# On a problem of El-Zahar and Erdős

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**Abstract**

Two subgraphs $A, B$ of a graph $G$ are *anticomplete* if they are vertex-disjoint and there are no edges joining them. Is it true that if $G$ is a graph with bounded clique number, and sufficiently large chromatic number, then it has two anticomplete subgraphs, both with large chromatic number? This is a question raised by El-Zahar and Erdős in 1986, and remains open. If so, then at least there should be two anticomplete subgraphs both with large minimum degree, and that is one of our results.

We prove two variants of this. First, a strengthening: we can ask for one of the two subgraphs to have large chromatic number: that is, for all $t, c \geq 1$ there exists $d \geq 1$ such that if $G$ has chromatic number at least $d$, and does not contain the complete graph $K_t$ as a subgraph, then there are anticomplete subgraphs $A, B$, where $A$ has minimum degree at least $c$ and $B$ has chromatic number at least $c$.

Second, we look at what happens if we replace the hypothesis that $G$ has sufficiently large chromatic number with the hypothesis that $G$ has sufficiently large minimum degree. This, together with excluding $K_t$, is *not* enough to guarantee two anticomplete subgraphs both with large minimum degree; but it works if instead of excluding $K_t$ we exclude the complete bipartite graph $K_{t,t}$. More exactly: for all $t, c \geq 1$ there exists $d \geq 1$ such that if $G$ has minimum degree at least $d$, and does not contain the complete bipartite graph $K_{t,t}$ as a subgraph, then there are two anticomplete subgraphs both with minimum degree at least $c$.

# 1 Introduction

We begin with some notation. If $G$ is a graph and $A \subseteq V(G)$, $G[A]$ denotes the subgraph induced on $A$. The chromatic number of $G$ is denoted by $\chi(G)$, the size of its largest clique is denoted by $\omega(G)$, and if $A \subseteq V(G)$, we sometimes write $\chi(A)$ for $\chi(G[A])$. If $A, B$ are subsets of $V(G)$, they are *anticomplete* if $A \cap B = \emptyset$ and there are no edges of $G$ between $A$ and $B$.

There is a well-known problem of El-Zahar and Erdős [2, 3]:

**1.1 Is the following true?** *For all integers $t, c \geq 1$, there exists $d \geq 1$, such that if $\chi(G) \geq d$ and $\omega(G) < t$, then there are anticomplete subsets $A, B \subseteq V(G)$ with $\chi(A), \chi(B) \geq c$.*

This remains open. El-Zahar and Erdős proved that under the same hypotheses, there are anticomplete subsets $A, B \subseteq V(G)$ with $\chi(A) \geq 3$ and $\chi(B) \geq c$, but there has been little further progress. (See [7] for results on an analogous question with infinite graphs and infinte chromatic number.) We remark that if we omit the hypothesis about $\omega(G)$, the result is no longer true, and a large complete graph is a counterexample.

Minimal graphs with large chromatic number have large minimum degree, and so if 1.1 is true, under the same hypotheses there should at least be anticomplete subsets $A, B \subseteq V(G)$ such that $G[A]$, $G[B]$ have minimum degree at least $c$. This is true, and can be strengthened: we can require that one of $G[A], G[B]$ has chromatic number at least $c$. We will prove:

**1.2** *For all integers $t, c \geq 1$, there exists $d \geq 1$, such that if $\chi(G) \geq d$ and $\omega(G) < t$, then there are anticomplete subsets $A, B \subseteq V(G)$ where $G[A]$ has minimum degree at least $c$ and $\chi(B) \geq c$.*

This suggests that a straightforward random graph counterexample to 1.1 is unlikely.

What if we relax the hypothesis that $\chi(G)$ is large, and just assume that $G$ has large minimum degree? With $\omega(G)$ bounded, can we still necessarily find anticomplete subsets $A, B \subseteq V(G)$ such that $G[A]$, $G[B]$ have minimum degree at least $c$? No: a large complete bipartite graph is a counterexample. For this question, it becomes natural to bound $\tau(G)$ rather than $\omega(G)$, where $\tau(G)$ is the largest integer $t$ such that $G$ contains $K_{t,t}$ as a subgraph. We will prove:

**1.3** *For all integers $t, c \geq 1$, there exists $d \geq 1$, such that if $G$ has minimum degree at least $d$ and $\tau(G) < t$, then there are anticomplete subsets $A, B \subseteq V(G)$ where $G[A], G[B]$ both have minimum degree at least $c$.*

Finally, we will examine a possible extension of 1.1 to tournaments.

# 2 Some preliminary results

We denote the number of vertices of a graph $G$ by $|G|$; and let us say the *denseness* of a non-null graph $G$ is $|E(G)|/|G|$. (In some papers this is called "density", but density is also frequently used to mean something else, so we prefer a different word.) The denseness of the null graph is zero. Also, we define the *minimum degree* of the null graph to be zero.

The next result is well-known and standard.

**2.1** *Let $d > 0$. Every graph of minimum degree at least $d$ has denseness at least $d/2$; and every graph of denseness at least $d$ has a subgraph with minimum degree at least $d + 1$.*

**Proof.** The first statement is trivial. For the second, let $G$ be a graph with denseness at least $d$, and let $H$ be a minimal subgraph of $G$ with denseness at least $d$. Thus $|E(H)| \geq d|H|$. If some vertex $v \in V(H)$ has degree at most $d$, then $|H| \geq 2$ (since $H$ has denseness at least $d$ and $d > 0$), so the graph $H'$ obtained by deleting $v$ is non-null and satisfies

$$|E(H')| \geq |E(H)| - d \geq d|H| - d = d|H'|,$$

contrary to the minimality of $H$. This proves 2.1. ∎

In view of 2.1, we can replace the conditions about minimum degree in 1.2 and 1.3 with conditions about denseness, and this is a little more convenient.

If $p \geq 1$ is an integer, let us say a *p-rock* of a graph $G$ is a set $A \subseteq V(G)$ such that

- $A \neq \emptyset$ and $|E(G[A])| \geq p|A|$
- subject to the above, $|A|$ is minimum; and
- subject to the two conditions above, $|E(G[A])|$ is maximum.

We will need:

**2.2** *Let $p \geq 1$ be an integer, let $G$ be a graph, and let $A$ be a $p$-rock of $G$. Then every vertex $v \in V(G)$ not in $A$ has at most $2p+1$ neighbours in $A$.*

**Proof.** For a contradiction, suppose that $v \in V(G) \setminus A$ has at least $2p+2$ neighbours in $A$. It follows that $|A| \geq 2p+2 \geq 4$. Choose $u \in A$ with minimum degree in $G[A]$, say degree $\delta$. Let $A' := (A \setminus \{u\}) \cup \{v\}$ and $A'' := A \setminus \{u\}$. Since $A$ is a $p$-rock and $A'$ has the same size as $A$, it follows that $G[A']$ does not have more edges than $G[A]$, and so $\delta \geq 2p+1$. But $G[A'']$ has minimum degree at least $\delta - 1 \geq 2p$, and so has at least $p|A''|$ edges, contradicting that $A$ is a $p$-rock. This proves 2.2. ∎

We remark that the bound of 2.2 is tight, because for instance $A$ might be a clique with $2p+1$ vertices. Our third lemma is rather obvious, but we will use it twice, so we might as well state it explicitly:

**2.3** *Let $H$ be a graph and $q \geq 1$ an integer. Then there is a partition of $E(H)$ into sets $M_0, \dots, M_n$ for some $n \geq 0$, such that*

- *there is a subset $X \subseteq V(H)$ with $|X| \leq 2q-2$ such that every edge in $M_0$ is incident with a vertex in $X$; and*
- *$M_1, \dots, M_n$ are all matchings, each with cardinality $q$.*

**Proof.** We use induction on $|E(H)|$. Suppose first that $H$ has no matching with cardinality $q$. Let $M$ a maximal matching of $H$; then every edge of $H$ has an end in $X$, where $X$ is the set of vertices incident with an edge of $M$, from the maximality of $M$. Since $|M| \leq q-1$ and hence $|X| \leq 2q-2$, we may set $M_0 = E(H)$ and $n = 0$, and the theorem holds. So we may assume that $H$ has a matching $M$ of cardinality $q$; but then the result follows from the inductive hypothesis applied to the graph obtained from $H$ by deleting the edges of $M$. This proves 2.3. ∎

Fourth, we need:

**2.4** *Let $H$ be a graph. Build a set $Z \subseteq V(H)$ by putting each vertex of $H$ in $Z$ independently with probability* $1/2$.

- *If $M$ is a matching of $H$, the probability that at most $|M|/8$ edges in $M$ have both ends in $Z$ is at most $e^{-|M|/32}$.*
- *Let $d \geq 0$, and for each $v \in V(H)$, let $0 \leq d_v \leq d$, and let $m := \sum_{v \in V(H)} d_v$; then the probability that $\sum_{v \in Z} d_v \leq m/4$ is at most $e^{-m/(8d)}$.*

**Proof.** The first statement is immediate from Hoeffding's inequality, since each edge of $M$ has both ends in $Z$ independently with probability $1/4$. For the second statement, since $D := \sum_{v \in Z} d_v$ is a sum of independent bounded random variables, and the expected value of $D$ is $m/2$, we can apply Hoeffding's inequality, and deduce that the probability that $D \leq m/4$ is at most

$$\exp\left(\frac{-m^2}{8\sum_{v \in V(H)} d_v^2}\right).$$

But $\sum_{v \in V(H)} d_v^2$ is at most $md$, since $\sum_{v \in V(H)} d_v = m$ and each $d_v \leq d$; so the probability that $D \leq m/4$ is at most $e^{-m/(8d)}$. This proves the second statement, and so proves 2.4. ∎

# 3 The main proofs

Now we prove 1.2, but before that, we give a sketch of its proof. In view of 2.1, it suffices to show that for all $t, c \geq 1$, if $G$ is a graph with $\omega(G) < t$ that does not contain anticomplete subsets $A, B \subseteq V(G)$ where $G[A]$ has denseness at least $c$ and $\chi(B) \geq c$, then $\chi(G)$ is bounded. We use induction on $t$, and so we can assume that for every vertex of $G$, its set of neighbours has bounded chromatic number. We can assume there is a $p$-rock $A$ (where $p = 32c$). Since $G[A]$ has large denseness, the set of vertices with no neighbour in $A$ has small chromatic number. If also $|A|$ is bounded, then since the set of neighbours of each vertex has bounded chromatic number, it follows that $\chi(G)$ is bounded as required; so we may assume that $|A|$ is at least any constant that we choose, which is convenient for future calculations. But $\chi(A)$ is bounded, from the minimality of $A$ in the definition of a $p$-rock; and so we can assume that $V(G) \setminus A$ has large chromatic number. From 2.2, each vertex in $V(G) \setminus A$ has at most $2p + 1$ neighbours in $A$, and so if we take a partition of $A$ into $4p + 2$ sets $A_1, \dots, A_{4p+2}$, each vertex in $V(G) \setminus A$ has no neighbour in at least half of these sets. So there is a choice of half the sets $A_1, \dots, A_{4p+2}$ (say $A_i$ $(i \in I)$), such that the set of vertices in $V(G) \setminus A$ with no neighbour in $\bigcup_{i \in I} A_i$ has large chromatic number.

This suggests that we should try to choose the partition $A_1, \dots, A_{4p+2}$ carefully, such that for every choice of half of them (say $A_i$ $(i \in I)$), there are many edges with both ends in $\bigcup_{i \in I} A_i$, enough that $\bigcup_{i \in I} A_i$ has large denseness. If we can do this, then we win, since, as we just saw, there is some choice of $I$ such that the set of vertices in $V(G) \setminus A$ with no neighbour in $\bigcup_{i \in I} A_i$ has large chromatic number, and that will give us the pair of anticomplete subsets that the theorem claims. Unfortunately, such a partition $A_1, \dots, A_{4p+2}$ need not exist, because perhaps some $X \subseteq A$ with $|X|$ bounded meets most of the edges of $G[A]$. But in that case, we choose the partition $A_1, \dots, A_{4p+2}$

such that for every choice of half of them (say $A_i$ ($i \in I$)), there are many edges with both ends in $X \cup \bigcup_{i \in I} A_i$; and this turns out to be just as good, and can be used in the same way.

That completes the sketch of the proof; now let us give the actual proof. We are proving 1.2, which we restate (in terms of denseness rather than minimum degree, which by 2.1 is equivalent):

**3.1** *For all integers $t, c \geq 1$, there exists $d \geq 1$, such that if $G$ is a graph with $\chi(G) \geq d$ and $\omega(G) < t$, then there are anticomplete subsets $A, B \subseteq V(G)$ where $G[A]$ has denseness at least $c$ and $\chi(B) \geq c$.*

**Proof.** We proceed by induction on $t$. If $t \leq 2$ we may take $d = 2$, because $\chi(G) \leq 1$ for every graph $G$ with $\omega(G) \leq 1$. Thus we may assume that $t \geq 3$, and the result holds for $t-1$. Choose $d' \geq 1$ such that for every graph $G$, if $\chi(G) \geq d'$ and $\omega(G) < t-1$, then there are anticomplete subsets $A, B \subseteq V(G)$ where $G[A]$ has denseness at least $c$ and $\chi(B) \geq c$.

Let $p = 32c$; choose $q$ such that $e^{-q/32} < 2^{-4p-3}$, and choose $d$ such that

$$d > \max\left(2p + 1 + 2qd' + 2^{4p+2}c, 8q^2d'/p + c\right).$$

We will show that $d$ satisfies the theorem.

Let $G$ be a graph with $\omega(G) < t$, such that there do not exist anticomplete subsets $A, B \subseteq V(G)$ where $G[A]$ has denseness at least $c$ and $\chi(B) \geq c$. We will prove that $\chi(G) < d$. From the inductive hypothesis, it follows that for every vertex $v$, its set of neighbours $N$ satisfies $\chi(N) \leq d'$. We may assume that $G$ has a non-null subgraph with minimum degree at least $d-1$, because otherwise $\chi(G) < d$ as required. Since $p \leq (d-1)/2$, there is a $p$-rock $A$ of $G$. We may assume that:

(1) $|A| \geq 8q^2/p$.

Suppose not. Then the set of vertices of $G$ with a neighbour in $A$ (this set includes $A$, from the minimality of $A$) has chromatic number at most $d'|A| \leq 8d'q^2/p$; and the set with no neighbour in $A$ (and that therefore do not belong to $A$) has chromatic number less than $c$, since it is anticomplete to $A$ and $p \geq c$. Thus $\chi(G) < 8d'q^2/p + c \leq d$ as required. This proves (1).

Let $F := E(G[A])$. By 2.3, $F$ may be partitioned into $M_0, M_1, \ldots, M_n$ for some $n \geq 0$, such that

- there exists $X \subseteq A$ with $|X| \leq 2q - 2$ such that every edge in $M_0$ has an end in $X$; and
- $M_1, \ldots, M_n$ are all matchings of cardinality $q$.

Let $\mathcal{I}$ be the set of all subsets of $\{1, \ldots, 4p+2\}$ with cardinality $2p+1$.

(2) *There is a partition of $A$ into $4p+2$ subsets $A_1, \ldots, A_{4p+2}$, such that for each $I \in \mathcal{I}$, at least $|F|/32 \geq p|A|/32$ edges have both ends in $X \cup \bigcup_{i \in I} A_i$.*

For each $v \in A$, choose $\phi(v) \in \{1, \ldots, 4p+2\}$, uniformly and independently at random. For $1 \leq i \leq 4p+2$ let $A_i$ be the set of all $v \in A$ with $\phi(v) = i$. Thus $A_1, \ldots, A_{4p+2}$ are pairwise disjoint sets with union $A$. We will show that with positive probability, the statement of (2) is satisfied. For each $I \in \mathcal{I}$ let $A_I := \bigcup_{i \in I} A_i$.

There are two cases, depending whether $|M_1 \cup \cdots \cup M_n| \geq |F|/2$ or not. Suppose first that $|M_1 \cup \cdots \cup M_n| \geq |F|/2$. For $I \in \mathcal{I}$ and $1 \leq j \leq n$, we say that $j$ is *bad* for $I$ if at most $|M_j|/8$ edges of $M_j$ have both ends in $A_I$. By the first statement of 2.4, since each vertex of $A$ belongs to $A_I$ independently with probability 1/2, and $|M_j| = q$, it follows that the probability that $j$ is bad for $I$ is at most $e^{-q/32}$. Consequently the expected number of values of $j \in \{1, \ldots, n\}$ such that $j$ is bad for some $I \in \mathcal{I}$ is at most

$$ne^{-q/32}|\mathcal{I}| \leq ne^{-q/32}2^{4p+2} \leq n/2.$$

Let $J$ be the set of $j \in \{1, \ldots, n\}$ such that $j$ is not bad for any $I \in \mathcal{I}$. It follows that $|J| \geq n/2$ with positive probability. If $|J| \geq n/2$, then

$$|\bigcup_{j \in J} M_j| \geq |M_1 \cup \cdots \cup M_n|/2 \geq |F|/4.$$

Moreover, for each $I \in \mathcal{I}$, at least $q/8$ edges of $M_j$ have both ends in $A_I$, for each $j \in J$; and so at least 1/8 of the edges of $\bigcup_{j \in J} M_j$ have both ends in $A_I$. Consequently, with positive probability at least $|F|/32$ edges of $G[A]$ have both ends in $A_I$, and hence in this case the claim is true.

Now we assume that $|M_1 \cup \cdots \cup M_n| \leq |F|/2$, and so $|M_0| \geq |F|/2$. For each $v \in A \setminus X$, let $d_v$ be the number of neighbours of $v$ in $X$, and let $d_v = 0$ for $v \in X$. Let $m = \sum_{v \in A} d_v = \sum_{v \in A \setminus X} d_v$. For each $I \in \mathcal{I}$, the probability that $\sum_{v \in A_I} d_v \leq m/4$ is at most $e^{-m/(8|X|)} \leq e^{-m/(16q)}$, by the second statement of 2.4, taking $d = |X|$. Since $|X| \leq 2q - 2$, there are at most $2q^2$ edges of $F$ with both ends in $X$. But $|F| \geq p|A| \geq 8q^2$ by (1), and at least half the edges in $F$ belong to $M_0$, and therefore have at least one end in $X$. It follows that at least $2q^2$ edges in $F$ have exactly one end in $X$, and so $m \geq 2q^2$. Consequently, for each $I \in \mathcal{I}$, the probability that $\sum_{v \in A_I} d_v \leq m/4$ is at most $e^{-q/8}$; and hence the probability that $\sum_{v \in A_I} d_v > m/4$ for each $I \in \mathcal{I}$ is at least $1 - 2^{4p+2}e^{-q/8} > 0$. We deduce that there is a partition of $A$ into $4p + 2$ subsets $A_1, \ldots, A_{4p+2}$, such that $\sum_{v \in A_I} d_v > m/4$ for each $I \in \mathcal{I}$. But $\sum_{v \in A_I} d_v$ is at most the number of edges that have both ends in $X \cup A_I$. This proves (2).

Choose $A_1, \ldots, A_{4p+2}$ as in (2), and as before, let $A_I := \bigcup_{i \in I} A_i$ for each $I \in \mathcal{I}$. Let $W_0$ be the set of vertices in $V(G) \setminus A$ with a neighbour in $X$. For each $I \in \mathcal{I}$, let $W_I$ be the set of vertices $v \in V(G) \setminus A$ with no neighbour in $X \cup A_I$. From 2.2, every vertex in $V(G) \setminus A$ has at most $2p + 1$ neighbours in $A$, and so $V(G) \setminus A$ is the union of $W_0$ and the sets $W_I$ ($I \in \mathcal{I}$). Since $G[A]$ has no non-null subgraph with minimum degree at least $2p + 1$ (from the minimality of $A$), it follows that $\chi(A) \leq 2p + 1$. Also, $\chi(W_0) \leq |X|d' \leq 2qd'$. Let $I \in \mathcal{I}$. Thus $G[X \cup A_I]$ has at least $p|A|/32$ edges (by the choice of $A_1, \ldots, A_{4p+2}$) and at most $|A|$ vertices, and therefore its denseness is at least $p/32 = c$. Since $G[X \cup A_I]$ is anticomplete to $W_I$, we may assume that $\chi(W_I) < c$, since otherwise the theorem holds. Since $|\mathcal{I}| \leq 2^{4p+2}$, it follows that

$$\chi(G) \leq 2p + 1 + 2qd' + 2^{4p+2}c < d,$$

as required. This proves 3.1. ∎

Now we turn to 1.3. Again, we first sketch the proof. As before, the result can be stated in terms of denseness. Given $t, c \geq 1$, we will show that if $G$ is a graph with $\tau(G) < t$ that does not contain two anticomplete subsets $A, B \subseteq V(G)$ where $G[A], G[B]$ both have denseness at least $c$, then $G$ has bounded denseness. Unlike the proof of 3.1, the proof does not use induction on $t$, but it does share some ideas with that proof.

Let $p := \max(32c, 4t)$. A $p$-rock $R$ of an induced subgraph $H$ of $G$ is *little* if $|R| \leq s$. The first idea is to choose a sequence $R_1, \ldots, R_k$ of disjoint sets, where each $R_i$ is a little $p$-rock of $G \setminus (R_1 \cup \cdots \cup R_{i-1})$ for each $i$, with $k$ maximum. We claim that $k \leq 2t$. The reason is, suppose that $k \geq 2t$. Only a bounded number of vertices have a neighbour in $t$ of the sets $R_1, \ldots, R_{2t}$ (because otherwise we could find a $K_{t,t}$ subgraph, a contradiction). So, if $G$ has large denseness, most edges of $G$ have no end in $R_1 \cup \cdots \cup R_{2t}$, and are anticomplete to one of $R_1, \ldots, R_{2t}$. On the other hand, for each $i$, not many edges are anticomplete to $R_i$, or else we could find the anticomplete pair $A, B$; and this gives a contradiction. So $k$ is bounded. Let $R = R_1 \cup \cdots \cup R_k$; and so $|R|$ is bounded. It follows that if $G$ has large denseness, then so does $G \setminus R$, and hence contains a $p$-rock $A$; and we know $A$ is not little, from the maximality of $k$.

Now the proof proceeds something like that of 3.1. We try to partition $A$ into $8p + 4$ subsets $A_1, \ldots, A_{8p+4}$, such that for every choice of half of them, say $A_i$ $(i \in I)$, the subgraph $G[\bigcup_{i \in I} A_i]$ has large denseness. If we can do this, then we win much as before. (Since we are working with denseness instead of chromatic number, we need that many *edges* of $G \setminus A$ are anticomplete to $\bigcup_{i \in I} A_i$, for some $I$, instead of a set of vertices with large chromatic number. This is why we move to $8p + 4$ instead of $4p + 2$.) In the previous proof, there was a problem here: there might be a subset $X \subseteq A$ of bounded size that meets many of the edges of $G[A]$. If such a set were to exist, it would be a serious headache since the method we used in the previous proof to handle it no longer applies. But in fact no such set $X$ exists, because if it did we could find a $K_{t,t}$ subgraph, by counting the edges between $X$ and $A \setminus X$.

That completes the sketch; now the proof itself. We are proving 1.3, which we restate in terms of denseness as follows:

**3.2** *For all integers $t, c \geq 1$, there exists $d \geq 1$, such that if $G$ has denseness at least $d$ and $\tau(G) < t$, then there are anticomplete subsets $A, B \subseteq V(G)$ where $G[A], G[B]$ both have denseness at least $c$.*

**Proof.** Define $p := \max(32c, 4t)$ and let $q$ be an integer with $e^{-q/32}2^{8p+4} \leq 1/2$. Choose $s$ with $st \geq 2q^2 + 2^{2q+1}q(t-1)$, and choose $d$ with

$$d > \max\left(p + 2st, 2ct + 2st + ts^t 2^{2t}, 2st + 2^{8p+4}c + 3p + 2\right).$$

We will show that $d$ satisfies the theorem.

Let $G$ be a graph with denseness at least $d$ and $\tau(G) < t$. Choose vertex-disjoint subsets $R_1, \ldots, R_k$ of $V(G)$ with $k$ maximum, such that for $1 \leq i \leq k$, $R_i$ is a $p$-rock of $G \setminus (R_1 \cup \cdots \cup R_{i-1})$ and $|R_i| \leq s$.

(1) *$k \leq 2t$.*

Suppose that $k \geq 2t$, and let $R_1 \cup \cdots \cup R_{2t} = R$. For $1 \leq i \leq 2t$ let $Z_i$ be the set of all vertices in $V(G) \setminus R_i$ that have no neighbour in $R_i$. Let $W$ be the set of all $v \in V(G) \setminus R$ that have a neighbour in $R_i$ for at least $t$ values of $i \in \{1, \ldots, 2t\}$. For each $I \subseteq \{1, \ldots, 2t\}$ with $|I| = t$, and each choice of $a_i \in R_i$ for each $i \in I$, there are fewer than $t$ vertices in $V(G) \setminus R$ adjacent to $a_i$ for each $i \in I$, since $\tau(G) < t$. For each $I$ there are at most $s^t$ choices of the vertices $a_i$ $(i \in I)$, and so there are at most $ts^t$ vertices in $V(G) \setminus R$ with a neighbour in $R_i$ for each $i \in I$. Since there are at most $2^{2t}$ choices of $I$, it follows that $|W| \leq ts^t 2^{2t}$. Thus $|R \cup W| \leq 2st + ts^t 2^{2t}$, and so at most $(2st + ts^t 2^{2t})|G|$ edges have an end in $R \cup W$. Since $G$ has at least $d|G|$ edges, there are at least $(d - (2st + ts^t 2^{2t}))|G|$ edges with neither end in $R \cup W$. For every such edge, say $uv$, since $u$ has a neighbour in at most $t - 1$ of $R_1, \ldots, R_{2t}$, and the same for $v$, there exists $i \in \{1, \ldots, 2t\}$ such that neither of $u, v$ has a neighbour in $R_i$, that is, $u, v \in Z_i$. Consequently there exists $i \in \{1, \ldots, 2t\}$ such that at least $(d - (2st + ts^t 2^{2t}))|G|/(2t)$ edges $uv$ of $G$ have both ends in $Z_i$. It follows that $G[Z_i]$ has denseness at least $(d - (2st + ts^t 2^{2t}))/(2t) \geq c$, and it is anticomplete to the $p$-rock $R_i$, and so the theorem holds. This proves (1).

Let $R = R_1 \cup \cdots \cup R_k$. Thus $|R| \leq 2st$ by (1). Consequently at most $2st|G|$ edges of $G$ have an end in $R$, and so the graph $G \setminus R$ has at least $(d - 2st)|G|$ edges. Since $d - 2st \geq p$, there is a rock $A$ of $G \setminus R$. From the maximality of $k$, $|A| > s$.

From 2.3, there is a partition of $E(G[A])$ into sets $M_0, \ldots, M_n$ for some $n \geq 0$, such that

- there is a subset $X \subseteq A$ with $|X| \leq 2q - 2$ such that every edge in $M_0$ is incident with a vertex in $X$; and
- $M_1, \ldots, M_n$ are all matchings, each with cardinality $q$.

(2) $|M_0| \leq 2t|A| \leq p|A|/2$, *and hence* $M_1 \cup \cdots \cup M_n$ *has cardinality at least* $p|A|/2$.

There are at most $2q^2$ edges in $E(G[A])$ with both ends in $X$, since $|X| \leq 2q$. We need to count the number of edges with exactly one end in $X$. For each subset $Y$ of $X$ with $|Y| = t$, there are at most $t - 1$ vertices adjacent to each vertex in $Y$, and so there are at most $2^{2q}(t - 1)$ vertices in $A \setminus X$ with at least $t$ neighbours in $X$. Hence there are at most $2^{2q}(t - 1)|X| \leq 2^{2q+1}q(t - 1)$ edges $uv$ of $G[A]$ with $u \in X$ and $v \in A \setminus X$ such that $v$ has at least $t$ neighbours in $X$. But there are at most $(t - 1)|A|$ edges $uv$ of $G[A]$ with $u \in X$ and $v \in A \setminus X$ such that $v$ has fewer than $t$ neighbours in $X$; so altogether there are at most

$$2q^2 + 2^{2q+1}q(t-1) + (t-1)|A| \leq \left((2q^2 + 2^{2q+1}q(t-1))/s + (t-1)\right)|A| \leq 2t|A| \leq p|A|/2$$

edges of $G[A]$ with an end in $X$, since $|A| \geq s$. This proves the first statement of (2). The second follows since $|E(G[A])| \geq p|A|$. This proves (2).

Let $\mathcal{I}$ be the set of all subsets of $\{1, \ldots, 8p + 4\}$ with cardinality $4p + 2$.

(3) *There is a partition of* $A$ *into* $8p + 4$ *subsets* $A_1, \ldots, A_{8p+4}$, *such that for each* $I \in \mathcal{I}$ *there are at least* $p|A|/32$ *edges of* $G[A]$ *that have both ends in* $\bigcup_{i \in I} A_i$.

For each $v \in A$, choose $\phi(v) \in \{1, \ldots, 8p + 4\}$, uniformly and independently at random. For $1 \leq i \leq 8p + 4$ let $A_i$ be the set of all $v \in A$ with $\phi(v) = i$. Thus $A_1, \ldots, A_{8p+4}$ are pairwise disjoint sets with union $A$. We will show that with positive probability, the statement of (3) is satisfied. For each $I \in \mathcal{I}$ let $A_I := \bigcup_{i \in I} A_i$.

For $I \in \mathcal{I}$ and $1 \leq j \leq n$, we say that $j$ is *bad* for $I$ if at most $q/8$ edges of $M_j$ have both ends in $A_I$. By the first statement of 2.4, since each vertex of $A$ belongs to $A_I$ independently with probability $1/2$, it follows that the probability that $j$ is bad for $I$ is at most $e^{-q/32}$. Consequently the expected number of values of $j \in \{1, \ldots, n\}$ such that $j$ is bad for some $I \in \mathcal{I}$ is at most

$$ne^{-q/32}|\mathcal{I}| \leq ne^{-q/32}2^{8p+4} \leq n/2.$$

Let $J$ be the set of $j \in \{1, \ldots, n\}$ such that $j$ is not bad for any $I \in \mathcal{I}$. It follows that $|J| \geq n/2$ with positive probability. Moreover, if $|J| \geq n/2$, then

$$|\bigcup_{j \in J} M_j| \geq |M_1 \cup \cdots \cup M_n|/2 \geq p|A|/4$$

by (2). But for each $I \in \mathcal{I}$, at least $q/8$ edges of $M_j$ have both ends in $A_I$, for each $j \in J$; and so at least $1/8$ of the edges of $\bigcup_{j \in J} M_j$ have both ends in $A_I$. Consequently, with positive probability at least $p|A|/32$ edges of $G[A]$ have both ends in $A_I$. This proves (3).

Choose $A_1, \ldots, A_{8p+4}$ as in (3), and as before, let $A_I := \bigcup_{i \in I} A_i$ for each $I \in \mathcal{I}$. For each $I \in \mathcal{I}$, let $W_I$ be the set of vertices in $V(G) \setminus (A \cup R)$ with no neighbour in $A_I$. Since for every edge $uv$ of $G \setminus R$ with $u, v \notin A$, $u$ has a neighbour in $A_i$ for at most $2p+1$ values of $i \in \{1, \ldots, 8p+4\}$ by 2.2, and the same for $v$, it follows that there exists $I \in \mathcal{I}$ with $u, v \in W_I$. But, since $G[A_I]$ has denseness at least $p/32 \geq c$ by (3), and is anticomplete to $W_I$, we may assume that $G[W_I]$ has denseness less than $c$, and so there are at most $c|G|$ edges of $G \setminus R$ with both ends in $W_I$. We will show that this leads to a contradiction. Since there are only at most $2^{8p+4}$ choices of $I$, there are at most $2^{8p+4}c|G|$ edges of $G \setminus R$ with neither end in $A$. But there are at most $(2p+1)|G|$ edges with one end in $A$ and the other in $V(G) \setminus (A \cup R)$, since every vertex in $V(G) \setminus (A \cup R)$ has at most $2p+1$ neighbours in $A$ by 2.2. Also, from the minimality of $A$ (in the definition of a rock), if we delete a vertex of $A$, the remainder induces a graph with fewer than $p(|A|-1)$ edges, and so $G[A]$ has fewer than

$$p(|A|-1) + |A| \leq (p+1)|A| \leq (p+1)|G|$$

edges. Altogether, then, $G \setminus R$ has fewer than

$$2^{8p+4}c|G| + (2p+1)|G| + (p+1)|G| < (d-2st)|G|$$

edges. But we already saw that $G \setminus R$ has at least $(d-2st)|G|$ edges, a contradiction. This proves 3.2. ∎

# 4 Tournaments

There is an interesting extension of 1.1 to tournaments. If $G$ is a tournament, a subset $X \subseteq V(G)$ is *acyclic* if $G[X]$ has no directed cycle; and $\chi(G)$ is the minimum $k$ such that $V(G)$ is the union of $k$ acyclic subsets. Again, we write $\chi(A)$ for $\chi(G[A])$ when $A \subseteq V(G)$. If $A, B \subseteq V(G)$ are disjoint, we say $A$ is *complete* to $B$ if every vertex in $B$ is adjacent from every vertex in $A$.

One might hope that:

**4.1 Conjecture:** *For all integers $c \geq 1$ there exists $d \geq 1$ such that if $G$ is a tournament and $\chi(G) \geq d$, there are disjoint $A, B \subseteq V(G)$, with $A$ complete to $B$, and both inducing tournaments with chromatic number at least $c$.*

We discuss this further in another paper [8], where we prove that it implies 1.1. (Indeed, very recently Klingelhoefer and Newman [6] have extended that result, proving that 4.1 is equivalent to 1.1.) We also prove the following two results (among others):

**4.2** *For all $c \geq 1$ there exists $d \geq 1$ such that if $G$ is a tournament with $\chi(G) \geq d$, then there exist disjoint $A, B \subseteq V(G)$ with $A$ complete to $B$, where $A$ is a cyclic triangle and $\chi(B) \geq c$.*

(A *cyclic triangle* is a three-vertex set inducing a directed cycle.) The second result concerns domination number. A tournament $G$ has *domination number $k$* if $k$ is minimum such that for some set $X \subseteq V(G)$ with $|X| = k$, every vertex in $V(G) \setminus X$ is adjacent from some vertex in $X$.

**4.3** *For every integer $c \geq 1$, there exists $d \geq 1$ such that if $G$ is a tournament with domination number at least $d$, then there are disjoint $A, B \subseteq V(G)$, such that $A$ is complete to $B$ and $\chi(A), \chi(B) \geq c$.*


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## References


[1] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebl, A. Scott, P. Seymour and S. Thomassé, "Tournaments and colouring", *J. Combinatorial Theory, Ser. B*, **103** (2013), 1–20.

[2] M. El-Zahar and P. Erdős, "On the existence of two non-neighboring subgraphs in a graph", *Combinatorica* **5** (1985), 295–300.

[3] P. Erdős, "Problems and results on chromatic numbers in finite and infinite graphs", in *Graph Theory and its Applications to Algorithms and Computer Science*, (Y. Alavi et al., eds.) J. Wiley and Sons, 1985, 201–213.

[4] A. Harutyunyan, T. Le, S. Thomass, and H. Wu, "Coloring tournaments: From local to global", *J. Combinatorical Theory, Ser. B*, **138** (2019), 166–171.

[5] W. Hoeffding, "Probability inequalities for sums of bounded random variables" *J. American Statistical Assoc.* **58** (1963), 13–30.

[6] F. Klingelhoefer and A. Newman, "Bounding the chromatic number of dense digraphs by arc neighborhoods", `arXiv:2307.04446`.

[7] E. C. Milner, "The use of elementary substructures in combinatorics", *Discrete Math.* **136** (1994), 243–252.

[8] T. Nguyen, A. Scott and P. Seymour, "Some results and conjectures on tournament structure", submitted for publication, `arXiv:2306.02364`.