A). (Figure 4(B) is used later in the section.)

Recall that T' is obtained from T by flipping edge AC within quadrilateral $ABCD$. For each $i \in \{1, 2, \ldots, \lceil \log_\alpha n \rceil\}$, let S_i be the subgraph of T with edge set $E(S_i) = \{e \in E(T) : \sigma_e = i\}$, and let S'_i be the subgraph of T' with edge set $E(S'_i) = \{e \in E(T') : \sigma_e = i\}$. Define $\widehat{S}_i$ and $\widehat{S}'_i$ as in the preceding paragraph (so



(A) The left-turn and right-turn labelling of a component $\hat{H}$ of $\widehat{S}$ with root node r. Labels are given in the form $(g(v), d(v))$ for all nodes v of $\widehat{S}$.

(B) The reduced tree $\tilde{H}$ corresponding to the component $\hat{H}$, together with the triangulation of a polygon to which $\tilde{H}$ is dual. The root edge of $\tilde{H}$ is dashed.

FIGURE 4. In both subfigures, left-turn edges are red and right-turn edges are blue.

$\widehat{S}_i$ is a subgraph of $\widehat{T}$ and $\widehat{S}'_i$ is a subgraph of $\widehat{T}'$). Let

$$G(T) := \sum_{i=1}^{\lceil \log_\alpha n\rceil} \sum_{v\in V(\widehat{S}_i)} g_{S_i}(v) \quad \text{and} \quad D(T) := \sum_{i=1}^{\lceil \log_\alpha n\rceil} \sum_{v\in V(\widehat{S}_i)} d_{S_i}(v)\,.$$

The following proposition implies that in cases (a), (b) and (c), flipping edge AC yields a change in parity of at least one of G and D.

Proposition 5. *If the scales of the edges AB, AC, AD, BC, BD and CD are as in cases (a), (b) or (c) above, then $G(T') = G(T) - 1$ or $D(T') = D(T) + 1$ (or both).*

Proof. Let $\sigma = \sigma_{AC}$. We first claim that for all $i \neq \sigma$, the contributions to $G(T)$ and to $D(T)$ from scale-i nodes are unchanged by the edge flip operation; that is,

$$\sum_{i\neq\sigma}\sum_{v\in V(\widehat{S}_i)} g_{S_i}(v) = \sum_{i\neq\sigma}\sum_{v\in V(\widehat{S}'_i)} g_{S'_i}(v) \quad \text{and} \quad \sum_{i\neq\sigma}\sum_{v\in V(\widehat{S}_i)} d_{S_i}(v) = \sum_{i\neq\sigma}\sum_{v\in V(\widehat{S}'_i)} d_{S'_i}(v)\,. \tag{2}$$

We prove these equalities in case (a); the other two cases are similar but easier.

The triangle containing the head of the edge $\hat{e}_{AB}$ dual to AB is ABC in T and is ABD in T'. The case (a) assumptions on the scales of the edges then imply that the head of $\hat{e}_{AB}$ has out-degree 0 and in-degree 1 in $\widehat{S}_{\sigma_{AB}}$. In particular, it is the root of its component of $\widehat{S}_{\sigma_{AB}}$. Moreover, $\hat{e}_{AB}$ is a right-turn edge in both T and T', since C and D are both larger than A and B.

Similarly, the triangle containing the head of the edge $\hat{e}_{CD}$ dual to CD is ACD in T and is BCD in T'. The assumptions on the scales of edges again imply that the head of $\hat{e}_{CD}$ has in-degree 1 and out-degree 0 within $\widehat{S}_{\sigma_{CD}}$, so is the root of its component of $\widehat{S}_{\sigma_{CD}}$. Moreover, $\hat{e}_{CD}$ is a left-turn edge in both T and T', since A and B are both smaller than C and D.

Since the structures of T and of T' are unaffected outside of the quadrilateral $ABCD$, the equalities in (2) follow in case (a).

We now restrict our attention to the scale σ. We write $g(\cdot) = g_{S_\sigma}(\cdot)$ and $d(\cdot) = d_{S_\sigma}(\cdot)$, and likewise $g'(\cdot) = g_{S'_\sigma}(\cdot)$ and $d'(\cdot) = d_{S'_\sigma}(\cdot)$. Note that all nodes not lying within the quadrilateral $ABCD$ either belong to both S_σ and S'_σ or belong to neither of S_σ and S'_σ.

The remainder of the proof boils down to inspection of Figures 5, 6 and 7. In case (a), observe (see Figure 5) that $g(u) = g'(u) = a+1$ and $d(u) = d'(u) = b+1$, which implies that $(g(q), d(q)) = (g'(q), d'(q))$ for all nodes q not lying within $ABCD$. Since $g(v) + g(x) = 2a + 1 = g(p) + g(z) + 1$ and $d(v) + d(z) = 2b = d(p) + d(z) - 1$, it follows that $G(T) = G(T') + 1$ and $D(T') = D(T) - 1$.

Figure 6 depicts the situation in case (b). In this case $d(u) = d'(u)$ and $d(w) = d'(w)$, which implies that $d(q) = d'(q)$ for all q not lying in $ABCD$. Since $d'(z) + d'(p) = d(v) + d(x) + 1$, it follows that $D(T') = D(T) + 1$.

Finally, Figure 7 relates to case (c). In this case $g(u) = g'(u)$, $g(y) = g'(y)$, and $g(v) + g(x) = g'(z) + g(p) + 1$, so the above argument implies that $G(T') = G(T) - 1$. □

We now turn our attention to cases (d) and (e). Consider any subgraph S of T, and let $\hat{H}$ be a connected component of $\widehat{S}$. Note that $\hat{H}$ is a rooted sub-binary

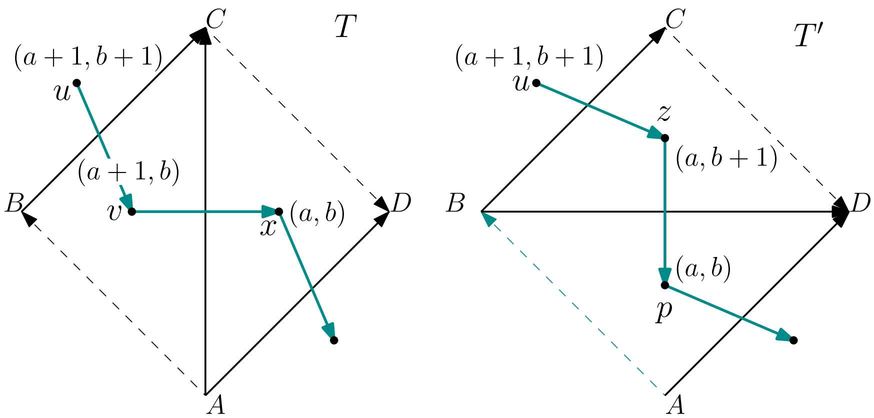


FIGURE 5. The left-turn and right-turn labels near quadrilateral $ABCD$ in T and T': case (a). Here $(g(x), d(x)) = (a, b)$, $(g(v), d(v)) = (a+1, b)$ and $(g(u), d(u)) = (a+1, b+1)$.

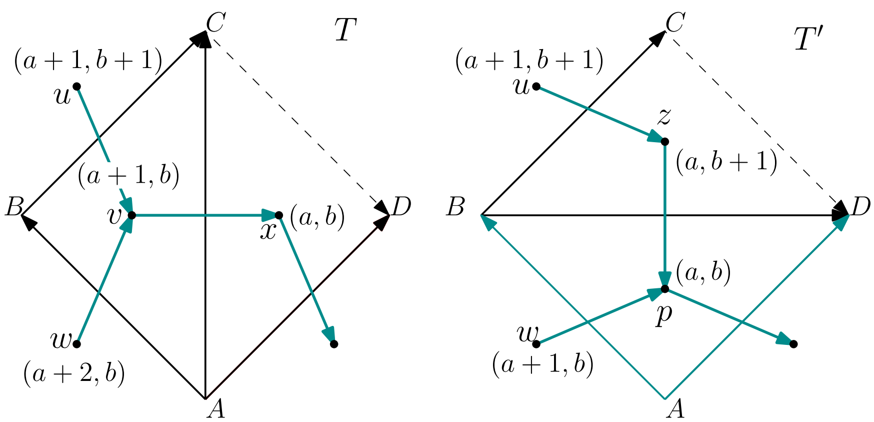


FIGURE 6. The left-turn and right-turn labels near $ABCD$ in T and T': case (b).

tree (that is, every node has degree at most three; see Figure 3(B)). Let $\tilde{H}$ be the tree obtained from $\hat{H}$ as follows (see Figure 4(A) and 4(B)). First, if the root r of $\hat{H}$ has exactly two children then add a new node $\tilde{r}$ incident only to r and reroot at $\tilde{r}$. Next, suppress each node of degree exactly two (that is, contract one edge incident to the node). We obtain a rooted binary tree $\tilde{H}$ called the *reduced tree* of $\hat{H}$.

Proposition 6. *For each $i \in \{1, 2, \ldots, \lceil \log_\alpha n \rceil\}$ and for each component $\hat{H}$ of $\widehat{S}_i$, the reduced tree $\tilde{H}$ has at most $2\alpha - 1$ leaves.*

Proof. Fix any node u of $\tilde{H}$ with in-degree zero, and consider the edge uv incident to u in $\hat{H}$. Then uv is dual to an edge AB with $\sigma_{AB} = i$ so with $\alpha^{i-1} < B - A \leqslant \alpha^i$. Now fix another node w of $\tilde{H}$ with in-degree zero, write wx for the edge incident to w in $\hat{H}$, and let CD be its dual edge. Then necessarily either $A < B \leqslant C < D$ or $C < D \leqslant B < A$.

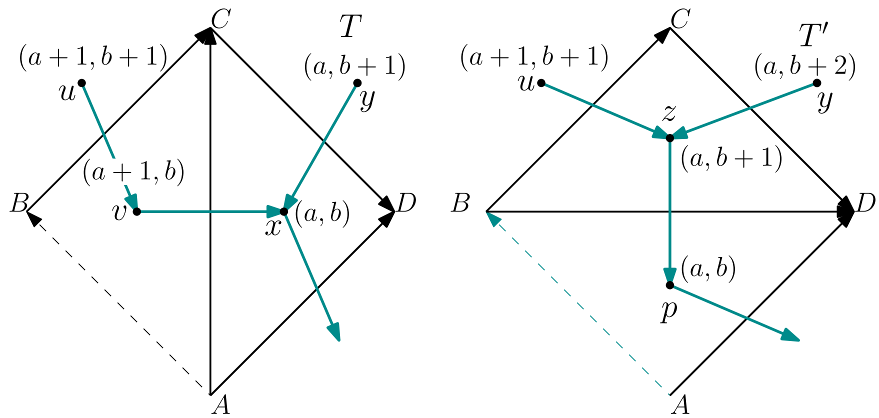


FIGURE 7. The left-turn and right-turn labels near $ABCD$ in T and T': case (c).

Consider an oriented edge yr where r is the root of $\hat{H}$. Writing EF for the edge dual to yr, then the observation of the preceding paragraph implies that $F-E>\alpha^{i-1}\cdot\ell$, where ℓ is the number of nodes with in-degree zero in the subtree rooted at y. On the other hand, $\sigma_{EF}=i$ so $F-E\leqslant\alpha^i$; so $\ell\leqslant\alpha-1$. If r has only one child (so is a leaf itself) this yields that $\tilde{H}$ has at most α leaves. If r has two children then each of their subtrees contains at most $\alpha-1$ leaves; in this case $\tilde{r}$ is also a leaf, so the total number of leaves is at most $2(\alpha-1)+1$. $\square$

For any subgraph S of the triangulation T, the embedding of $\widehat{T}$ in the plane induces a total order of the connected components of $\widehat{S}$, given by the order their roots are visited by a clockwise tour around the contour of $\widehat{T}$ starting from the head of the root edge ρ. For each $1\leqslant i\leqslant\lceil\log_\alpha n\rceil$, list the components of $\widehat{S}_i$ in the order just described as $H_{i,1},\ldots,H_{i,\ell}$, where $\ell=\ell(T,i)$ is the number of such components. Then, for $1\leqslant j\leqslant\ell(T,i)$ let $\tilde{H}_{i,j}$ be the reduced tree of $H_{i,j}$.

Each tree from $(\tilde{H}_{i,j}, i\leqslant\lceil\log_\alpha n\rceil, j\leqslant\ell(T,i))$ is a dual to a unique triangulation $\tilde{T}_{i,j}$ of a polygon, as in Figure 4(B). Proposition 6 implies that $\tilde{T}_{i,j}$ belongs to an associahedron $\mathcal{A}_k$ for some $k\leqslant 2\alpha-1$. Let ϕ be a proper colouring of the disjoint union of $(\mathcal{A}_k, k\leqslant 2\alpha-1)$, with colours $\{1,\ldots,\chi(\mathcal{A}_{2\alpha-1})\}$, and define

$$I(T):=\left(\sum_{i=0}^{\lceil\log_\alpha n\rceil}\sum_{j=1}^{\ell(T,i)}\phi(\tilde{T}_{i,j})\right)\mod\chi(\mathcal{A}_{2\alpha-1})\,.$$

Proposition 7. *In cases (d) and (e), we have $I(T)\neq I(T')$.*

Proof. Write $vx=\hat{e}$ for the dual edge of AC in T, and $zp=\hat{e}_{BD}$ for the dual edge of BD in T'. Figures 8 and 9 illustrate cases (d) and (e) respectively.

The clockwise contour exploration of a rooted plane tree is a walk around the outside of the tree which starts and finishes at the root, keeping the unbounded face to its left at all times. This walk traverses each edge exactly twice, and records the vertices it visits in sequence, with repetition. In cases (d) and (e), for the clockwise contour explorations of $\widehat{T}$ and of $\widehat{T}'$, there are (possibly empty) strings $P_1,\ldots,P_5$ so that the sequences recorded by the contour explorations of

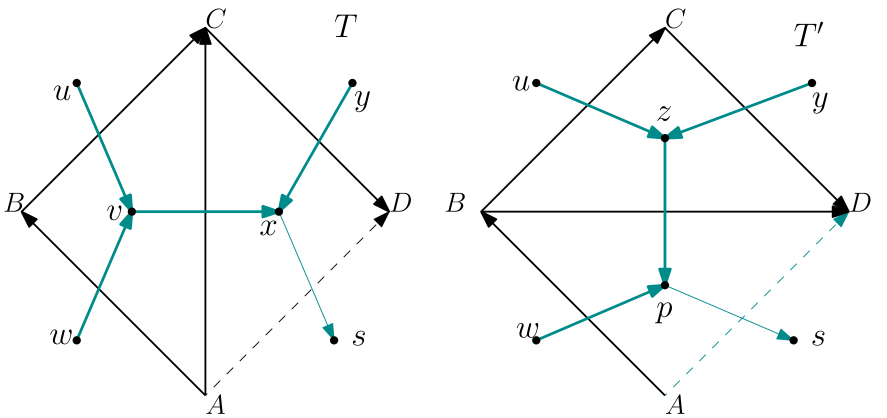


FIGURE 8. The structure near the quadrilateral $ABCD$ in T and T' case (d). The dashed edges have scale $\sigma_{AD} > \sigma$, all other edges of the triangulations shown in the figure have scale σ.

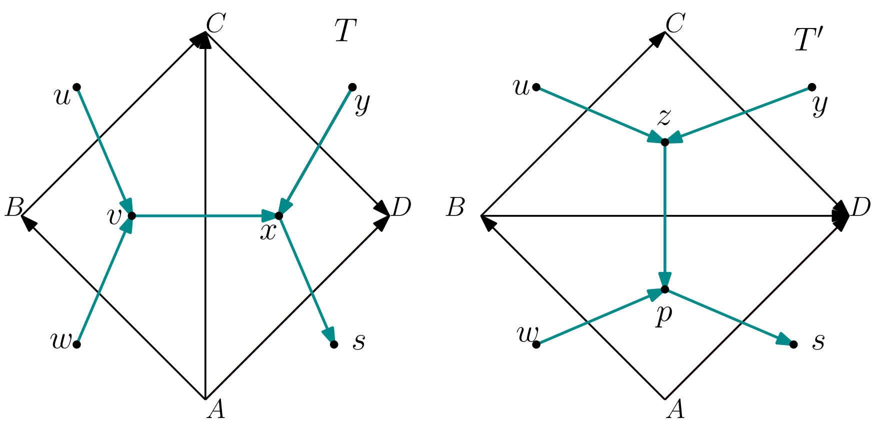


FIGURE 9. The structure near the quadrilateral $ABCD$ in T and T' in case (5). All edges of the triangulations shown in the figure have scale σ.

$\widehat{T}$ and of $\widehat{T}'$ are respectively of the form

$$P_1 sxvwP_2wvuP_3uvxyP_4yxsP_5 \quad \text{and} \quad P_1spwP_2wpzuP_3uzyP_4yzpsP_5\,;$$

see Figures 8 and 9.

The contour explorations of $\widehat{T}$ and $\widehat{T}'$ agree until they visit dual vertices lying within $ABCD$. It follows that if $H_{\sigma,j}$ is the component of S_σ containing $\hat{e}_{AC}$, then the component of S'_σ containing $\hat{e}_{BD}$ is $H'_{\sigma,j}$.

In case (d), since x has two children in $H_{\sigma,j}$, by construction it is the unique child of the root of $\tilde{H}_{\sigma,j}$. It is thus natural to identify s with the root of $\tilde{H}_{\sigma,j}$. We may likewise identify s with the root of $\tilde{H}'_{\sigma,j}$, since p has two children in $H'_{\sigma,j}$. After the addition of s as a root, the nodes v, x, p and z all have degree 3, so none of these nodes are suppressed when constructing $\tilde{H}_{\sigma,j}$ and $\tilde{H}'_{\sigma,j}$ from $H_{\sigma,j}$ and $H'_{\sigma,j}$.

In case (e), nodes v, x, p and z all have degree 3, and the edges xs and ps belong to $H_{\sigma,j}$ and $H'_{\sigma,j}$, respectively.

It follows from the two preceding paragraphs that in cases (d) and (e), we may view v and x as nodes of both $H_{\sigma,j}$ and $\tilde{H}_{\sigma,j}$, and p and z as nodes of both $H'_{\sigma,j}$ and $\tilde{H}'_{\sigma,j}$. It is then clear that flipping the edge AC in the triangulation T corresponds precisely to flipping the corresponding edge in $\tilde{T}_{\sigma,j}$ to form $\tilde{T}'_{\sigma,j}$.

Since $\tilde{T}_{\sigma,j}$ and $\tilde{T}'_{\sigma,j}$ are related by a single edge flip, and ϕ is a proper colouring, it follows that $\phi(\tilde{T}_{\sigma,j}) \neq \phi(\tilde{T}'_{\sigma,j})$. Since all other components of S_σ, and more generally of each $(S_i, 1 \leqslant i \leqslant \lceil \log_\alpha n \rceil)$, are unchanged when moving from T to T', the result follows. □

Proof of Theorem 1. Consider the graph G with vertex set $\mathbb{N}^4$ where distinct vertices (ℓ, m, r, z) and (ℓ', m', r', z') are adjacent if $|\ell'-\ell|+|m'-m|+|r'-r|+|z'-z| \leqslant 4$. This graph has maximum degree at most 320 (see [25]), so is 321-colourable. (We are not optimizing constants.) Let $\kappa : \mathbb{N}^4 \to \{1, \ldots, 321\}$ be a proper colouring of G.

For each triangulation T of $\mathcal{A}_n$, let $g(T) := G(T) \bmod 2$ and $d(T) := D(T) \bmod 2$; colour T by the 5-tuple

$$\psi(T) := \big(\kappa((\ell_T, m_T, r_T, z_T)),\, c(T),\, g(T),\, d(T),\, I(T)\big). \tag{3}$$

We now show that ψ is a proper colouring of $\mathcal{A}_n$.

Consider adjacent vertices T and T' in $\mathcal{A}_n$. In all situations covered by Proposition 2, the vectors $(\ell_{T'}, m_{T'}, r_{T'}, z_{T'})$ and (ℓ_T, m_T, r_T, z_T) are different, and thus adjacent in G, since each of $|\ell_{T'} - \ell_T|$, $|m_{T'} - m_T|$, $|r_{T'} - r_T|$ and $|z_{T'} - z_T|$ is at most 1. Thus $\kappa((\ell_{T'}, m_{T'}, r_{T'}, z_{T'})) \neq \kappa((\ell_T, m_T, r_T, z_T))$. In the situations covered by Proposition 4, we have $c(T) \neq c(T')$. By Proposition 5, in cases (a)–(c), either $g(T) \neq g(T')$ or $d(T) \neq d(T')$ or both. Finally, in cases (d) and (e), by Proposition 7 we have $I(T) \neq I(T')$.

Thus $\psi(T) \neq \psi(T')$, implying for any integer $\alpha \geqslant 3$,

$$\chi(\mathcal{A}_n) \leqslant 321 \cdot \lceil 3 \log_\alpha n \rceil \cdot 2 \cdot 2 \cdot \chi(\mathcal{A}_{2\alpha-1}).$$

Taking $\alpha = 3$ yields $\chi(\mathcal{A}_{2\alpha-1}) = \chi(\mathcal{A}_5) = 3$, since $\mathcal{A}_5$ is a 5-cycle. It follows that

$$\chi(\mathcal{A}_n) \leqslant 12 \cdot 321 \cdot \lceil 3 \log_3 n \rceil \in O(\log n)\,.$$ □

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