Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás' conjectures

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Abstract

Gyárfás conjectured in 1985 that for all k, ℓ, every graph with no clique of size more than k and no odd hole of length more than ℓ has chromatic number bounded by a function of k, ℓ. We prove three weaker statements:

- Every triangle-free graph with sufficiently large chromatic number has an odd hole of length different from five;

- For all ℓ, every triangle-free graph with sufficiently large chromatic number contains either a 5-hole or an odd hole of length more than ℓ;

- For all k, ℓ, every graph with no clique of size more than k and sufficiently large chromatic number contains either a 5-hole or a hole of length more than ℓ.

1 Introduction

All graphs in this paper are finite, and without loops or parallel edges. A *hole* in a graph G is an induced subgraph which is a cycle of length at least four, and an *odd hole* means a hole of odd length. (The *length* of a path or cycle is the number of edges in it, and we sometimes call a hole of length n an *n-hole*.) In 1985, A. Gyárfás [2] made a sequence of three famous conjectures:

1.1 Conjecture: *For every integer k there exists $n(k)$ such that every graph G with no clique of cardinality more than k and no odd hole has chromatic number at most $n(k)$.*

1.2 Conjecture: *For all integers k, ℓ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

1.3 Conjecture: *For all integers k, ℓ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no odd hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

In a recent paper [4], two of us proved the first conjecture. Note that the first two conjectures are special cases of the third. In the case of the third conjecture, we might as well assume that $k \geq 2$, and $\ell \geq 3$ and is odd. Thus it follows from [4] that conjecture 1.3 holds for all pairs (k, ℓ) when $\ell = 3$. No other cases have been settled at the time of writing this paper, and the cases when $k = 2$ are presumably the simplest to attack next. Here we settle the first open case, when $k = 2$ and $\ell = 5$. (Since this paper was submitted for publication, we have proved the second conjecture [1], and two of us proved the third [5] when $k = 2$; part of the proof of the latter uses results of this paper, however, so this paper is not completely redundant.)

The conjecture 1.3 when $(k, \ell) = (2, 5)$ asserts that all pentagonal graphs have bounded chromatic number, where we say a graph is *pentagonal* if every induced odd cycle in it has length five (and in particular, it has no triangles). Pentagonal graphs might all be 4-colourable as far as we know (the 11-vertex Grötzsch graph is pentagonal and not 3-colourable), but at least they do indeed all have bounded chromatic number. The following is our main result:

1.4 *Every pentagonal graph is* 58000-*colourable.*

The proof of 1.4 occupies almost the whole paper. (Much of the proof needs just that G is triangle-free and has no odd hole of length more than ℓ, for any fixed ℓ, and so we have written it in this generality wherever we could.) We prove:

- if G has no triangle and no odd hole of length more than ℓ, and for every vertex v the set of vertices with distance at most two from v has chromatic number at most some k, then $\chi(G)$ is bounded by a function of k and ℓ;

- if G is pentagonal, and $\chi(G)$ is large, then there is an induced subgraph with large chromatic number in which for every vertex v the set of vertices with distance at most two from v has chromatic number at most 5.

Together these imply that every pentagonal graph has bounded chromatic number. Both of these are consequences of a lemma, a variant of a theorem of [4], asserting roughly that for all ℓ, if G is triangle-free and has no odd hole of length more than ℓ, and $\chi(G)$ is large, then there is an induced

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subgraph H such that for some vertex v_0 of H, if we partition $V(H)$ by distance in H from v_0, then all these "level sets" are stable except for one with large χ. We prove this lemma first, and then apply it to prove the two bulleted statements in later sections.

At the end of this paper, we prove two further special cases of conjectures 1.2 and 1.3: we show that conjecture 1.2 holds if in addition we assume that G contains no 5-hole, and that conjecture 1.3 holds if in addition we assume that G contains no triangle and no 5-hole More precisely, we prove the next two results, where $\omega(G)$ denotes the size of the largest clique of G:

1.5 *Let $\ell \geq 2$ be an integer, and let G be a triangle-free graph with no 5-hole and no odd hole of length more than $2\ell + 1$. Then $\chi(G) \leq (\ell + 1)4^{\ell-1}$.*

1.6 *Let $\ell \geq 3$ be an integer, and let G be a graph with no 5-hole and no hole of length more than ℓ. Then*
$$\chi(G) \leq (2\ell - 2)^{2^{\omega(G)}}.$$

The last was proved (but not published) by the second author some time ago, and improves on [3].

2   Lollipops

In [2], Gyárfás gave a neat proof that for any fixed path P, all graphs with no induced subgraph isomorphic to P and with bounded clique number also have bounded chromatic number, and in this section we use basically the same proof for a lemma that we need later. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we sometimes write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity. If $x \in V(G)$ and $Y \subseteq V(G)$, the *distance* in G of x from Y (or of Y from x) is the length of the shortest path containing x and a vertex in Y. Let us say a *lollipop* in a graph G is a pair (C, T) where $C \subseteq V(G)$ and T is an induced path of G with vertices $t_1\text{-}\cdots\text{-}t_k$ in order, say, with $k \geq 2$, satisfying:

- $V(T) \cap C = \emptyset$;

- $G[C]$ is connected;

- t_k has a neighbour in C; and

- $t_1, \ldots, t_{k-1}$ have no neighbours in C.

With this notation, the *cleanliness* of a lollipop (C, T) in G is the maximum l such that $t_1, \ldots, t_l$ all have distance (in G) at least three from C (or 0 if t_1 has distance two from C). It follows that the cleanliness is at most $k - 2$. We call t_1 the *end* of the lollipop. If (C, T) and (C', T') are lollipops in G, we say the second is a *licking* of the first if $C' \subseteq C$, and they have the same end, and T is a subpath of T', and $V(T') \subseteq V(T) \cup C$ (and consequently the cleanliness of (C', T') is at least that of (C, T)). We observe first:

2.1 *Let (C, T) be a lollipop in G, and let $C' \subseteq C$ be non-null, such that $G[C']$ is connected. Then there is a path T' of G such that (C', T') is a licking of (C, T).*

Proof. Let T be t_1-$\cdots$-t_k, where (C,T) has end t_1. If t_k has a neighbour in C' then we may take $T = T'$, so we assume not. Since t_k has a neighbour in C, there is a path P of G with one end t_k, and with $V(P) \subseteq C \cup \{t_k\}$, such that the other end of P has a neighbour in C'. Choose such a path P with minimum length. Then $V(P) \cap C' = \emptyset$, and $P' = T \cup P$ is an induced path. No vertex of P' has a neighbour in C' except its last, and so $(C', T \cup P)$ is a licking of (C,T) as required. This proves 2.1. ∎

For a vertex v of G, we denote the set of neighbours of v in G by $N(v)$ or $N_G(v)$, and for $r \geq 1$, we denote the set of vertices at distance exactly r from v by $N^r(v)$ or $N_G^r(v)$. We need the following:

2.2 *Let $h, \kappa \geq 0$ be integers. Let G be a graph such that $\chi(N^2(v)) \leq \kappa$ for every vertex v; and let (C,T) be a lollipop in G, with $\chi(C) > h\kappa$. Then there is a licking (C', T') of (C,T), with cleanliness at least h more than the cleanliness of (C,T), and such that $\chi(C') \geq \chi(C) - h\kappa$.*

Proof. We proceed by induction on h. If $h = 0$ we may take $(C', T') = (C,T)$; so we assume that $h > 0$, and that the result holds for $h - 1$. Let (C,T) have cleanliness c say (where possibly $c = 0$), and let T have vertices t_1-$\cdots$-t_k in order, where t_1 is the end. Thus t_i has distance at least three from C for $1 \leq i \leq c$, and so $k \geq c+2$. Since $\chi(N^2(t_{c+1})) \leq \kappa$, it follows that $\chi(C \setminus N^2(t_{c+1})) \geq \chi(C) - \kappa$, and so there is a component C'' of $C \setminus N^2(t_{c+1})$ with

$$\chi(C'') \geq \chi(C) - \kappa > (h-1)\kappa \geq 0.$$

By 2.1, there exists T'' such that (C'', T'') is a licking of (C,T). Since t_{c+1} has distance at least three from C'', it follows that (C'', T'') has cleanliness at least $c + 1$. From the inductive hypothesis, there is a licking (C', T') of (C'', T'') and hence of (C,T) that satisfies the theorem. This proves 2.2. ∎

3   Stable levelling

Let G be a graph. A *levelling* $\mathcal{L}$ in G is a sequence $(L_0, L_1, \ldots, L_k)$ of disjoint subsets of $V(G)$, with the following properties:

- $|L_0| = 1$;

- for each i with $1 \leq i \leq k$, every vertex in L_i has a neighbour in L_{i-1}; and

- for $0 \leq i, j \leq k$ with $|j - i| > 1$, there are no edges between L_i and L_j.

The levelling $\mathcal{L}$ is called *stable* if each of the sets $L_0, \ldots, L_{k-1}$ is stable (we do not require L_k to be stable). For $1 \leq i \leq k$, a *parent* of $v \in L_i$ is a neighbour u of v in L_{i-1} (and we also say v is a *child* of u).

The next result is a variant of a theorem proved in [4]; we could use that theorem directly, but the modification here works better numerically. Let the *odd hole number* of G be the length of the longest induced odd cycle in G (or 1, if G is bipartite). If $(L_0, \ldots, L_k)$ is a stable levelling, we call L_k its *base*.

3.1 *Let G be a triangle-free graph with odd hole number at most $2\ell+1$, such that $\chi(N^2(v)) \le \kappa$ for every vertex v. Let $(L_0, L_1, \ldots, L_k)$ be a levelling in G. Then there is a stable levelling in G with base of chromatic number at least $(\chi(L_k) - (\ell-1)\kappa)/2$.*

Proof. We may assume $\ell \ge 1$, since otherwise G is bipartite and the result is trivial. Also we may assume that $\chi(L_k) > (2\ell-1)\kappa$, because otherwise the stable levelling (L_0, L_1) satisfies the theorem. We proceed by induction on $|V(G)|$, and so we may assume:

- $V(G) = L_0 \cup L_1 \cup \cdots \cup L_k$;

- $G[L_k]$ is connected; and

- for $0 \le i < k$ and every vertex $u \in L_i$, there exists $v \in L_{i+1}$ such that u is its only parent (for if not, we may replace L_i by $L_i \setminus \{u\}$).

Let $L_0 = \{s_0\}$, and inductively for $1 \le i \le k$, choose $s_i \in L_i$ such that s_{i-1} is its only parent. Then s_0-s_1-$\cdots$-s_k is an induced path S say.

Now s_{k-2} has no neighbour in L_k, so $(L_k, s_{k-2}$-$s_{k-1})$ is a lollipop. By 2.2, there is a licking of this lollipop, say (C', T'), with cleanliness at least $2\ell - 1$ and with $\chi(C') \ge \chi(L_k) - (2\ell-1)\kappa$. Let the first $2\ell - 1$ vertices of T' be s_{k-2}-s_{k-1}-t_1-$\cdots$-$t_{2\ell-3}$.

Let $N(S)$ be the set of vertices of G not in S but with a neighbour in S. If $v \in L_i \cap N(S)$, then v is adjacent to exactly one of s_i, s_{i-1} and has no other neighbour in S; because every neighbour of v belongs to one of L_{i-1}, L_i, L_{i+1}, and G is triangle-free, and v is not adjacent to s_{i+1} since s_i is the only parent of s_{i+1}. So every vertex in $L_i \cap N(S)$ has one of two possible types. We say the *type* of a vertex $v \in L_i \cap N(S)$ is α where $\alpha = 1$ or 2 depending whether v is adjacent to s_{i-1} and not to s_i, or adjacent to s_i and not to s_{i-1}.

Let us fix a type α. Let $V(\alpha)$ be the minimal subset of $V(G) \setminus V(S)$ such that

- every vertex in $N(S)$ of type α belongs to $V(\alpha)$; and

- for every vertex $v \in V(G) \setminus (V(S) \cup N(S))$, if some parent of v belongs to $V(\alpha)$ then $v \in V(\alpha)$.

Consequently, for every vertex $v \in V(\alpha)$, there is a path starting at v and ending at some vertex in $N(S)$ of type α, such that each vertex of the path (except v) is the parent of the previous vertex, and no vertex of the path belongs to $N(S)$ except the last.

There are only two types α, and so there is a type α such that $\chi(V(\alpha) \cap C') \ge \chi(C')/2 > 0$. Let C be the vertex set of a component of $G[V(\alpha) \cap C']$ with maximum chromatic number, so

$$\chi(C) \ge \chi(C')/2 \ge (\chi(L_k) - (2\ell-1)\kappa)/2.$$

By 2.1, there is a path T such that (C, T) is a licking of (C', T').

Let $J_k = C$, and for $i = k-1, k-2, \ldots, 1$ choose $J_i \subseteq V(\alpha) \cap L_i$ minimal such that every vertex in $J_{i+1} \setminus N(S)$ has a neighbour in J_i. It follows from the cleanliness of (C', T') that $J_{k-1} \cap N(S) = \emptyset$, and no vertex in J_{k-1} is adjacent to any of $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2\ell-3}$.

(1) *For $1 \le i \le k-1$, if $v \in J_i$ and v is nonadjacent to s_i, then there is an induced path P_v between v and s_i of length at least $2l - 3 + 2(k-i)$ with interior in $L_{i+1} \cup \cdots \cup L_k$, such that*

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- *if $i \leq k-2$, no vertex in J_i different from v has a neighbour in the interior of P_v*

- *if $i = k-1$, and $u \in J_i \setminus \{v\}$ has a neighbour in the interior of P_v, then the induced path between u, s_{k-1} with interior in $V(P_v)$ has length at least $2\ell - 1$.*

Since $v \in J_i$, v has a neighbour in $J_{i+1} \setminus N(S)$ with no other parent in J_i; and so there is a path $v = p_i\text{-}p_{i+1}\text{-}\cdots\text{-}p_k$ such that

- $p_j \in J_j$ for $i \leq j \leq k$

- $p_j \notin N(S)$ for $i < j \leq k$

- p_{j-1} is the only parent of p_j in J_{j-1} for $i < j \leq k$.

Since $p_{k-1} \in J_{k-1}$, and no vertex in J_{k-1} is adjacent to any of $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-3}$, it follows that there is an induced path from p_{k-1} to s_{k-1} with interior in L_k containing all of $t_1, \ldots, t_{2l-3}$ and at least one more vertex of L_k, and therefore with length at least $2\ell - 1$. Its union with the path $p_i\text{-}\cdots\text{-}p_{k-1}$ and the path $s_{k-1}\text{-}s_{k-2}\text{-}\cdots\text{-}s_i$ is an induced path between v and s_i, of length at least $2\ell - 3 + 2(k-i)$. If $u \in J_i \setminus \{v\}$ and has a neighbour in the interior of P_v, then since u is nonadjacent to all of $s_{i+1}, \ldots, s_{k-1}, p_{i+1}, \ldots, p_{k-1}$ (because u has no neighbour in $L_{i+2} \cup \cdots \cup L_k$, and s_{i+1} has a unique parent s_i, and p_{i+1} has no parent in J_i except p_i), it follows that $i = k-1$; and since no vertex in J_{k-1} is adjacent to any of $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-3}$, this proves (1).

For $1 \leq i \leq k$ and for every vertex $v \in J_i$, either $v \in N(S)$ or it has a parent in J_{i-1}; and so there is a path $v = r_i\text{-}r_{i-1}\text{-}\cdots\text{-}r_h$ for some $h \leq i$, such that $r_j \in J_j$ for $h \leq j \leq i$, and $r_h \in N(S)$, and $r_j \notin N(S)$ for $h+1 \leq j \leq i$. Since r_h has a neighbour in S, one of

$$r_i\text{-}r_{i-1}\text{-}\cdots\text{-}r_h\text{-}s_{h-1}\text{-}s_h\text{-}s_{h+1}\text{-}\cdots\text{-}s_i,$$

$$r_i, r_{i-1}\text{-}\cdots\text{-}r_h\text{-}s_h\text{-}s_{h+1}\text{-}\cdots\text{-}s_i$$

is an induced path (the first if $\alpha = 1$ and the second if $\alpha = 2$). We choose some such path and call it R_v. Note that for all $v \in J_1 \cup \cdots \cup J_k$, the path R_v has even length if $\alpha = 1$, and odd length otherwise.

(2) *For $0 \leq i \leq k-1$, J_i is stable.*

Suppose that $u, v \in J_i$ are adjacent. Since G is triangle-free and u, v have the same type, not both $u, v \in N(S)$. Suppose that $u \in N(S)$, and hence $v \notin N(S)$. Since $N(S) \cap J_{k-1} = \emptyset$ it follows that $i \leq k-2$. Consequently u has no neighbour in the interior of P_v, where P_v is as in (1), and so $P_v \cup R_v$, $s_i\text{-}P_v\text{-}v\text{-}u\text{-}R_u\text{-}s_i$ are both holes of length at least $2\ell + 2$, of different parity, which is impossible. So $u, v \notin N(S)$. We claim that there is a path P of length at least $2\ell - 1$, from one of u, v to s_i, with interior in $L_{i+1} \cup \cdots \cup L_k$, such that the other (of u, v) has no neighbour in its interior. For if u has no neighbour in the interior of P_v then we may take $P = P_v$, where P_v is as in (1); and if u has such a neighbour, let P be the induced path between u and s_i with interior a subset of the interior of P_v. Note that in the second case, v has no neighbour in the interior of P, since G is triangle-free. This proves that the desired path P exists; say from v to s_i. Now the union of P and R_v is a hole of length at least $2\ell + 2$, and so P, R_v have the same parity. But the union of

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P and the path v-u-R_u-s_i is also a hole, of length at least $2\ell + 3$, and since R_u, R_v have the same parity this is impossible. This proves (2).

If $\alpha = 1$ let $M_i = \{s_i\} \cup J_i$ for $0 \leq i \leq k$, and if $\alpha = 2$ let $M_0 = \{s_1\}, M_i = \{s_{i+1}\} \cup J_i$ for $1 \leq i < k$, and $M_k = J_k$. In each case $(M_0, \ldots, M_k)$ is a levelling satisfying the theorem. This proves 3.1. ∎

We deduce:

3.2 *Let G be pentagonal, and let $n \geq 1$ be an integer. If $\chi(G) \geq 10n - 9$, there is a stable levelling in G with base of chromatic number at least n.*

Proof. Let G' be a component of G with $\chi(G') = \chi(G)$. Choose $v_0 \in V(G')$, and for $i \geq 0$ let L_i be the set of vertices in G' with distance i from v_0. There exists k such that $\chi(L_k) \geq \chi(G)/2$ and hence $\chi(L_k) \geq 5n - 4$. Now $(L_0, \ldots, L_k)$ is a levelling in G. By 3.1, taking $\ell = 2$ and $\kappa = n - 1$, either

- there is a vertex v with $\chi(N^2(v)) \geq n$, and hence there is a levelling (M_0, M_1, M_2) with $\chi(M_2) \geq n$, necessarily stable, or

- there is a stable levelling $(M_0, \ldots, M_k)$ in G with $\chi(M_k) \geq (\chi(L_k) - 3(n-1))/2 \geq n - 1/2$.

In either case the theorem holds. ∎

4 Reducing to bounded radius

Let $(L_0, \ldots, L_k)$ be a levelling. If $0 \leq i \leq j \leq k$ and $u \in L_i$ and $v \in L_j$, and there is a path between u, v of length $j - i$ with one vertex in each of $L_i, L_{i+1}, \ldots, L_j$, we say that u is an *ancestor* of v and v is a *descendant* of u.

4.1 *Let G be a triangle-free graph with odd hole number at most $2\ell + 1$. For $r = 2, 3$, let $\chi(N^r(v)) \leq \kappa_r$ for every vertex v. Then $\chi(G) \leq (12\ell - 6)\kappa_2 + 4\kappa_3 + 8$.*

Proof. Suppose that $\chi(G) > (12\ell - 6)\kappa_2 + 4\kappa_3 + 8$. There is a levelling in G with base of chromatic number at least $\chi(G)/2$, and so by 3.1, there is a stable levelling $(L_0, \ldots, L_k)$ in G with

$$\chi(L_k) \geq \chi(G)/4 - (\ell - 1/2)\kappa_2 > (2\ell - 1)\kappa_2 + \kappa_3 + 2.$$

We may choose it in addition such that $G[L_k]$ is connected, and for $0 \leq i < k$ every vertex in L_i has a descendant in L_k. Since $\chi(L_k) > 1$ it follows that $k > 1$. Choose $a_{k-2} \in L_{k-2}$. Let X_1 be the set of descendants of a_{k-2} in L_k; thus $X_1 \neq \emptyset$, and $\chi(X_1) \leq \kappa_2$, and since $\chi(L_k) > \kappa_2$, there is a component C_1 of $G[L_k \setminus X_1]$ with

$$\chi(C_1) \geq \chi(L_k) - \kappa_2 > (2\ell - 2)\kappa_2 + \kappa_3 + 2.$$

Since $G[L_k]$ is connected and $X_1 \neq \emptyset$, there exists $a_k \in X_1$ with a neighbour in C_1. Let a_{k-1} be a parent of a_k and child of a_{k-2}.

Let X_2 be the set of neighbours of a_k in C_1; then X_2 is stable and nonempty, and since $\chi(C_1) > 1$, there is a component C_2 of $C_1 \setminus X_2$ with

$$\chi(C_2) \geq \chi(C_1) - 1 > (2\ell - 2)\kappa_2 + \kappa_3 + 1,$$

and a neighbour $b_k \in L_k$ of a_k with a neighbour in C_2. Let b_{k-1} be a parent of b_k. Thus b_{k-1}, a_{k-2} are nonadjacent since $X_1 \cap C_1 = \emptyset$. Also b_{k-1}, a_{k-1} are nonadjacent since L_{k-1} is stable, and b_{k-1}, a_k are nonadjacent since G is triangle-free, and similarly a_{k-1}, b_k are nonadjacent. Consequently a_{k-2}-a_{k-1}-a_k-b_k-b_{k-1} is an induced path of G.

Let X_3 be the set of all children of b_{k-1}; then since X_3 is stable, and $\chi(C_2) > 1$, it follows that there is a component C_3 of $C_2 \setminus X_3$ with

$$\chi(C_3) \geq \chi(C_2) - 1 > (2\ell - 2)\kappa_2 + \kappa_3,$$

and a child c_k of b_{k-1} with a neighbour in C_3, taking $c_k = b_k$ if b_k has a neighbour in C_3. Thus $(C_3, b_{k-1}$-$c_k)$ is a lollipop. By 2.2, since $\chi(C_3) > (2\ell - 2)\kappa_2$, there is a licking (C_4, T) of $(C_3, b_{k-1}$-$c_k)$, with cleanliness at least $2\ell - 2$, such that

$$\chi(C_4) \geq \chi(C_3) - (2\ell - 2)\kappa_2 > \kappa_3.$$

Let T have vertices t_1-t_2-t_3-$\cdots$-t_m say, where $m \geq 2\ell$ and $t_1 = b_{k-1}$ and $t_2 = c_k$. Note that if $b_k \neq c_k$ then b_k has no neighbour in C_3 and in particular b_k has no neighbour in T except t_1.

Let X_4 be the set of all vertices of C_4 with distance three from b_{k-1}. Since $\chi(X_4) \leq \kappa_3$, and $\chi(C_4) - \kappa_3 > 0$, there is a component C_5 of $C_4 \setminus X_4$. By 2.1, there is a licking (C_5, S) say of (C_4, T). Let S have vertices t_1-$\cdots$-t_n say where $n \geq m$. Let $t_{n+1} \in V(C_5)$ be adjacent to t_n, and let d_{k-1} be a parent of t_{n+1}. Choose i with $1 \leq i \leq n + 1$ minimum such that d_{k-1} is adjacent to t_i. Note that d_{k-1} is nonadjacent to all of $t_1, \ldots, t_{2\ell-2}$ since (C_4, T) has cleanliness at least $2\ell - 2$ and hence so does (C_5, S); and so $i > 2\ell - 2$. Let P be the path t_2-t_3-$\cdots$-t_i-d_{k-1}. This path P is induced and has length $i - 1 \geq 2\ell - 2$.

Choose parents b_{k-2}, d_{k-2} of b_{k-1}, d_{k-1}. Since Since t_{n+1} is in C_5, it follows that b_{k-1}, d_{k-1} have distance at least three; and consequently $b_{k-2} \neq d_{k-2}$, and b_{k-2} is nonadjacent to d_{k-1}, and b_{k-1} is nonadjacent to d_{k-2}. Now $a_{k-2} \neq d_{k-2}$, since a_{k-2} has no descendant in C_1, and d_{k-2} has a descendant t_{n+1} in C_5 and hence in C_1. For the same reason a_{k-2} is nonadjacent to d_{k-1}, and in particular $a_{k-1} \neq d_{k-1}$.

Since $L_0, \ldots, L_{k-3}$ are stable, there is an induced path between b_{k-2}, d_{k-2} of even length with interior in $L_0 \cup \cdots \cup L_{k-3}$, and its union with the path b_{k-2}-b_{k-1}-t_2-P-d_{k-1}-d_{k-2} is a hole of length at least $2\ell + 3$, which consequently has even length; and so P has odd length. Now there is an even induced path Q between a_{k-1}, d_{k-1} with interior in $L_0 \cup \cdots \cup L_{k-2}$, not containing any neighbour of b_{k-1}; for if a_{k-1}, d_{k-2} are adjacent then the path a_{k-1}-d_{k-2}-d_{k-1} satisfies our requirements, and otherwise any even induced path between a_{k-2}, d_{k-2} with interior in $L_0 \cup \cdots \cup L_{k-3}$ (extended by the edges $a_{k-1}a_{k-2}$ and $d_{k-1}d_{k-2}$) provides the desired path. If $b_k \neq c_k$ then

$$a_{k-1}\text{-}a_k\text{-}b_k\text{-}b_{k-1}\text{-}c_k\text{-}P\text{-}d_{k-1}\text{-}Q\text{-}a_{k-1}$$

is an odd hole of length at least $2\ell + 4$, while if $b_k = c_k$ then

$$a_{k-1}\text{-}a_k\text{-}b_k\text{-}P\text{-}d_{k-1}\text{-}Q\text{-}a_{k-1}$$

is an odd hole of length at least $2\ell + 2$, in either case a contradiction. This proves 4.1. ∎

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Next we prove a variant of 4.1 in which κ_3 is eliminated, the following. The proof is almost the same, but differs in a couple of key places, and we felt it best to write it out completely, despite the duplication.

4.2 *Let G be a triangle-free graph with odd hole number at most $2\ell + 1$. Let $\chi(N^2(v)) \leq \kappa_2$ for every vertex v. Then $\chi(G) \leq (40\ell + 28)\kappa_2 + 40$.*

Proof. Suppose that $\chi(G) > (40\ell + 28)\kappa_2 + 40$. Choose a levelling $(M_0, \ldots, M_k)$ in G with base of chromatic number at least $\chi(G)/2$, and let $H = G[M_k]$.

(1) *There exists $v \in M_k$ and a parent $v_{k-1} \in M_{k-1}$ of v, and a levelling $(L_0, \ldots, L_3)$ in H with $L_0 = \{v\}$, such that L_3 is disjoint from $N_G^1(v_{k-1}) \cup N_G^2(v_{k-1})$, and $\chi(L_3) > (2\ell + 4)\kappa_2 + 2$.*

Choose $v \in V(H)$ with $\chi(N_H^3(v))$ maximum; $\chi(N_H^3(v)) = \kappa_3$ say. By 4.1 applied in H,

$$\chi(H) \leq (12\ell - 6)\kappa_2 + 4\kappa_3 + 8,$$

and since $\chi(H) \geq \chi(G)/2$, it follows that

$$\kappa_3 \geq \chi(G)/8 - (3\ell - 3/2)\kappa_2 - 2 > (2\ell + 5)\kappa_2 + 3.$$

Let $v_{k-1} \in M_{k-1}$ be a parent of v. Now the set of neighbours of v_{k-1} is stable, and $\chi(N_G^2(v)) \leq \kappa_2$, and so there exists $L_3 \subseteq N_H^3(v)$ disjoint from $N_G^1(v_{k-1}) \cup N_G^2(v_{k-1})$ with

$$\chi(L_3) \geq \kappa_3 - \kappa_2 - 1 > (2\ell + 4)\kappa_2 + 2.$$

Thus $(\{v\}, N_H^1(v), N_H^2(v), L_3)$ is the desired levelling in H. This proves (1).

It follows that v_{k-1} has no neighbour in L_1, since G is triangle-free, and has no neighbour in $L_2 \cup L_3$, since L_3 is disjoint from $N_G^1(v_{k-1}) \cup N_G^2(v_{k-1})$. In addition we may choose $L_0, \ldots, L_3$ such that $H[L_3]$ is connected, and every vertex in L_1 has a descendant in L_3. Note that L_2 might not be stable, but L_1 is stable since G is triangle-free. Choose $a_1 \in L_1$. Let X_1 be the set of descendants of a_1 in L_3; thus $X_1 \neq \emptyset$, and $\chi(X_1) \leq \kappa_2$, and since $\chi(L_3) > \kappa_2$, there is a component C_1 of $G[L_3 \setminus X_1]$ with

$$\chi(C_1) \geq \chi(L_3) - \kappa_2 > (2\ell + 3)\kappa_2 + 2.$$

Since $G[L_3]$ is connected and $X_1 \neq \emptyset$, there exists $a_3 \in X_1$ with a neighbour in C_1. Let $a_2 \in L_2$ be adjacent to a_1, a_3.

Let X_2 be the set of neighbours of a_3 in C_1; then X_2 is stable and nonempty, and since $\chi(C_1) > 1$, there is a component C_2 of $C_1 \setminus X_2$ with $\chi(C_2) \geq \chi(C_1) - 1 > (2\ell + 3)\kappa_2 + 1$, and a neighbour $b_3 \in L_3$ of a_3 with a neighbour in C_2. Let $b_2 \in L_2$ be adjacent to b_3. Thus b_2, a_1 are nonadjacent since $X_1 \cap C_1 = \emptyset$. Also b_2, a_3 are nonadjacent since H is triangle-free. (But b_2, a_2 might be adjacent.) Consequently one of a_1-a_2-a_3-b_3-b_2, a_1-a_2-b_2 is an induced path of H with even length.

Let X_3 be the set of all children of b_2; then since X_3 is stable, and $\chi(C_2) > 1$, it follows that there is a component C_3 of $C_2 \setminus X_3$ with

$$\chi(C_3) \geq \chi(C_2) - 1 > (2\ell + 3)\kappa_2,$$

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and a child c_3 of b_2 with a neighbour in C_3, taking $c_3 = b_3$ if b_3 has a neighbour in C_3. Thus $(C_3, b_2\text{-}c_3)$ is a lollipop. By 2.2 applied in G (not just in H), since $\chi(C_3) > (2\ell - 2)\kappa_2$, there is a licking (C_4, T) of $(C_3, b_2\text{-}c_3)$, with cleanliness at least $2\ell - 2$ in G, such that

$$\chi(C_4) \geq \chi(C_3) - (2\ell - 2)\kappa_2 > 5\kappa_2.$$

Let T have vertices $t_1\text{-}t_2\text{-}t_3\text{-}\cdots\text{-}t_m$ say, where $m \geq 2\ell$ and $t_1 = b_2$ and $t_2 = c_3$. Note that if $b_3 \neq c_3$ then b_3 has no neighbour in C_3 and in particular b_3 has no neighbour in T except t_1.

Let $b_1 \in L_1$ be adjacent to b_2; and let X_4 be the set of all vertices of C_4 with distance two in G from one of a_1, a_2, a_3, b_1, b_3. Since $\chi(X_4) \leq 5\kappa_2$, and $\chi(C_4) - 5\kappa_2 > 0$, there is a component C_5 of $C_4 \setminus X_4$. By 2.1, there is a licking (C_5, S) say of (C_4, T). Let S have vertices $t_1\text{-}\cdots\text{-}t_n$ say where $n \geq m$. Let $t_{n+1} \in V(C_5)$ be adjacent to t_n, and let $d_{k-1} \in M_{k-1}$ be a parent of t_{n+1}. (Note that d_{k-1} belongs to M_{k-1}, not to L_2; this is where this proof differs essentially from the proof of 4.1.) Choose i with $1 \leq i \leq n+1$ minimum such that d_{k-1} is adjacent to t_i. Note that d_{k-1} is nonadjacent to all of $t_1, \ldots, t_{2\ell-2}$ since (C_4, T) has cleanliness in G at least $2\ell - 2$ and hence so does (C_5, S); and so $i > 2\ell - 2$. Note also that d_{k-1} is nonadjacent to all of $a_1, a_2, a_3, b_1, b_2, b_3$, from the definition of C_5.

The path $c_3 = t_2\text{-}t_3\text{-}\cdots\text{-}t_i\text{-}d_{k-1}$ is induced and has length $i - 1 \geq 2\ell - 2$. Now there is an induced path between d_{k-1}, v with interior a subset of $M_0 \cup M_1 \cup \cdots \cup M_{k-2} \cup \{v_{k-1}\}$; and the union of this path with the previous one is an induced path P of length at least $2\ell - 1$ between t_2 and v. Note that none of $a_1, a_2, a_3, b_1, b_2, b_3$ have neighbours in the interior of P. Now the union of P and the path $c_3\text{-}b_2\text{-}b_1\text{-}v$ is a hole of length at least $2\ell + 2$, and so is even; and hence P has odd length. Let Q be the path

- $c_3\text{-}b_2\text{-}a_2\text{-}a_1\text{-}v$ if b_2, a_2 are adjacent;

- $c_3\text{-}b_2\text{-}b_3\text{-}a_3\text{-}a_2\text{-}a_1\text{-}v$ if b_2, a_2 are nonadjacent and $c_3 \neq b_3$; and

- $c_3\text{-}a_3\text{-}a_2\text{-}a_1\text{-}v$ if b_2, a_2 are nonadjacent and $c_3 = b_3$.

In each case Q is between c_3, v, and has even length, at least four. The union of P and Q is therefore an odd hole of length at least $2\ell + 3$, a contradiction. This proves 4.2. ∎

5 The Grötzsch graph

Let G be a graph, and H an induced subgraph of G. We say a levelling $(L_0, \ldots, L_k)$ in G is *over H* if $V(H) \subseteq L_k$. For $n \geq 1$ an *n-covering (in G, over H)* is a sequence of graphs $H = G_0, G_1, \ldots, G_n = G$, such that for $1 \leq i \leq n$ there is a stable levelling in G_i over G_{i-1}. For $n \geq 1$, let us say a graph H is *n-coverable* if there is an n-covering over H in some pentagonal graph G (and in particular, H itself is pentagonal).

The *Grötzsch graph* has vertex set $\{a_1, \ldots, a_5, b_1, \ldots, b_5, c\}$, where $a_1\text{-}a_2\text{-}\cdots\text{-}a_5\text{-}a_1$ is a cycle, a_i, b_i are both adjacent to a_{i-1} and a_{i+1} for $1 \leq i \leq 5$ (reading subscripts modulo 5), and c is adjacent to $b_1, \ldots, b_5$. We call the 5-hole $a_1\text{-}a_2\text{-}\cdots\text{-}a_5\text{-}a_1$ its *rim* and c its *apex*.

5.1 *The Grötzsch graph is not 1-coverable.*

Proof. Suppose it is, and let G be pentagonal, with a stable levelling $(L_0, \ldots, L_k)$, such that $G[L_k]$ has an induced subgraph H isomorphic to the Grötzsch graph. Let $V(H)$ be labelled as above. We may assume that $L_k = V(H)$, and L_{k-1} is minimal such that every vertex in $V(H)$ has a neighbour in L_{k-1}. For each $v \in L_{k-1}$, let $H(v)$ denote the set of neighbours of v in $V(H)$. Consequently:

(1) *For each $v \in L_{k-1}$, there exists $u \in H(v)$ with no neighbour in L_{k-1} except v.*

We call such a vertex u a *dependent* of v. If $u, v \in L_{k-1}$, by a *u-v gap* we mean an induced path P of G, with one end in $H(u)$ and the other in $H(v)$, and with no other vertex in $H(u) \cup H(v)$ (a vertex in $H(u) \cap H(v)$ forms a 1-vertex gap.) Thus a u-v gap is the interior of an induced path between u and v.

(2) *For all $u, v \in L_{k-1}$, no u-v gap has length three.*

For suppose some u-v gap has length three; then there is an induced path between u, v of length five, with interior in L_k. But u, v have neighbours in L_{k-2}, and so are joined by an induced path of even length with interior in the top of the levelling; and the union of these two paths is an odd hole of length at least seven, which is impossible.

(3) *For every four-vertex induced path u_1-u_2-u_3-u_4 of H, if $v, v' \in L_{k-1}$ and $u_1 \in H(v)$ and $u_4 \in H(v')$, then either one of $u_1, u_2 \in H(v')$, or one of $u_3, u_4 \in H(v)$.*

Because $H(v), H(v')$ are stable sets since G is triangle-free; and from (2) this path is not a u-v gap; and the claim follows.

(4) $|H(v_0)| \geq 2$ *for all $v_0 \in L_{k-1}$ with $c \in H(v_0)$.*

For suppose that $H(v_0) = \{c\}$. Then by (1), c has no other neighbour in L_{k-1}. So for every four-vertex induced path of H ending at c, say u_1-u_2-u_3-c, and for all $v \in L_{k-1}$ with $u_1 \in H(v)$, (3) implies that $u_3 \in H(v)$ (because $u_1, u_2 \notin H(v_0)$ since $|H(v_0)| = 1$, and $c \notin H(v)$ since c is a dependent of v_0). Choose $v_1 \in L_{k-1}$ with $a_1 \in H(v_1)$. From a_1-a_5-b_1-c it follows that $b_1 \in H(v_1)$, and similarly $b_3, b_4 \in H(v_1)$. Since $H(v_1)$ is stable, and the set $\{a_1, b_1, b_3, b_4\}$ is a maximal stable set of H, it follows that $H(v_1) = \{a_1, b_1, b_3, b_4\}$. Choose $v_3 \in L_{k-1}$ with $a_3 \in H(v_3)$; then (from the symmetry of H taking a_3 to a_1) it follows that $H(v_3) = \{a_3, b_3, b_1, b_5\}$. But b_4-a_5-a_4-b_5 is a v_1-v_3 gap contradicting (2). This proves (1).

(5) $|H(v_0)| = 3$ *for all $v_0 \in L_{k-1}$ with $c \in H(v_0)$.*

No stable set of H containing c has cardinality more than three, so we just need to show that $|H(v_0)| \neq 2$. Suppose not; then from the symmetry of H, we may assume that $H(v_0) = \{c, a_1\}$. One of c, a_1 is a dependent of v_0.

  Suppose first that c is a dependent of v_0. Choose $v_5 \in L_{k-1}$ with $a_5 \in H(v_5)$. From a_5-a_4-b_3-c and (3) it follows that $b_3 \in H(v_5)$, and from a_5-a_4-b_5-c that $b_5 \in H(v_5)$. Since a_2, b_3 are adjacent it follows that $a_2 \notin H(v_5)$; choose $v_2 \in L_{k-1}$ with $a_2 \in H(v_2)$. Then from the symmetry of H exchanging a_2, a_5 and fixing a_1, it follows $b_2, b_4 \in H(v_2)$. From a_5-b_1-c-b_2 and (3) it follows that $b_2 \in H(v_5)$

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(since $a_5, b_1 \notin H(v_2)$ because they both have neighbours in $H(v_2)$, and $c \notin H(b_5)$ because it is a dependent of v_1). From the same symmetry, $b_5 \in H(v_2)$; and so $a_3 \notin H(v_2)$ and $a_4 \notin H(v_5)$. But then a_5-a_4-a_3-a_2 is a v_5-v_2 gap, contrary to (2).

This shows that c is not a dependent of v_0, and so a_1 is its dependent. Choose $v_3 \in L_{k-1}$ with $a_3 \in H(v_3)$; then $b_5 \in H(v_3)$ from a_3-a_4-b_5-a_1, and $a_5 \in H(v_3)$ from a_3-b_4-a_5-a_1. Now $a_4 \notin H(v_3)$; choose $v_4 \in L_{k-1}$ with $a_4 \in H(v_4)$, and then similarly $a_2, b_2 \in H(v_4)$. But then a_5-b_1-c-b_2 is a v_3-v_2 gap, a contradiction. This proves (5).

In view of (5) and the symmetry we may assume henceforth that $H(v_0) = \{a_5, a_2, c\}$. One of a_5, a_2, c is a dependent of v_0. Suppose first that c is a dependent of v_0. Choose $v_3 \in L_{k-1}$ with $a_3 \in H(v_3)$; then $b_5 \in H(v_3)$ from a_3-a_4-b_5-c, and $b_3 \in H(v_3)$ from a_3-a_4-b_3-c. Similarly, let $a_4 \in H(v_4)$; then $b_2, b_4 \in H(v_4)$. From b_4-a_5-a_1-b_5 it follows that $a_5 \in H(v_3)$, and similarly $a_2 \in H(v_4)$. But then a_5-b_1-c-b_2 is a v_3-v_4 gap, a contradiction.

From the symmetry between a_2, a_5, we may therefore assume that a_5 is a dependent of v_0. Let $b_2 \in H(v_2)$; then $a_4 \in H(v_2)$ from b_2-a_3-a_4-a_5, and $b_4 \in H(v_2)$ from b_2-a_3-b_4-a_5. Also, $a_2 \in H(v_2)$ from a_4-b_5-a_1-a_2. Let $a_3 \in H(v_3)$; then $a_1 \in H(v_3)$ from a_3-b_2-a_1-a_5, and $c \in H(v_3)$ from a_1-b_5-c-b_4. But then a_4-a_5-b_1-c is a v_2-v_3 gap, a contradiction. This proves 5.1. ∎

6 Radius two

In this section we prove a bound on $\chi(N^2(v))$ for 2-coverable graphs, to allow us to apply 4.1. We begin with:

6.1 *Let $(L_0, \ldots, L_k)$ be a stable levelling in a pentagonal graph G, and let P be a 5-hole of $G[L_k]$. Choose $S \subseteq L_{k-1}$ minimal such that every vertex in P has a neighbour in S. Then*

- *$|S| = 3$;*

- *we can label the vertices of P as p_1-$\cdots$-p_5-p_1 in order, and label the elements of S as a, b, c, such that the edges of G between S and $V(P)$ are $ap_1, ap_3, bp_2, bp_4, cp_5$ and possibly cp_3;*

- *there exists $z \in L_{k-2}$ adjacent to every vertex in S.*

Proof. We begin by proving the first two assertions. Each vertex in S has at most two neighbours in P, because its neighbours form a stable set. Suppose that every vertex in S has exactly two neighbours in P. We may assume that $a \in S$ is adjacent to p_1, p_3; then choose $b \in S$ adjacent to p_2. It follows that b is adjacent to one of p_4, p_5, say p_4. Choose $c \in S$ adjacent to p_5; then c might also be adjacent to one of p_2, p_3, and from the symmetry we may assume it is not adjacent to p_2; and so $S = \{a, b, c\}$, and the first two assertions of the theorem hold. We may therefore assume that some vertex in S, say c, has only one neighbour in P, say p_5. From the minimality of S, no other vertex in S is adjacent to p_5. Choose $a \in S$ adjacent to p_3. If a has no more neighbours in P, then the path a-p_3-p_2-p_1-p_5-c can be completed via an even path joining a, c with interior in $L_0 \cup \cdots \cup L_{k-2}$ to an odd hole of length at least seven, which is impossible. So a has another neighbour in P, and since a is not adjacent to p_5 it is adjacent to p_1. Similarly, choose $b \in S$ adjacent to p_2; then b is also adjacent to p_4. From the minimality of S, $S = \{a, b, c\}$ and again the first two assertions hold.

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For the third assertion, choose $Z \subseteq L_{k-2}$ minimal containing a neighbour of each member of S. Suppose that there are distinct $z_1, z_2 \in Z$. From the minimality of Z, there exist $s_1, s_2 \in S$ such that for $1 \le i, j \le 2$, z_i is adjacent to s_j if and only if $i = j$. But from the second assertion of the theorem, there is a three-edge path joining s_1, s_2 with interior in $V(P)$, say s_1-p_1-p_2-s_2, where $p_1 p_2$ is an edge of P. Then z_1-s_1-p_1-p_2-s_2-z_2 is an induced path, and can be completed to an odd hole of length at least seven via an even induced path joining z_1, z_2 with interior in $L_0 \cup \cdots \cup L_{k-3}$, which is impossible. Thus $|Z| = 1$, and so the third assertion holds. This proves 6.1. ∎

We also need the following lemma.

6.2 *Let G be pentagonal, and let $(L_0, \ldots, L_k)$ be a stable covering in G of a graph H. Let $z \in V(H)$, and A be the set of all vertices $v \in N_H^2(z)$ such that every neighbour of v in L_{k-1} is adjacent to z. Then $\chi(A) \le 2$.*

Proof. Suppose that $\chi(A) > 2$; then there is a 5-hole P of $G[A]$. Choose a minimal subset S of $N_H^1(z)$ such that every vertex in p has a neighbour in S; then by 6.1 we may assume that $S = \{a, b, c\}$, where the edges between S and $V(P)$ are $ap_1, ap_3, bp_2, bp_4, cp_5$ and possibly cp_3. Choose $v \in L_{k-1}$ adjacent to p_5; by hypothesis, v is adjacent to z. Choose $a', b' \in L_{k-1}$ adjacent to a, b respectively. Consequently a', b' are not adjacent to z, and so have no neighbours in $V(P)$; and in particular, a', b' are different from v (although possibly $a' = b'$). There is an even induced path between v, a' with interior in $L_0 \cup \cdots \cup L_{k-2}$, and so the odd path v-p_5-p_4-p_3-a-a' is not induced, since its union with the previous path would form an odd hole of length at least seven. But a' has no neighbour in P (because $V(P) \subseteq A$), and v is not adjacent to a (because G is triangle-free) and v is not adjacent to a' (because L_{k-1} is stable), and it follows that v is adjacent to p_3. The same arguments applied to the path v-p_5-p_1-p_2-b-b' show that v is adjacent to p_2; yet not both of these are true since G is triangle-free, a contradiction. This proves 6.2. ∎

We deduce:

6.3 *If H is a 2-coverable graph and $z \in V(H)$ then $\chi(N_H^2(z)) \le 5$.*

Proof. Since H is 2-coverable, there is a 1-coverable graph G and a stable levelling $(L_0, \ldots, L_k)$ in G over H. Let A be the set of all vertices v in $N_H^2(z)$ such that every neighbour of v in L_{k-1} is adjacent to z, and let $B = N_H^2(z) \setminus A$. By 6.2, $\chi(A) \le 2$, so we may assume (for a contradiction) that $\chi(B) > 3$.

Choose $z_0 \in L_{k-1}$ adjacent to z. Since $N_G(z_0)$ is stable, it follows that $\chi(B \setminus N_G(z_0)) \ge 3$; and so there is a 5-hole P with $V(P) \subseteq B$, such that z_0 has no neighbours in P. Let $S_1 \subseteq N_H(z)$ be minimal such that every vertex in P has a neighbour in S_1. Each vertex in P has a neighbour in L_{k-1} nonadjacent to z, and so there exists a minimal subset S_2 of $L_{k-1} \setminus N_G(z)$ such that every vertex in P has a neighbour in S_2. By 6.1, $|S_1| = |S_2| = 3$.

(1) *If $a_1 \in S_1$ and $a_2 \in S_2$ are joined by a three-edge path with interior in $V(P)$, then a_1, a_2 are adjacent. In particular, if $a_1 \in S_1$ and $a_2 \in S_2$ both have two neighbours in $V(P)$ and have a common neighbour in $V(P)$ then they have the same neighbours in $V(P)$.*

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Let a_1, a_2 be adjacent to p_1, p_2 respectively, where $p_1 p_2$ is an edge of P. If a_1, a_2 are not adjacent, then the path z_0-z-a_1-p_1-p_2-a_2 is induced, and can be completed to an odd hole of length at least seven via an even induced path between z_0, a_2 with interior in $L_0 \cup \cdots \cup L_{k-2}$, which is impossible. This proves the first claim of (1). For the second, suppose that a_1, a_2 have a common neighbour in $V(P)$; then they are nonadjacent, and so cannot be joined by a three-edge path with interior in $V(P)$, by the first claim. This proves (1).

Let $S_i = \{a_i, b_i, c_i\}$ for $i = 1, 2$. By 6.1, for $i = 1, 2$ we may assume that a_i, b_i each have two neighbours in $V(P)$, and have no common neighbour in $V(P)$. So one of a_2, b_2, say a_2, is adjacent to a neighbour of a_1 in $V(P)$, and hence a_1, a_2 have the same neighbours in $V(P)$, by the second claim of (1). Therefore b_2 and b_1 have a common neighbour in $V(P)$, and so by the same argument, b_1, b_2 have the same neighbours in $V(P)$. If c_1 has two neighbours in $V(P)$, then it has a common neighbour in $V(P)$ with one of a_2, b_2, and so by the second claim of (1) it has the same neighbours in $V(P)$ as one of a_2, b_2, and hence the same as one of a_1, b_1, which is impossible by the minimality of S_1. Thus c_1 has exactly one neighbour in P, and similarly c_2 has exactly one neighbour in P, and the same neighbour as c_1.

We may therefore assume that for $i = 1, 2$, a_i is adjacent to p_2, p_4 and b_i to p_3, p_5, and c_i to p_1. By the first claim of (1), it follows that a_1 is adjacent to b_2, c_2, and b_1 to a_2, c_2, and c_1 to a_2, b_2. But then the subgraph induced on

$$\{p_1, p_2, p_4, p_5, a_1, b_1, c_1, a_2, b_2, c_2, z\}$$

is isomorphic to the Grötzsch graph (with rim a_2-c_1-p_1-p_5-b_1-a_2 and apex a_1), contradicting 5.1 since G is 1-coverable. This proves 6.3. ∎

Now we complete the proof of 1.4, which we restate:

6.4 *Every pentagonal graph is* 58000-*colourable.*

Proof. Define $n_1 = 581$, $n_2 = 10n_1 - 9$, and $n_3 = 10n_2 - 9$. Suppose that there is a pentagonal graph G_3 with $\chi(G_3) \geq n_3$. By 3.2, there is a stable levelling in G_3 over some graph G_2 with $\chi(G_2) \geq n_2$. Similarly there is a stable levelling in G_2 over some G_1 with $\chi(G_1) \geq n_1$. By 6.3, $\chi(N_{G_1}^2(v)) \leq 5$ for every vertex v of G_1. By 4.2 with $l = 2$ and $\kappa_2 = 5$ it follows that $\chi(G_1) \leq (40l + 28)\kappa_2 + 40 = 580$, a contradiction. Thus there is no such G_3, and hence every pentagonal graph has chromatic number at most $n_3 - 1 = 58000$. This proves 6.4. ∎

7   7-holes

The previous result shows that triangle-free graphs with large chromatic number contain odd holes of length at least seven. But what if we ask for a hole of length *exactly* seven? We need to study this for an application in a future paper. Of course, the result is not true any more; graphs with large chromatic number can have large girth. But we can still rescue something, by modifying the foregoing proofs, with an additional hypothesis. We need to assume that in every induced subgraph with large chromatic number, there is a vertex v with $\chi(N^2(v))$ large.

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More precisely, let $\mathbb{N}$ denote the set of non-negative integers, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. We say a graph G is ϕ-balled if for every non-null induced subgraph H of G, there is a vertex $v \in V(H)$ such that $\chi(H) \leq \phi(\chi(N_H^2(v)))$. We claim:

7.1 *For every non-decreasing function* $\phi : \mathbb{N} \to \mathbb{N}$, *every triangle-free* ϕ-balled graph with no 7-hole has chromatic number at most $\phi(\phi(\phi(2\phi(1) + 1)))$.

Proof. As we said, the proof is a modification of the previous arguments (replacing the "pentagonal" condition in the definition of n-coverable and in 6.1, 6.2 by a "no 7-hole" condition), so we just sketch it. First, we modify the definition of a levelling; now we only consider levellings (L_0, L_1, L_2) with at most three members. We define n-coverable as before (with this modification, and with "no 7-hole" replacing "pentagonal"). 5.1 remains true under this modification (to see this, observe that in the proof of 5.1, if $k = 2$ then all the odd holes of length at least seven found in that proof are in fact 7-holes.) Also 6.1 still holds. To go further we need the following.

(1) *If* G *is triangle-free and* ϕ-balled, and $\chi(G) > \phi(1)$, *then* G *has a 5-hole.*

Choose a vertex v such that $\chi(G) \leq \phi(\chi(N^2(v)))$. Since $\chi(G) > \phi(1)$ and ϕ is non-decreasing, it follows that $\chi(N^2(v)) > 1$ and in particular some two vertices in $N^2(v)$ are adjacent; and since G is triangle-free it follows that G has a 5-hole as required. This proves (1).

With the aid of (1), we can resuscitate a version of 6.2, assuming that G has no 7-hole. At the start of the proof, we have a set A with $\chi(A) \geq 3$, and we deduce that there is a 5-hole P in $G[A]$. This is no longer valid, because there might instead be a hole of length greater than seven. But if we replace 3 by $\phi(1) + 1$, then we can apply (1) to get a 5-hole, and the remainder of the proof works. The new version of 6.2 says that $\chi(A) \leq \phi(1)$, with A as before. Similarly we can use (1) to obtain a form of 6.3; the new version says that if H has no 7-hole then $\chi(N_H^2(z)) \leq 2\phi(1) + 1$.

The proof of 6.4 becomes easier, because we no longer need 3.2 or 4.2. From 6.3, and the definition of ϕ-balled, every 2-coverable triangle-free ϕ-balled graph with no 7-hole has chromatic number at most $\phi(2\phi(1) + 1)$. Consequently every 1-coverable such graph has chromatic number at most $\phi(\phi(2\phi(1) + 1))$; and so every triangle-free ϕ-balled graph with no 7-hole has chromatic number at most $\phi(\phi(\phi(2\phi(1) + 1)))$. This proves 7.1. ∎

8 Long holes

In this section we prove 1.5 and 1.6. The first is implied by the next result with $m = 2$:

8.1 *Let* $\ell \geq m \geq 2$ *be integers, and let* G *be a triangle-free graph with no odd hole of length at most* $2m + 1$ *and no odd hole of length more than* $2\ell + 1$. *Then* $\chi(G) < (3 + 4\ell)4^{\ell - m} - 4\ell$.

Proof. We proceed by induction on $\ell - m$. If $m = \ell$ then G is bipartite and the result is true, so we assume that $m < \ell$. Suppose that $\chi(G) \geq (3 + 4\ell)4^{\ell - m} - 4\ell$. Then we may choose a levelling in G with base of chromatic number at least $\chi(G)/2 \geq (6 + 8\ell)4^{\ell - m - 1} - 2\ell$. Since G has no odd cycle of length at most five, it follows that $N^2(v)$ is stable for every vertex v; and so by 3.1 with

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$\kappa = 1$, there is a stable levelling $(L_0, L_1, \ldots, L_k)$ in G with $\chi(L_k) \geq (3 + 4\ell)4^{\ell - m - 1} - 2\ell$, and we may choose it such that $G[L_k]$ is connected. It follows that $k \geq 3$. For $0 \leq i \leq k$ choose $s_i \in L_i$ such that s_0-s_1-$\cdots$-s_k is a path. Since $\chi(L_k) > 2l$ and $(L_k, s_{k-2}\text{-}s_{k-1})$ is a lollipop, 2.2 with $\kappa = 1$ implies that there is a licking (C, T_1) of this lollipop with

$$\chi(C) \geq \chi(L_k) - 2\ell = (3 + 4\ell)4^{\ell - m - 1} - 4\ell$$

and cleanliness at least 2ℓ. From the inductive hypothesis, there is a $(2m + 3)$-hole P in C, with vertices p_1-$\cdots$-p_{2m+3}-p_1 say. By 2.1 there is a licking (P, T) of (C, T_1). Let T have vertices

$$s_{k-2}\text{-}s_{k-1}\text{-}t_1\text{-}\cdots\text{-}t_r$$

say; thus t_r has a neighbour in P, and since the lollipop (P, T) has cleanliness at least 2ℓ, it follows that $r \geq 2\ell$ and each of $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2\ell-2}$ has distance at least three from $V(P)$.

Now since G has no odd cycle of length less than $2m + 3$, it follows that every vertex of G not in P either has at most one neighbour in P, or has exactly two neighbours in P with distance two in P. We may therefore assume that t_r is adjacent to p_1 and to no other vertex of P except possibly p_{2m+2}. For $i = 3, 4$, choose $a_i \in L_{k-1}$ adjacent to p_i. It follows that a_3, a_4 are nonadjacent to $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-2}$. Since $L_0, \ldots, L_{k-1}$ are stable, for $i = 3, 4$ there is an even induced path R_i between a_i and s_{k-1} with interior in $L_0 \cup \cdots \cup L_{k-2}$.

(1) *a_4 has a neighbour in $V(T)$.*

Because suppose not. Then $R_4 \cup T$ is an induced path from a_4 to t_r, of length at least $r + 2 \geq 2l + 2$. But there is an odd induced path and an even induced path between a_4 and t_r with interior in $V(P)$ (since a_4 has no neighbours in P except p_4 and possibly p_2, p_6, and t_r has no neighbours in P except p_1 and possibly p_{2m+2}; one of a_4-p_4-p_3-p_2-p_1-t_4, a_4-p_2-p_1-t_r is the desired odd path, and the even path goes the other way around P.) But then the union of one of these paths with $R_4 \cup Q$ is an odd hole of length at least $2l + 4$, which is impossible. This proves (1).

Choose $i \leq r$ minimum such that t_i is adjacent to one of a_4, a_3. By (1), such a choice is possible. Since a_3, a_4 are nonadjacent to $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2\ell-2}$, it follows that $i > 2\ell - 2$. Since G has no odd cycle of length at most five, t_i is not adjacent to both a_3, a_4; let t_i be adjacent to a_h and not to a_j, where $\{h, j\} = \{3, 4\}$. Let Q be a minimal path between a_h, s_{k-1} with interior in $V(T)$. It follows that Q has length at least 2ℓ. Consequently $Q \cup R_h$ is a hole of length at least $2\ell + 2$, and so it is even; and hence Q is even. Now a_h has no neighbour in R_j, since a_h is not adjacent to the parent of a_j (because G has no 5-holes) and a_h is nonadjacent to s_{k-2} (because (P, T) is a lollipop of cleanliness at least one). Thus

$$a_h\text{-}p_h\text{-}p_j\text{-}a_j\text{-}R_j\text{-}s_{k-1}\text{-}Q\text{-}a_h$$

is an odd hole of length at least $2\ell + 5$, which is impossible. This proves 8.1. ∎

Finally we turn to the proof of 1.6. It follows from the next result.

8.2 *Let $\ell \geq 3$ and $\kappa \geq 1$ be integers, and let G be a graph with no hole of length more than ℓ, such that $\chi(N(v)), \chi(N^2(v)) \leq \kappa$ for every vertex v. Then $\chi(G) \leq (2\ell - 2)\kappa$.*

Proof. Suppose not; then there is a levelling $(L_0, \ldots, L_k)$ in G with $\chi(L_k) > (\ell-1)\kappa$. Let C' be the vertex set of a component C' of $G[L_k]$ with $\chi(C') > (\ell-1)\kappa$. Since $\ell - 1 > 1$, it follows that $k \geq 2$. For $i = k-2, k-1$ choose $s_i \in L_i$, such that s_{k-2}, s_{k-1} are adjacent and s_{k-1} has a neighbour in C'. Since $\chi(C') > (\ell-1)\kappa$ and $(V(C'), s_{k-2}\text{-}s_{k-1})$ is a lollipop, by 2.2 there is a licking (C, T) of it with cleanliness at least $\ell - 1$ and with $\chi(C) \geq \chi(C') - (\ell-1)\kappa > 0$. Choose $a \in L_{k-1}$ with a neighbour in C. Now a might have neighbours in T, but since (C, T) has cleanliness at least $\ell - 1$, a is nonadjacent to the first $\ell - 1$ vertices of T. Let P be an induced path between s_{k-1} and a with interior in $V(T) \cup C$; thus P has length at least $l - 1$. But a, s_{k-1} are joined by an induced path with interior in $L_0 \cup \cdots \cup L_{k-2}$, and the union of this path with P is a hole of length at least $\ell + 1$, a contradiction. This proves 8.2. ∎

We deduce 1.6, which we restate, slightly strengthened.

8.3 *Let $\ell \geq 3$ be an integer, and let G be a graph with no 5-hole and no hole of length more than ℓ. Then*
$$\chi(G) \leq (2\ell - 2)^{2^{\omega(G)-1}-1}.$$

Proof. We proceed by induction on $\omega(G)$. If $\omega(G) = 1$ the result is true, so we assume $\omega(G) > 1$. Let
$$n = (2\ell - 2)^{2^{\omega(G)-2}-1}.$$
From the inductive hypothesis, every induced subgraph H of G with $\omega(H) < \omega(G)$ is n-colourable.

(1) *For every vertex v of G, $\chi(N(v)) \leq n$, and $\chi(N^2(v)) \leq n^2$.*

The graph $G[N(v)]$ contains no clique of size $\omega(G)$, and so is n-colourable. Let $A_1, \ldots, A_n$ be a partition of $N(v)$ into n stable sets, and for $1 \leq i \leq n$ let B_i be the set of vertices in $N^2(v)$ with a neighbour in A_i. Suppose that there is a clique C of cardinality $\omega(G)$ with $C \subseteq B_i$ for some i. Choose $a \in A_i$ with as many neighbours in C as possible; then there exists $c' \in C$ nonadjacent to a, since G has no $(\omega(G) + 1)$-clique. Choose $a' \in A_i$ adjacent to c'; then from the choice of a, there exists $c \in C$ adjacent to a and not to a'. But then the subgraph induced on $\{v, a, a', c, c'\}$ is a 5-hole, which is impossible. Thus there is no such clique C, and so $\chi(A_i) \leq n$. Since this holds for all i, it follows that $\chi(N^2(v)) \leq n^2$. This proves (1).

From (1) and 8.2, it follows that
$$\chi(G) \leq (2\ell - 2)n^2 = (2\ell - 2)(2\ell - 2)^{2^{\omega(G)-1}-2} = (2\ell - 2)^{2^{\omega(G)-1}-1}.$$

This proves 8.3. ∎

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