Judicious partitions of graphs

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0. Introduction

The problem of finding good lower bounds on the size of the largest bipartite subgraph of a given graph has received a fair amount of attention. In particular, improving a result of Erdős ([10]; see also [11] for related problems), Edwards [9] proved the essentially best possible assertion that every graph with n vertices and m edges has a bipartite subgraph with at least $m/2 + (n-1)/4$ edges. More recently, Andersen et al [1] and Erdős et al [12] gave lower bounds for the size of the largest k-partite subgraph of a given graph, and Shearer [18] and Ngoc and Tuza [15] gave bounds for the lowest bipartite subgraph of a triangle free graph. Various algorithms for finding large k-partite subgraphs have been considered in [16], [17] and [15].

In this paper we consider a naturally related question. Given a graph G, we again consider partitions $V_1, \ldots, V_k$ of $V(G)$ into k sets. We ask, however, for the minimal value of $\max_{1\le i\le k} e(G[V_i])$. Thus we seek a partition of $V(G)$ in which *every* class induces relatively few edges, in contrast to the problem of finding the largest k-partite subgraph of a given graph G, which asks for a partition in which the total number of edges induced by the classes is small.

As we shall see, the nature of the problem depends on the size of the graph. Our first aim in this paper is to prove a bound valid for all graphs: although the bound is best possible, the only graphs on which it is attained are very small. Our other aim is to prove a much better and essentially best possible bound for graphs with many edges.

In §1 we shall give our exact result: for any k, and for any graph G, there is a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that $e(G[V_i]) \le e(G)/\binom{k+1}{2}$ for $i = 1, \ldots, k$. For a given value of k. this inequality is best possible. The improvement for graphs with many edges will be given in §2: as we shall see, the upper bound can almost be halved. In fact, we can even demand that $e(G[V_i])$ be fairly close to $e(G)/k^2$, for $i = 1, \dots, k$, and that $e(V_i, V_j)$ be fairly close to $2e(G)/k^2$, for $1 \le i < j \le k$. The section also contains sharper results in the case when G is regular and in the case when we restrict only $e(G[V_i])$, for $i = 1, \dots, k$.

We use standard notation, as in [5], say. For a graph G and a set $W \subset V(G)$ we write $G[W]$ for the subgraph of G induced by W and, when it is unambiguous, $e(W)$ for $e(G[W])$. For disjoint sets $W_1, W_2 \subset V(G)$ we write $e(W_1, W_2)$ for $|\{xy : x \in W_1,\ y \in W_2,\ xy \in E(G)\}|$.

1. A universal bound

Given a graph G, it is easy to find a bipartition $V(G) = V_1 \cup V_2$ for which both $e(V_1)$ and $e(V_2)$ are small. Indeed every graph G has a so-called *unfriendly partition* ,that is a partition of $V(G)$ into sets V_1 and V_2 so that $|\Gamma(x) \cap V_2| \ge |\Gamma(x) \cap V_1|$ for $x \in V_1$ and $|\Gamma(x) \cap V_1| \ge |\Gamma(x) \cap V_2|$ for $x \in V_2$. As we shall note below, such a partition has $\max\{e(V_1), e(V_2)\} \le e(G)/3$. More general restrictions for bipartitions have been considered in [8] and [4], and the analogous problem for infinite graphs has been studied in [2] and [19].

The aim of this section is to prove the following result.

Theorem. *For any positive integer k and any graph G, we can partition the vertex set of G into k sets $V_1, \dots, V_k$ so that*

$$e(V_i) \le \frac{2}{k(k+1)} e(G)$$

for $i = 1, \dots, k$. This is best possible for all values of k.

Proof. The bound is easily seen to be best possible by considering K_{k+1}, the complete graph on $k+1$ vertices. Any partition of this into k parts gives one part with at least two vertices and thus at least one edge, which is $\frac{2}{k(k+1)} e(G)$.

We prove that any graph G can be partitioned into k sets, each inducing a subgraph with at most $\frac{2}{k(k+1)} e(G)$ edges, by induction on k.

For $k = 1$ the theorem is trivial. If $k = 2$, then let $V(G) = V_1 \cup V_2$ be a partition with $e(V_1, V_2)$ maximal. We may assume $e(V_1) \ge e(V_2)$. Now if $x \in V_1$ then $|\Gamma(x) \cap V_1| \le$

$|\Gamma(x) \cap V_2|$, since otherwise $(V_1 \setminus \{x\}, V_2 \cup \{x\})$ contradicts our choice of (V_1, V_2). Thus

$$e(V_1, V_2) = \sum_{x \in V_1} |\Gamma(x) \cap V_2| \geq \sum_{x \in V_1} |\Gamma(x) \cap V_1| = 2e(V_1).$$

Therefore

$$e(G) \geq e(V_1) + e(V_1, V_2) \geq 3e(V_1). \tag{1}$$

Since $e(V_1) \geq e(V_2)$ we have $e(V_i) \leq e(G)/3$ for $i = 1, 2$, as required. Note that we have $e(V_1) + e(V_2) \leq e(V_1, V_2)$, so $e(V_1) + e(V_2) \leq e(G)/2$.

Now let $k \geq 3$, and assume that the theorem is true for smaller values of k. Let $V(G) = V_1 \cup \cdots \cup V_k$ be a partition minimizing $\sum_{i=1}^k e(V_i)$. We assume $e(V_1) \geq e(V_2) \geq \cdots \geq e(V_k)$. If $x \in V_1$ then $|\Gamma(x) \cap V_1| \leq |\Gamma(x) \cap V_i|$ for $i = 2, \ldots, k$, or else moving x from V_1 to V_i gives a better partition. Thus, as before, for $1 = 2, \ldots, k$, we must have

$$e(V_1, V_i) \geq 2e(V_1). \tag{2}$$

Let $G' = G[V_2 \cup \cdots \cup V_k]$. We are done if

$$e(G) \geq \binom{k+1}{2} e(V_1),$$

which is true if

$$e(G') \geq \binom{k+1}{2} e(V_1) - e(V_1) - \sum_{i=2}^{k} e(V_1, V_i).$$

Now, by (2),

$$\begin{aligned} \binom{k+1}{2} e(V_1) - e(V_1) - \sum_{i=2}^{k} e(V_1, V_i) &\leq \frac{k(k+1)}{2} e(V_1) - e(V_1) - 2(k-1)e(V_1) \\ &= \binom{k-1}{2} e(V_1). \end{aligned}$$

Thus we are done if

$$e(G') \geq \binom{k-1}{2} e(V_1),$$

and so we may assume that

$$e(G') < \binom{k-1}{2} e(V_1). \tag{3}$$

Since the theorem holds for $k = 2$, we can partition V_1 into $V_1' \cup V_2'$ so that

$$\max\{e(V_1'), e(V_2')\} \leq e(V_1)/3;$$

furthermore, by our inductive hypothesis, we can partition G' into $k-2$ sets $V_3', \ldots, V_k'$ in such a way that, for $i = 3, \ldots, k$,

$$e(V_i') \leq e(G')/\binom{k-1}{2}.$$

We claim that this partition will do.

To see this, note that (3) implies that, $i = 3, \ldots, k$,

$$e(V_i') < e(V_1), \tag{4}$$

and so, by (2)-(4), we have

$$\begin{aligned} e(G) &= e(V_1) + \sum_{i=2}^{k} e(V_1, V_k) + e(G') \\ &> e(V_i') + 2(k-1)e(V_i') + \binom{k-1}{2} e(V_i') \\ &= \binom{k+1}{2} e(V_i'). \end{aligned}$$

It remains only to check that $\max\{e(V_1'), e(V_2')\} \leq \frac{2}{k(k+1)} e(G)$. Now $e(V_i') \leq e(V_i)/3$, for $i = 1, 2$, so it is enough to prove that

$$e(V_1) \leq \frac{6}{k(k+1)} e(G). \tag{5}$$

We claim that, for $1 < i < j$,

$$e(V_i, V_j) \geq e(V_1)/2. \tag{6}$$

Indeed, otherwise we can partition V_1 into $V_1' \cup V_i'$ so that $e(V_1') + e(V_i') \leq e(V_1)/2$ and by setting $V_j' = V_i \cup V_j$, and $V_l' = V_l$ for $l \neq 1, i, j$ we get a better partition. Therefore, by (2) and (6),

$$\begin{aligned} e(G) &\geq e(V_1) + \sum_{i=2}^{k} e(V_1, V_i) + \sum_{1<i<j} e(V_i, V_j) \\ &\geq e(V_1) + 2(k-1)e(V_1) + \frac{1}{2}\binom{k-1}{2} e(V_1) \\ &= \frac{k^2 - 5k + 2}{4} e(V_1). \end{aligned}$$

This implies (5), since

$$\frac{k^2-5k+2}{4} > \frac{k(k+1)}{6}$$

for all positive integers k. Thus the partition $(V_1', \dots, V_k')$ will do, as claimed. ■

Although the bound in Theorem 1 is best possible, it can be improved for every graph other than K_{k+1}. Indeed, for $k = 2$, it is clear that we have equality in (1) only for $G = K_3$; for larger values of k we can work through the rest of the proof inductively. In fact, if G has many edges then one can do considerably better than Theorem 1; this will be done in the next section.

2. Bounds for large graphs

What can one hope to prove as an upper bound for the number of edges in each of the k vertex classes of our partition, if our graph has many edges?

Given a graph G, let us pick a random partition $V_1, \dots, V_k$ of $V(G)$. Clearly we have $\mathbb{E}(\sum_{i=1}^k e(V_i)) = e(G)/k$, and so $\sum_{i=1}^k e(V_i) \le e(G)/k$ for some $V_1, \dots, V_k$. Bounding $\max_{1\le i\le k} e(V_i)$ is more difficult. Ideally, we would like $e(V_i)$ to be close to $e(G)/k^2$ for $i = 1, \dots, k$. We cannot in general hope for more, as can be seen by considering the complete graph on n vertices for n large, or by making use of random regular graphs. In fact, for a given value of k, and r sufficiently large, almost every r-regular graph is such that we cannot improve the bound in Theorem 1 by more than about a factor of 2. As we shall see, this can in fact be achieved if the graph has many edges.

The proofs of our results in this section will be based on the following immediate consequence of the Azuma-Hoeffding inequality ([3], [13]; see also [6], [7], [14]).

Theorem 2. *Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_k\}$, let $X_1, \dots, X_n$ be independent random variables taking values in $[k] = \{1, \dots, k\}$, and let $X = (X_1, \dots, X_n)$. Suppose $f\colon [k]^n \to \mathbb{N}$ satisfies*

$$|f(Y) - f(Y')| \le d(v_i) \tag{7}$$

whenever the vectors Y and Y' differ only in the ith coordinate. Then for any $t > 0$,

$$\begin{aligned}\mathbb{P}(X - \mathbb{E}(X) \geq t) &\leq \exp(-2t^2/\sum_{i=1}^{k} d(v_i)^2)\\ \mathbb{P}(X - \mathbb{E}(X) \leq -t) &\leq \exp(-2t^2/\sum_{i=1}^{k} d(v_i)^2).\end{aligned}$$

■

Note that in Theorem 2 the random variable X is just a random k-colouring of $V(G)$, and f is a function defined on the set of k-colourings of G. In our applications below, the X_i will all be uniformly distributed unless otherwise stated.

Our first result guarantees a partition $V(G) = \bigcup_{i=1}^{k} V_i$ in which the $e(V_i)$ and $e(V_i, V_j)$ are neither too small nor too large, i.e. they are all close to their expectations in a random partition.

Theorem 3. *Let G be a graph with n vertices, m edges and maximal degree Δ. Then we can partition the vertex set of G into k sets $V_1, \dots, V_k$ so that*

$$\left| e(V_i) - \frac{m}{k^2} \right| \leq R \tag{8}$$

and

$$\left| e(V_i, V_j) - \frac{2m}{k^2} \right| \leq R \tag{9}$$

for $i \neq j$, where

$$R = \min\left\{ (\Delta m \log(2k^2))^{1/2},\ \frac{2\Delta}{k} + (6m)^{4/5}(\log(2k^2))^{2/5} \right\}.$$

Proof. We prove first that we may take $R \leq (\Delta m \log(2k^2))^{1/2}$. Let $X = (X_1, \dots, X_n)$ be a random k-colouring of $V(G) = \{v_1, \dots, v_n\}$, and let $C_r = \{v_j : X_j = r\}$, for $r = 1, \dots, k$. We define functions

$$f_i = e(C_i),$$

for $i = 1, \dots, k$, and

$$g_{ij} = e(C_i, C_j),$$

for $1 \leq i < j \leq k$. Note that $\mathbb{E}f_i = m/k^2$ for $i = 1, \dots, k$ and $\mathbb{E}g_{ij} = 2m/k^2$ for $1 \leq i < j \leq k$, and all these functions satisfy the condition (7) of Theorem 2. Then in order to prove the first part of the theorem, it is enough to show that, for some X, all the f_i and g_{ij} are within $(\Delta m \log(2k^2))^{1/2}$ of their expectation. Indeed, applying Theorem 2,

$$\begin{aligned}
\mathbb{P}\left(|f_i - \mathbb{E}f_i| \geq (\Delta m \log(2k^2))^{1/2}\right) &< 2\exp\left(-2(\Delta m \log(2k^2))\Big/\sum_{i=1}^{n} d(v_i)^2\right) \\
&\leq 2\exp\left(-2\Delta m \log(2k^2)/\Delta \sum_{i=1}^{n} d(v_i)\right) \\
&= \frac{1}{k^2}.
\end{aligned}$$

for $i = 1, \ldots, k$. Similarly,

$$\mathbb{P}\left(|g_{ij} - \mathbb{E}g_{ij}| \geq (\Delta m \log(2k^2))^{1/2}\right) < \frac{1}{k^2}.$$

Thus a random k-colouring X fails (8) for a given colour with probability strictly less than $1/k^2$, and fails (9) for a given pair of colours with probability strictly less than $1/k^2$, so the probability that X fails at all is strictly less than $k(1/k^2) + \binom{k}{2}(1/k^2) = 1$. Therefore, there some colouring that works for all colours and pairs of colours.

For the second part, we prove a slightly stronger bound, namely that we may take

$$R \leq \frac{2\Delta}{k} + (2m)^{4/5}(\log(2k^2))^{2/5} + 2(2m)^{2/5}(\log(2k^2))^{1/5} + 1.$$

We may assume that $V(G) = \{v_1, \ldots v_n\}$ satisfies $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$.

Let r be minimal such that

$$\frac{r^2}{2} \geq \left(\frac{1}{2}\log(2k^2)\right)^{1/2} \left(\sum_{i=r+1}^{n} d(v_i)^2\right)^{1/2}. \tag{10}$$

Note that

$$\begin{aligned}
(r-1)^4 &\leq 2\log(2k^2) \sum_{i=r}^{n} d(v_i)^2 \\
&\leq 2\log(2k^2) d(v_r) \sum_{i=r}^{n} d(v_i) \\
&\leq 2(\log(2k^2))2m^2/r,
\end{aligned}$$

since $d(v_r) \leq 2m/r$. Thus

$$r - 1 \leq (2m)^{2/5}(\log(2k^2))^{1/5}.$$

Let S_1 be the r vertices of highest degree, and $S_2 = V(G) \setminus S_1$. Then $e(S_1) \le \binom{r}{2}$: let $m_0 = m - e(S_1)$. We can partition S_1 into classes $C_1, \ldots, C_k$ in such a way that

$$\max_{i \neq j} \left| \sum_{v \in C_i} |\Gamma(v) \cap S_2| - \sum_{v \in C_j} |\Gamma(v) \cap S_2| \right| \le \Delta.$$

Let $X = (X_{r+1}, \ldots, X_n)$ be a random k-colouring of $\{v_{r+1}, \ldots, v_n\}$, and $D_j = \{v_i : X_i = j\}$, for $j = 1, \ldots, k$. We define functions

$$f_i = e(C_i, D_i) + e(D_i)$$

for $i = 1, \ldots, k$ and

$$g_{ij} = e(C_i \cup D_i, C_j \cup D_j) - e(C_i, C_j)$$

for $1 \le i < j \le k$. It is easily seen that

$$\left| \mathbb{E}(f_i) - \frac{m_0}{k^2} \right| \le \frac{\Delta}{k}$$

for $i = 1, \ldots, k$, and

$$\left| \mathbb{E}(g_{ij}) - \frac{2m_0}{k^2} \right| \le \frac{2\Delta}{k}.$$

for $1 \le i < j \le k$, and that all the f_i and g_{ij} satisfy condition (7) in Theorem 2. Therefore, applying the theorem, we get

$$\mathbb{P}\left(|f_i - \mathbb{E}f_i)| > \frac{r^2}{2} \right) < 2\exp\left(-2(r^2/2)^2 / \sum_{i=r+1}^{n} d(v_i)^2 \right) \le \frac{1}{k^2},$$

and, similarly,

$$\mathbb{P}\left(|g_{i,j} - \mathbb{E}(g_{ij})| > \frac{r^2}{2} \right) < \frac{1}{k^2}.$$

Thus some colouring must satisfy

$$\left| e(C_i \cup D_i) - \frac{m}{k^2} \right| \le \frac{\Delta}{k} + r^2,$$

for $i = 1, \ldots, k$, and

$$\left| e(C_i \cup D_i, C_j \cup D_j) - \frac{2m}{k^2} \right| \le \frac{2\Delta}{k} + r^2,$$

for $1 \le i < j \le k$. ∎

If G is a regular graph then we can prove a somewhat stronger assertion.

Theorem 4. *If G is a regular graph, say every vertex has degree d, then we may take*

$$R = \left(\frac{1}{2} nd \log(2k^2)\right)^{1/2}$$

in Theorem 3.

Proof. Using the notation of the first part of the proof of Theorem 3, we have

$$\begin{aligned}\mathbb{P}\left(|f_i - \frac{m}{k^2}| \geq \left(\frac{1}{2} nd \log(2k^2)\right)^{1/2}\right) &< 2 \exp\left(-2(\frac{1}{2} nd \log(2k^2))^{1/2} / \sum_{i=1}^{n} d(v_i)^2\right) \\ &= 2 \exp(-nd \log(2k^2)^{1/2}/nd) \\ &= 1/k^2,\end{aligned}$$

for each f_i. The argument is similar for g_{ij}. ■

Finally, we can prove a sharper result when we bound $e(V_i)$, $i = 1, \ldots, k$, only from above.

Theorem 5. *Let G be a graph with n vertices, m edges and maximal degree Δ. Then we can partition the vertex set of G into k sets $V_1, \ldots, V_k$ so that*

$$e(V_i) \leq \frac{m}{k^2} + R$$

for $i = 1, \ldots, k$, where

$$R = \min\left\{(\Delta m \log k)^{1/2},\ (3m)^{4/5} (\log k)^{2/5}\right\}.$$

Proof. We prove first that we may take $R \leq (\Delta m \log k)^{1/2}$. Let $X = (X_1, \ldots, X_n)$ be a random k-colouring of $V(G) = \{v_1, \ldots, v_n\}$ and let $C_i = \{v_j : X_j = i\}$, for $i = 1, \ldots, k$. We define

$$f_i = e(C_i)$$

for $i = 1, \ldots, k$. Then applying Theorem 2, we get

$$\begin{aligned}\mathbb{P}(f_i \geq \frac{m}{k^2} + (\Delta m \log k)^{1/2}) &< \exp\left(-2(\Delta m \log k) \Big/ \sum_{i=1}^{n} d(v_i)^2\right) \\ &\leq \exp\left(-\Delta m \log k \Big/ \Delta \sum_{i=1}^{n} d(v_i)\right) \\ &= 1/k.\end{aligned}$$

The probability that a random colouring will fail for a given class is smaller than $1/k$, so some colouring must work for every class.

As in Theorem 3, we prove a slightly stronger result for the second part, namely that we may take

$$R \leq (2m)^{4/5}(\log k)^{2/5} + 2(2m)^{2/5}(\log k)^{1/5} + 1.$$

This is sharper than the bound above, except for small values of m, when we can apply Theorem 1. Let r be minimal so that

$$\frac{r^2}{2} \geq (\frac{1}{2}\log k)^{1/2}\left(\sum_{i=r+1}^{n} d(v_i)^2\right)^{1/2}. \tag{11}$$

Let S_1 be the r vertices of highest degree and $S_2 = V(G) \setminus S_1$. We k-colour S_2 randomly by giving each vertex colour 1 with probability p and colour i with probability q, for $i = 2, \ldots, k$, where $p + (k-1)q = 1$. Let $X = (X_{r+1}, \ldots, X_n)$ be our random k-colouring and $C_1, \ldots, C_k$ be the colour classes. Define

$$f_1 = e(C_1) + e(S_1, C_1)$$

and

$$f_i = e(C_i),$$

for $i = 2, \ldots, k$. Note that f_i satisfies condition (7) in Theorem 2, for $i = 1, \ldots, k$. Let $m_0 = m - e(S_1)$ and $A = e(S_1, S_2)$. Then

$$\mathbb{E}(f_1) = p^2(m_0 - A) + pA$$

and

$$\mathbb{E}(f_i) = q^2(m_0 - A),$$

for $i = 2, \ldots, k$. From (11) we know that

$$\mathbb{P}\left(f_i > \mathbb{E}(f_i) + \frac{r^2}{2}\right) < \exp\left(-2(r^2/2)^2 / \sum_{i=r+1}^{n} d(v_i)^2\right) \\ \leq \frac{1}{k}$$

Hence, in order to prove the theorem, it suffices to show that with an appropriate choice of p and q, the expectation $\mathbb{E}(f_i)$ is at most m/k^2, for $i = 1, \ldots, k$. In fact, we shall show that, with suitable p and q,

$$\mathbb{E}(f_i) \leq m_0/k^2,$$

where $m_0 = m - e(S_1)$. We choose p and q to satisfy

$$p^2(m_0 - A) + pA = q^2(m_0 - A). \tag{12}$$

Then in order to have $\mathbb{E}(f_i) \leq m_0/k^2$ for $i = 1, \ldots, k$, it is enough that

$$q^2(m_0 - A) \leq m_0/k^2. \tag{13}$$

Setting $a = A/m_0$, we can rewrite (12) and (13) as

$$p^2(1-a) + pa = q^2(1-a) \tag{14}$$

$$k^2q^2(1-a) \leq 1. \tag{15}$$

Relation (14) is satisfied by

$$q = \frac{(k-1)(2-a) - \alpha}{2(k^2-2k)(1-a)},$$

where $\alpha = [(k-1)^2(2-a)^2 - 4(k^2-2k)(1-a)]^{1/2}$. Then from (15) we see that it is enough to prove that

$$\frac{2(k-1)^2(2-a)^2 - 2(k-1)(2-a)\alpha - 4(k^2-2k)(1-a)}{4(k-2)^2(1-a)} \leq 1;$$

in other words, that

$$(k-1)(2-a)^2 - 4(k-2)(1-a) \leq (2-a)\alpha.$$

Squaring this, we get that it suffices to have

$$4(2-a)^2(k^2-2k)(1-a) + 16(k-2)^2(1-a)^2 \leq 8(k-1)(k-2)(2-a)^2(1-a);$$

that is

$$(2-a)^2k + 4(k-2)(1-a) \le 2(k-1)(2-a)^2,$$

which is equivalent to

$$4(k-2)(1-a) \le (k-2)(2-a)^2,$$

which is always true for $k \ge 2$.

Therefore there is some k-colouring $C_1 \cup S_1, C_2, \ldots, C_p$ of $V(G)$ with at most

$$\binom{r}{2} + \left(\mathbb{E}(f_i) + \frac{r^2}{2}\right) < \frac{m}{k^2} + r^2$$

edges in each colour class.

This will do, since from (11) we get

$$r \le (2m)^{2/5}(\log k)^{1/5} + 1.$$

■

Let us note that in Theorems 3, 4 and 5, we could also demand that the vertex classes in our partition $V(G) = V_1 \cup \cdots \cup V_k$ all be roughly the same size. This would change only the constants in the error term R.

With more work, it would be easy to improve all the constants in this section. It should also be possible to improve the bounds for graphs of medium size: at present, Theorem 1 gives a good bound for graphs with few edges, and the theorems in this section give good bounds for graphs with many edges. It seems likely, for instance, that for any i, and for large enough k, for any graph G with more than $\binom{k+1}{2} + i$ edges, we could ask for a partition $V(G) = V_1 \cup \cdots \cup V_k$ with

$$e(V_i) \le \frac{e(G)}{\binom{k+1}{2} + i},$$

for $i = 1, \ldots, k$.

It should also be possible to produce weighted versions of all the above theorems. For $k = 2$, it is easily proved that if $a, b > 0$, then for any graph G there is a partition $V(G) = V_1 \cup V_2$ such that $e(V_1) \le ae(G)/(a+2b)$ and $e(v_2) \le be(G)/(2a+b)$. Higher values of k are more complicated. We also have the following weighted analogue of Theorem 5.

Let $p_1, \ldots, p_k$ be positive reals satisfying $\sum_{i=1}^{k} p_i = 1$. Let G be a graph with n vertices, m edges and maximal degree Δ. Then we can partition the vertex set of G into k sets $V_1, \ldots, V_k$ so that $e(V_i) \leq (p_i m/k) + R$, where R is an error term similar to that in Theorem 5. This assertion can be proved by slightly modifying the proof of that theorem; the details are left to the reader.

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