Bipartite graphs with no K_6 minor

Maria Chudnovsky[1]
Princeton University, Princeton, NJ 08544

Alex Scott[2]
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[3]
Princeton University, Princeton, NJ 08544

Sophie Spirkl[4]
University of Waterloo, Waterloo, Ontario N2L3G1, Canada

February 19 2022; revised April 21, 2022

[1]Supported by NSF DMS-EPSRC grant DMS-2120644.
[2]Research supported by EPSRC grant EP/V007327/1.
[3]Supported by AFOSR grant A9550-19-1-0187, and by NSF grant DMS-1800053.
[4]We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number RGPIN-2020-03912].Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence RGPIN-2020-03912].

Abstract

A theorem of Mader shows that every graph with average degree at least eight has a K_6 minor, and this is false if we replace eight by any smaller constant. Replacing average degree by minimum degree seems to make little difference: we do not know whether all graphs with minimum degree at least seven have K_6 minors, but minimum degree six is certainly not enough. For every $\varepsilon > 0$ there are arbitrarily large graphs with average degree at least $8 - \varepsilon$ and minimum degree at least six, with no K_6 minor.

But what if we restrict ourselves to bipartite graphs? The first statement remains true: for every $\varepsilon > 0$ there are arbitrarily large bipartite graphs with average degree at least $8 - \varepsilon$ and no K_6 minor. But surprisingly, going to minimum degree now makes a significant difference. We will show that every bipartite graph with minimum degree at least six has a K_6 minor. Indeed, it is enough that every vertex in the larger part of the bipartition has degree at least six.

1 Introduction

The graphs with no K_5 minor are well understood. A theorem of Wagner [6] gives an explicit construction for all such graphs: they can all be built by piecing together planar graphs and copies of one eight-vertex graph by a sum operation that we do not describe here. Consequently, every graph with $n \geq 3$ vertices and more than $3n-6$ edges has a K_5 minor. (All graphs in this paper are finite and have no loops or parallel edges.) This is tight: there are graphs with n vertices and with exactly $3n-6$ edges that have no K_5 minor. Indeed, one can make such graphs that are almost 6-regular: for infinitely many values of n there is an n-vertex planar graph (which therefore has no K_5 minor) with all vertices of degree six except for twelve of degree five. In summary:

- all graphs with average degree at least six contain K_5 minors, and this is false if we replace six by any smaller real number;
- all graphs with minimum degree at least six have K_5 minors, and this is false if we replace six by any smaller integer;
- this is all still true even if we insist that maximum degree is at most six.

What if we look just at bipartite graphs? One can make n-vertex bipartite graphs with no K_5 minor that have $3n-9$ edges (the complete bipartite graph $K_{3,n-3}$ – in fact this is the only such graph, which can easily be shown by induction using 1.1). So the situation for average degree is virtually unchanged: average degree six is enough to guarantee a K_6 minor, and no smaller constant works. But $K_{3,n-3}$ has vertices with degree much larger than the average, and also vertices with degree much smaller than average (if three is much smaller than six). So what happens if we insist that maximum degree is close to the average degree, or minimum degree is large?

It turns out that:

1.1 *Every non-null bipartite graph with minimum degree at least four has a K_5 minor.*

This can be derived from Wagner's construction [6], although the proof is rather long and we omit it. The result is already known: it was stated (in a stronger form, replacing "bipartite" by "girth at least four") in a lecture by János Barát [1], as joint work with David Wood, and also (without proof) in an early version of the paper [2] (unfortunately it was removed in a later version of the paper). When excluding K_5, imposing a bound on maximum degree is perhaps not so interesting: there are n-vertex bipartite graphs with average degree at least four and maximum degree at most five, with no K_5 minor. (For example, take five disjoint copies of $K_{3,5}$, and for $1 \leq i \leq 5$ let v_i be a vertex with degree three from the ith copy. Now add two more vertices both adjacent to each of $v_1, \ldots, v_5$. To make bigger examples, take disjoint unions.) Perhaps average degree at least five and maximum degree at most six will guarantee a K_5 minor in a bipartite graph, but we have not worked this out.

In this paper, we ask what happens for K_6 minors. A theorem of Mader [4] says:

1.2 *For $n \geq 4$, every n-vertex graph with more than $4n-10$ edges has a K_6 minor.*

There are graphs with n vertices and $4n-10$ edges with minimum degree at least six that have no K_6 minor: for instance, take a planar graph on $n-1$ vertices with $3(n-1)-6$ edges and minimum degree five, and add a new vertex adjacent to everything. So we need average degree at least eight to guarantee a K_6-minor; no smaller constant works. Again, we might ask what happens if we insist that maximum degree is close to average degree, or minimum degree is large. We have found K_6-minor-free graphs with minimum degree six and maximum degree at most nine; and K_6-minor-free graphs with minimum degree five, maximum degree seven, and average degree arbitrarily close to 98/15 (we omit the details). But as far as we know, both the following are open:

1.3 Conjecture: *Every non-null 6-regular graph has a K_6 minor.*

(Indeed, as far as we know, every non-null graph with minimum degree at least six and maximum degree at most eight has a K_6 minor.)

1.4 Conjecture: *Every non-null graph with minimum degree at least seven has a K_6 minor.*

There is a well-known conjecture of Jørgensen [3] that is related:

1.5 Conjecture: *Every 6-connected graph with no K_6 minor can be made planar by deleting some vertex, and therefore has a vertex of degree at most six.*

But in this paper we will restrict ourselves to bipartite graphs. Still no constant smaller than eight works as a bound on average degree to guarantee a K_6 minor, since the complete bipartite graph $K_{4,n-4}$ has $4n-16$ edges and has no K_6 minor. But what about minimum degree? We will show:

1.6 *Every non-null bipartite graph with minimum degree at least six has a K_6 minor.*

We do not know whether "six" can be replaced by "five" in 1.6. Minimum degree is more difficult than average degree to work with inductively, and fortunately there is a strengthening of 1.6 that is more amenable to induction:

1.7 *Let G admit a bipartition (A, B) with $|A| \geq |B| > 0$, such that every vertex in A has degree at least six. Then G has a K_6 minor.*

We remark that 1.7 becomes false if we replace "six" by "five"; we will show this in the next section.

This was also motivated by one of the steps in the proof of [5] that every graph with no K_6 minor is five-colourable. Let G be a minor-minimal graph with no K_6 minor that is not five-colourable, if such a graph exists; then in [5], section 12 was devoted to showing that G has a matching with at least $(|G|-1)/2$ edges. If not, then by Tutte's theorem, there is a set $X \subseteq V(G)$ such that $G \setminus X$ has more than $|X|$ odd components, and it was known that G is six-connected, and so each of these components has an edge to at least six vertices in X. By contracting these components to single vertices we obtain a bipartite graph satisfying the hypotheses of 1.7, which would be a contradiction, since G has no K_6 minor. In [5], Mader's theorem [4] was used in place of 1.7, with additional analysis of the components that had only six or seven neighbours in X. But it should be added that 1.7 is not going to shorten the proof of the main theorem of [5]; the proof of 1.7 is considerable longer than section 12 of [5]. It will take up almost all the paper, but we begin with proving the statements for K_5 mentioned above.

2 Some definitions, and the results for K_5

Let us be more precise. If $X \subseteq V(G)$, $G \setminus X$ is the graph obtained from G by deleting X, and $G[X] = G \setminus (V(G) \setminus X)$ denotes the subgraph of G induced on X. A graph H is a *minor* of G if H can be obtained by edge-contraction from a subgraph of G. (We repeat that graphs in this paper have no loops or parallel edges, so any loops or parallel edges produced by edge-contraction should be deleted.) We will only be concerned with complete graph minors. Let us say a *cluster* in G is a set of disjoint subsets $X_1, \ldots, X_k$ of $V(G)$, such that $G[X_i]$ is connected for $1 \leq i \leq k$, and for $1 \leq i < j \leq k$ there is an edge of G between X_i, X_j; and a *t-cluster* means a cluster of cardinality t. Thus G contains the complete graph K_t as a minor if and only if G contains a t-cluster.

A word on taking minors of bipartite graphs: we start with a graph with a bipartition (A, B), choose a subset $X \subseteq V(G)$ that induces a connected subgraph, and contract X to a single vertex. As we said, if this produces parallel edges we delete them, since we only work with simple graphs in this paper. But there is another issue: the graph we obtain by contraction might not be bipartite, and we want to produce a bipartite graph at the end, so we in general we must delete some of the edges incident with the new vertex. We could explicitly list the edges that we need to delete, but since we will apply this operation many times, let us set up a more convenient method. Let us say we *contract X into A* if we first contract X to a single vertex, x say, and then delete all edges between x and A. Thus the graph we produce has a bipartition $((A \setminus X) \cup \{x\}, B \setminus X)$. "Contracting into B" is defined similarly.

Let us see first:

2.1 *For $t = 1, 2, 3, 4$, if G admits a bipartition (A, B) with $|A| \geq |B| > 0$ such that every vertex in A has degree at least $t - 1$ then G has a K_t minor.*

Proof. We may assume that every vertex in A has degree exactly $t - 1$, by deleting edges, and we may assume that $|A| = |B|$, by deleting $|A| - |B|$ vertices from A. For $t \leq 2$ the result is clear. For $t = 3$, the graph has $2|A| = |G|$ edges and so has a cycle, and hence a K_3 minor.

Next let $t = 4$; we proceed by induction on $|A|$. We may assume that G has a vertex of degree at most two, b say (necessarily $b \in B$), because otherwise it has a K_4 minor. If b has degree zero we may delete it, and if it has degree one we may delete it and its neighbour, and in either case the result follows from the inductive hypothesis. So we assume that b has two neighbours a_1, a_2. If there are at least four vertices in $B \setminus \{b\}$ with a neighbour in $\{a_1, a_2\}$, we may contract $\{a_1, b, a_2\}$ into A and apply the inductive hypothesis; so we assume that a_1, a_2 have exactly the same neighbours b, b_1, b_2, b_3. If some vertex different from a_1, a_2 is adjacent to both b_1, b_2 then G has a K_4 minor: and otherwise we may contract $\{b_1, b_2, a_1, a_2, b\}$ into B and apply the inductive hypothesis. This proves 2.1. ∎

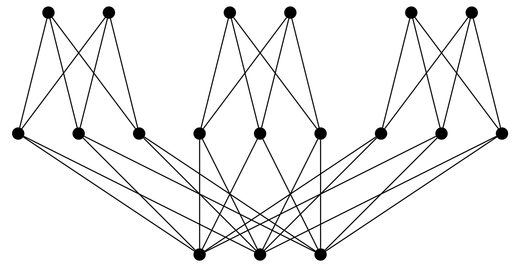

Figure 1: Counterexample to 2.1 with $t = 5$.

Since all bipartite graphs with minimum degree at least four have K_5 minors, one might hope that 2.1 would hold with $t = 5$, but that is false. Here is a counterexample (see figure 1). Let H be the graph obtained from $K_{3,5}$ by deleting three edges that form a matching. Now take k copies of H, say $H_1, \dots, H_k$, and for $1 \le i \le k$ let a_i, b_i, c_i be the three vertices of H_i that have degree two. Let G be obtained from the disjoint union of $H_1, \dots, H_k$ by making the identifications $a_1 = \cdots = a_k$, $b_1 = \cdots = b_k$ and $c_1 = \cdots = c_k$. Then G admits a bipartition (A, B) with $|A| = 3k$ and $|B| = 2k+3$, and every vertex in A has degree four, and G has no K_5 minor. Thus taking $k \ge 3$ we obtain a counterexample to 2.1 with $t = 5$. By taking $k = 4$ instead, and then adding a new vertex adjacent to every vertex in A, we obtain a graph that shows that we cannot replace "six" by "five" in 1.7.

We have:

2.2 *If G admits a bipartition (A, B) with $|A| \ge |B| > 0$ such that every vertex in A has degree at least five then G has a K_5 minor.*

This is turn is a consequence of 1.7 as we show now.

Proof of 2.2, assuming 1.7. Suppose that G admits a bipartition (A, B) with $|A| \ge |B| > 0$ such that every vertex in A has degree at least five and G has no K_5 minor. We may assume that $|A| = |B|$. Choose $b \in B$. Now take k copies of G, say $G_1, \dots, G_k$, and let b_i be the vertex of G_i that corresponds to b. Let H be obtained from the disjoint union of $G_1, \dots, G_k$ by identifying $b_1, \dots, b_k$. Then H has no K_5 minor, and has a bipartition (C, D) with $|C| = k|A|$ and $|D| = k(|B|-1)+1 = k(|A|-1)+1$, and every vertex in C has degree at least five. Now add one more vertex d to H adjacent to every vertex in C; then the graph we produce admits a bipartition $(C, D \cup \{d\})$ where every vertex in C has degree at least six, and it has no K_6 minor (because H has no K_5 minor). So if we choose k such that $k|A| \ge k(|A|-1)+2$, that is, $k \ge 2$, we obtain a contradiction to 1.7. This proves 2.2. ∎

3 Some lemmas

Let us begin on the proof of 1.7. Thus, let G be a graph that admits a bipartition (A, B) with $|A| \ge |B| > 0$ such that every vertex in A has degree at least six; we need to show that G admits a 6-cluster. Let us say G is a *candidate* if admits a bipartition (A, B) with $|A| \ge |B| > 0$ such that every vertex in A has degree at least six, and G has no 6-cluster. We need to show that there is no candidate. We call (A, B) the *bipartition* of the candidate. If G is a candidate with $|G| + |E(G)|$ minimum, we say it is a *minimal candidate.*

Let us begin with some easy observations.

3.1 *Let G be a minimal candidate with bipartition (A, B). Then*

- $|A| = |B|$;
- *every vertex in A has degree exactly six;*
- *if $X \subseteq B$ is nonempty then $G \setminus X$ has at most $|X|$ components;*
- *if $X \subseteq A$ is nonempty then $G \setminus X$ has at most $|X|$ components; and*
- *if $X \subseteq A$ is nonempty and $G \setminus X$ has exactly $|X|$ components then at most one of them has more than one vertex.*

Proof. If $|A| > |B|$ we could delete a vertex in A and obtain a smaller candidate; and if some vertex in A has degree more than six we could delete an edge incident with it to obtain a smaller candidate, in either case contradicting minimality.

For the third bullet, let $X \subseteq B$ be nonempty, and let $G_1, \dots, G_k$ be the components of $G \setminus X$. For $1 \leq i \leq k$, let $|V(G) \cap A| = p_i$ and $|V(G) \cap B| = q_i$. Then for $1 \leq i \leq k$, since $G \setminus V(G_i)$ is not a candidate, it follows that $|A| - p_i < |B| - q_i$, and so $p_i \geq q_i + 1$ since $|A| = |B|$; but then

$$|A| = p_1 + \cdots + p_k \geq q_1 + \cdots + q_k + k = |B| - |X| + k,$$

and since $|A| = |B|$ it follows that $k \leq |X|$. This proves the third bullet.

For the fourth bullet, let $X \subseteq A$ be nonempty, and let $G_1, \dots, G_k$ be the components of $G \setminus X$. For $1 \leq i \leq k$, let $|V(G) \cap A| = p_i$ and $|V(G) \cap B| = q_i$. For $1 \leq i \leq k$, since G_i is not a candidate, it follows that $p_i \leq q_i - 1$; but

$$|A| = p_1 + \cdots + p_k + |X| \leq q_1 + \cdots + q_k - k + |X| = |B| - k + |X|,$$

and since $|A| = |B|$ it follows that $k \leq |X|$. This proves the fourth bullet.

Finally, in the same notation, suppose that $k = |X|$, and G_1, G_2 both have at least two vertices. Thus $p_i \leq q_i - 1$ for $1 \leq i \leq k$, and since $k = |X|$, it follows that $p_i = q_i - 1$ for $1 \leq i \leq k$. From the third and fourth bullets it follows that G is two-connected, and so there are two vertex-disjoint paths of G, say R, S each with first vertex in $V(G_1)$ and last vertex in $V(G_2)$, and each with no other vertices in $V(G_1) \cup V(G_2)$. Consequently R, S each have first vertex in $V(G_1) \cap B$ and last vertex in $V(G_2) \cap B$. By contracting $V(R)$ and $V(S)$ into B we see that G contains as a minor the graph obtained from $G_1 \cup G_2$ by identifying the ends of R and identifying the ends of S. But this graph admits a bipartition with parts of cardinalities $p_1 + p_2$ and $q_1 + q_2 - 2 = p_1 + p_2$, and so it is a smaller candidate, a contradiction. This proves the fifth bullet and so proves 3.1. ∎

3.1 has a useful corollary:

3.2 *Let G be a minimal candidate with bipartition (A, B), and let $X \subseteq A$ or $X \subseteq B$ with $|X| = 4$. Then there do not exist five connected subgraphs $Y_1, \dots, Y_5$ of $G \setminus X$, pairwise vertex-disjoint, such that for $1 \leq i \leq 5$, every vertex in X has a neighbour in Y_i.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$. Suppose that such $Y_1, \dots, Y_5$ exist, and choose them with maximal union. By the third and fourth bullets of 3.1 they are not all components of $G \setminus X$, and so from the maximality of their union, some two of them are joined by an edge, say Y_4, Y_5. But then there is a 6-cluster

$$\{V(Y_1) \cup \{x_1\}, V(Y_2) \cup \{x_2\}, V(Y_3) \cup \{x_3\}, V(Y_4), V(Y_5), \{x_4\}\},$$

which is impossible. This proves 3.2. ∎

Here is another way of using the minimality of the candidate. Let $a_1, \dots, a_p \in A$ be distinct and $b_1, \dots, b_q \in B$ be distinct. The *cover graph* H (with respect to $a_1, \dots, a_p, b_1, \dots, b_q$) is the graph with vertex set $\{b_1, \dots, b_q\}$ in which two distinct vertices u, v are adjacent if there is a vertex $w \in A \setminus \{a_1, \dots, a_p\}$ adjacent in G to both u, v. We denote the chromatic number of H by $\chi(H)$. A partition of $V(H) = \{b_1, \dots, b_q\}$ into sets that are stable in H is a *colouring* of H, and a partition $\{Y_1, \dots, Y_k\}$ of $V(H)$ is *feasible* if there are pairwise disjoint subsets $X_1, \dots, X_k$ of $\{a_1, \dots, a_p\}$ such that $G[X_i \cup Y_i]$ is connected for $1 \le i \le k$. (Note that the sets X_i might be empty.)

3.3 *Let G be a minimal candidate with bipartition (A, B), and let $a_1, \dots, a_p \in A$ be distinct and $b_1, \dots, b_q \in B$, with cover graph H. Then no colouring of H of cardinality at most $q - p$ is feasible.*

Proof. Suppose that the colouring $\{Y_1, \dots, Y_k\}$ of H is feasible, where $k \le q - p$, and let $X_1, \dots, X_k$ be the corresponding subsets of $\{a_1, \dots, a_p\}$. By contracting each of the sets $X_i \cup Y_i$ into B, we obtain a graph with a bipartition (C, D) say, where $|C| \ge |A| - p$ and $|D| = |B| - q + k \le |B| - p \le |C|$; and every vertex in C has degree at least six, since each of $Y_1, \dots, Y_k$ is stable in H, and so this graph is a candidate, which is impossible from the minimality of G. This proves 3.3. ∎

A special case of 3.3 is used so frequently that it is worth stating explicitly:

3.4 *Let G be a minimal candidate with bipartition (A, B). If $b_1, b_2 \in B$ are distinct and have a common neighbour in A then they have at least two common neighbours in A.*

Proof. Let $a \in A$ be adjacent to b_1, b_2. If b_1, b_2 have no other common neighbour, then the covering graph of a, b_1, b_2 admits a colouring of cardinality one, which is therefore feasible, contrary to 3.3. ∎

4 Excluding $K(3,5,0)$- and $K(4,4,1)$-subgraphs

We will prove a series of results about minimal candidates, which eventually allow to show that there is no such graph. Most of these result are of the form "If G is a minimal candidate, then G has no subgraph of the following type", where the types describe subgraphs that become smaller and simpler as the sequence goes on. For instance, one of our result will say that there do not exist two vertices in A and six vertices in B such that each of the first is adjacent to each of the second. We need some notation to describe these "types". For integers $p, q, r \ge 0$ with $r \le \min(p, q)$, let us say a subgraph H of G is a *$K(p, q, r)$-subgraph* if it consists of p vertices $a_1, \dots, a_p \in A$ and q vertices $b_1, \dots, b_q \in B$, where the pairs $a_1b_1, a_2b_2, \dots, a_rb_r$ are nonadjacent, and otherwise each a_i is adjacent to each b_j. Thus H is obtained from a complete bipartite graph $K_{p,q}$ by deleting a matching with r edges; but it matters that the p vertices belong to A and the q belong to B, and not the other way around.

In this section we will prove that a minimal candidate has no $K(3, 5, 0)$-subgraph and no $K(4, 4, 1)$-subgraph. We begin with:

4.1 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(4, 4, 0)$-subgraph.*

Proof. Suppose that $a_1, \dots, a_4 \in A$ are adjacent to $b_1, \dots, b_4 \in B$. Let $Z = \{a_1, \dots, a_4, b_1, \dots, b_4\}$. For each component C of $G \setminus Z$, let $N(C)$ denote the set of vertices in Z with a neighbour in $V(C)$.

(1) *For each component C of $G \setminus Z$, $\{a_1, a_2, a_3, a_4\} \not\subseteq N(C)$, and $\{b_1, b_2, b_3, b_4\} \not\subseteq N(C)$.*

This is immediate from two applications of 3.2, setting $X = \{a_1, a_2, a_3, a_4\}$ and $X = \{b_1, b_2, b_3, b_4\}$.

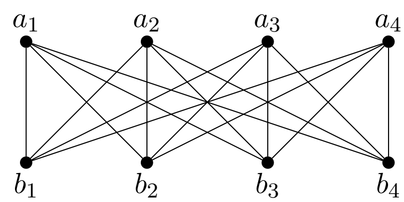

Figure 2: $K(4, 4, 0)$-subgraph.

Since a_1, a_2, a_3, a_4 have degree six, and so each of them belongs to $N(C)$ for some component C of $G \setminus Z$, it follows from (1) that there are at least two such components.

(2) *For each component C of $G \setminus Z$, $|N(C) \cap A| \neq 1$.*

Suppose that $N(C) \cap A = \{a_1\}$ say. Let $b \in V(C)$ be adjacent to a_1; thus $b \in B$. By (1) we may assume that $b_1 \notin N(C)$. But then b, b_1 have a unique common neighbour, contrary to 3.4. This proves (2).

(3) *For each component C of $G \setminus Z$, one of $|N(C) \cap A|, |N(C) \cap B| \geq 2$.*

If $|N(C) \cap A| \leq 1$ then $N(C) \cap A = \emptyset$ by (2), and so by the third bullet of 3.1, taking $X = N(C)$, it follows that $|N(C) \cap B| \geq 2$. This proves (3).

(4) *$|N(C) \cap B| \leq 1$ and $|N(C) \cap A| \in \{2, 3\}$ for every component C of $G \setminus Z$.*

Suppose that $|N(C) \cap B| \geq 2$, and let $b_1, b_2 \in N(C)$ say. By (1), there is a component $C' \neq C$ of $G \setminus Z$ with $N(C') \cap A \neq \emptyset$, and hence with $|N(C') \cap A| \geq 2$ by (2). Let $a_1, a_2 \in N(C')$ say. Then there is a 6-cluster

$$\{\{a_1\}, V(C') \cup \{a_2\}, \{b_1\}, V(C) \cup \{b_2\}, \{a_3, b_3\}, \{a_4, b_4\}\},$$

a contradiction. This proves that $|N(C) \cap B| \leq 1$ for every component C of $G \setminus Z$; and so $|N(C) \cap A| \in \{2, 3\}$ for every component C of $G \setminus Z$ by (3) and (1). This proves (4).

(5) *If $v \in B \setminus Z$ has a neighbour in $\{a_1, a_2, a_3, a_4\}$ then it has at least two such neighbours.*

Let C be the component of $G \setminus Z$ that contains v. We may assume that v is adjacent to a_1, and by (1), we may assume that $b_1 \notin N(C)$. By 3.4, b_1, v have another common neighbour, which must be in $\{a_2, a_3, a_4\}$ since $b_1 \notin N(C)$. This proves (5).

(6) *If $u, v \in B \setminus Z$ have a common neighbour in $\{a_1, a_2, a_3, a_4\}$ and belong to different components of*

$G \setminus Z$ then they have at least two common neighbours in $\{a_1, a_2, a_3, a_4\}$. Consequently, if C, C' are distinct components of $G \setminus Z$ then $|N(C) \cap N(C') \cap A| \neq 1$.

The first claim follows from 3.4 applied to u, v; and the second is a consequence. This proves (6).

(7) *$|N(C) \cap A| = 2$ for each component C of $G \setminus Z$.*

Suppose not; then by (4) $|N(C) \cap A| = 3$, and we may assume that $N(C) \cap A = \{a_1, a_2, a_3\}$. Let C' be a component of $G \setminus Z$ with $a_4 \in N(C')$. By (4), $N(C) \cap N(C') \neq \emptyset$, and so by (6), $|N(C') \cap A| = 3$ and we may assume that $a_2, a_3, a_4 \in N(C')$. Since a_2 has a neighbour in each of C, C' and is also adjacent to b_1, b_2, b_3, b_4, it has no more neighbours, and the same holds for a_3. Consequently if $C'' \neq C, C'$ is a component of $G \setminus Z$ then $N(C'') \cap A \subseteq \{a_1, a_4\}$, and so equality holds by (4), contrary to (6). Thus C, C' are the only components of $G \setminus Z$. Hence a_1 has two neighbours $d_1, d_2 \in V(C)$, and a_2, a_3 each have exactly one neighbour in $V(C')$. By (5), each of d_1, d_2 is adjacent to two of a_1, a_2, a_3, a_4, and so we may assume that d_1 is adjacent to a_2 and not to a_3. Similarly there exists $d' \in V(C')$ adjacent to a_2 and not to a_3. But then d_1, d' have a unique common neighbour, contrary to (6). This proves (7).

By (4), b_1, b_2, b_3, b_4 belong to different components of $G \setminus \{a_1, a_2, a_3, a_4\}$, and so by 3.1, these are the only components of $G \setminus \{a_1, a_2, a_3, a_4\}$, and three of them have only one vertex. Consequently we may assume that b_2, b_3, b_4 have degree four in G, and $b_1 \in N(C)$ for each component C of $G \setminus Z$. By (7) and (6), we may assume that for each component C of $G \setminus Z$, $N(C) \cap A = \{a_1, a_2\}$ or $\{a_3, a_4\}$. Let G_1 be the union of the components C with $N(C) \cap A = \{a_1, a_2\}$, and define G_2 similarly for $\{a_3, a_4\}$. Thus a_1, a_2 each have two neighbours in $V(G_1)$, and a_3, a_4 each have two in $V(G_2)$. Hence by contracting $\{a_1, b2, a_3\}$ and $\{a_2, b_3, b_4\}$ into A we obtain a smaller candidate, a contradiction. This proves 4.1. ∎

4.2 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(3,6,0)$-subgraph.*

Proof. Suppose that $a_1, a_2, a_3 \in A$ are all adjacent to each of $b_1, \ldots, b_6 \in B$. Let H be the cover graph with respect to $a_1, a_2, a_3, b_1, \ldots, b_6$.

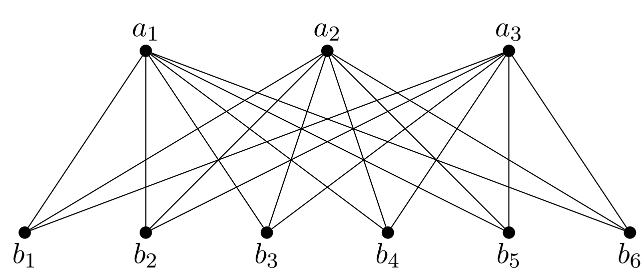

Figure 3: $K(3,6,0)$-subgraph.

(1) *If b_1, b_2, b_3 are pairwise adjacent in H, there is a vertex $a \neq a_1, a_2, a_3$ adjacent to all of b_1, b_2, b_3, and no other vertex in $A \setminus \{a_1, a_2, a_3\}$ is adjacent to any two of b_1, b_2, b_3.*

Since $b_1b_2 \in E(H)$, there exists $c_3 \in A \setminus \{a_1, a_2, a_3\}$ adjacent to b_1, b_2; and similarly there exists c_1 adjacent to b_2, b_3 and c_2 adjacent to b_3, b_1. If c_1, c_2, c_3 are all different, there is a 6-cluster

$$\{\{c_2, b_1\}, \{c_3, b_2\}, \{c_1, b_3\}, \{a_1, b_4\}, \{a_2, b_5\}, \{a_3, b_6\}\},$$

a contradiction. So c_1, c_2, c_3 cannot be chosen all different. In particular we may assume that some $c \in A \setminus \{a_1, a_2, a_3\}$ is adjacent to b_1, b_2, b_3. If some other vertex $d \in A \setminus \{a_1, a_2, a_3\}$ is adjacent to two of b_1, b_2, b_3, say to b_1, b_2, there is a 6-cluster

$$\{\{b_1, d\}, \{b_2\}, \{c, b_3\}, \{a_1, b_4\}, \{a_2, b_5\}, \{a_3, b_6\}\},$$

a contradiction. This proves (1).

Every colouring of H of cardinality three is feasible (as we can add one a_i to each vertex class), so $\chi(H) \geq 4$ by 3.3. Consequently either H consists of an induced cycle of length five together with one more vertex adjacent to every vertex of the cycle, or H has a clique of size four. In either case there are four vertices of H such that five of the six pairs of them are adjacent in H. We may assume that $b_1b_2, b_1b_3, b_1b_4, b_2b_3, b_2b_4$ are all edges of H. By (1) there exists $c \in A \setminus \{a_1, a_2, a_3\}$ adjacent to b_1, b_2, b_3, and $d \in A \setminus \{a_1, a_2, a_3\}$ adjacent to b_1, b_2, b_4; and by (1) again, $c = d$. Thus c is adjacent to b_1, b_2, b_3, b_4, and so $G[\{a_1, a_2, a_3, c, b_1, b_2, b_3, b_4\}]$ is a $K(4,4,0)$-subgraph, contrary to 4.1. This proves 4.2. ∎

If P is a path, we denote by P^* the set of vertices in the interior of P, that is, the vertices that have degree two in P.

4.3 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(3,5,0)$-subgraph.*

Proof. Suppose that $a_1, a_2, a_3 \in A$ are all adjacent to each of $b_1, \ldots, b_5 \in B$.

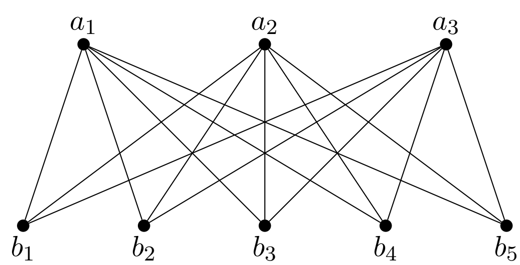

Figure 4: $K(3,5,0)$-subgraph.

Each of a_1, a_2, a_3 has exactly one neighbour different from $b_1, \ldots, b_5$, and they are not all equal since G has no $K(3,6,0)$-subgraph by 4.2. So some vertex is adjacent to exactly one of a_1, a_2, a_3; say b_6 is adjacent to a_1 and not to a_2, a_3. By 3.4, for $1 \leq i \leq 5$ b_i, b_6 have a common neighbour different from a_1. Choose a set X of neighbours of b_6, with $a_1, a_2, a_3 \in X$, minimal such that $b_1, \ldots, b_5$ each have a neighbour in X. Consequently for each $x \in X$ there exists $i \in \{1, \ldots, 5\}$ such that x is the unique neighbour of b_i in X.

(1) *Every vertex different from a_1, a_2, a_3 with two neighbours in $\{b_1, \ldots, b_5\}$ is in X.*

Suppose that $a \in A \setminus \{a_1, a_2, a_3\}$ is adjacent to a_1, a_2 say, and $a \notin X$. Then there is a 6-cluster

$$\{\{b_1\}, \{a, b_2\}, \{b_6, b_3\} \cup X, \{a_1\}, \{a_2, b_4\}, \{a_3, b_5\}\},$$

a contradiction. This proves (1).

(2) *Some vertex different from a_1, a_2, a_3 has three neighbours in $\{b_1, \ldots, b_5\}$.*

Let H be the cover graph with respect to $a_1, a_2, a_3, b_1, \ldots, b_5$. By 3.3, $\chi(H) \geq 3$. So either it is a cycle of length five, or it has a triangle. (A *triangle* means a clique with cardinality three.) Suppose first that H is a cycle of length five, with edges $b_1b_2, b_2b_3, b_3b_4, b_4b_5, b_5b_1$ say. Some vertex $d_{1,2} \neq a_1, a_2, a_3$ is adjacent in G to b_1, b_2, from the definition of H, and it is nonadjacent to b_3, b_4, b_5 since H is a cycle. Define $d_{2,3}$ and so on similarly. By (1), each of these five vertices is in the set X; but then none of $b_1, \ldots, b_5$ has a unique neighbour in X, contrary to the minimality of X.

It follows that H has a triangle, say with vertices b_1, b_2, b_3. Some vertex $d_{1,2} \neq a_1, a_2, a_3$ is adjacent in G to b_1, b_2, from the definition of H; define $d_{2,3}, d_{3,1}$ similarly. Suppose that $d_{1,2}, d_{2,3}, d_{3,1}$ are all different. Then there is a 6-cluster

$$\{\{d_{1,2}, b_1\}, \{d_{2,3}, b_2\}, \{d_{3,1}, b_3\}, \{a_1\}, \{a_2, b_4\}, \{a_3, b_5\}\},$$

a contradiction. So two of $d_{1,2}, d_{2,3}, d_{3,1}$ are equal. This proves (2).

Let a_4 be adjacent to b_1, b_2, b_3 say. It is nonadjacent to b_4, b_5 since G has no $K(4,4,0)$-subgraph by 4.1. By 3.3 the cover graph with respect to $a_1, a_2, a_3, a_4, b_1, \ldots, b_5$ has chromatic number at least two, and so has an edge. Choose a_5 different from $a_1, \ldots, a_4$ with two neighbours in $\{b_1, \ldots, b_5\}$. By (1), a_4, a_5 are both adjacent to b_6.

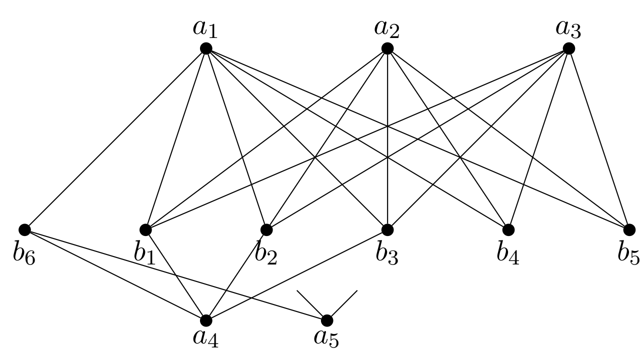

Figure 5: For the last part of the proof of 4.3. a_5 is adjacent to two of $b_1, \ldots, b_5$.

Up to symmetry there are three cases: a_5 is adjacent to b_1, b_2; a_5 is adjacent to b_1, b_4; and a_5 is adjacent to b_4, b_5.

First, if a_5 is adjacent to b_1, b_2, there is a 6-cluster

$$\{\{b_1\}, \{a_5, b_2\}, \{a_4, b_3\}, \{a_1\}, \{a_2, b_4\}, \{a_3, b_5\}\},$$

a contradiction. If a_5 is adjacent to b_1, b_4, there is a 6-cluster

$$\{\{b_1\}, \{a_4, b_2\}, \{a_5, b_4, b_6\}, \{a_1\}, \{a_2, b_3\}, \{a_3, b_5\}\},$$

a contradiction. So a_5 is adjacent to b_4, b_5.

By 3.1, the graph $G \setminus \{a_1, a_2, a_3, a_4, a_5\}$ has at most five components; and so some two of $b_1, \dots, b_6$ belong to the same component. So there is a path P of G between two of $b_1, \dots, b_6$ with no other vertices in $\{a_1, \dots, a_5, b_1, \dots, b_6\}$. Let P have ends b_i, b_j say. The subgraph induced on $\{a_4, a_5, b_1, \dots, b_6\}$ is a tree, and its union with P includes a cycle that contains P; and in all cases we can use this cycle to make three of $b_1, \dots, b_5$ adjacent and thereby produce a K_6 minor. In detail (up to symmetry these are the only possibilities):

- If $(i,j) = (1,2)$, there is a 6-cluster $\{\{b_1\}, \{P^* \cup \{b_2\}\}, \{a_4, b_3\}, \{a_1\}, \{a_2, b_4\}, \{a_3, b_5\}\}$.
- If $(i,j) = (1,4)$, there is a 6-cluster $\{\{b_1\}, \{a_4, b_2\}, P^* \cup \{b_4, b_6\}, \{a_1\}, \{a_2, b_3\}, \{a_3, b_5\}\}$.
- If $(i,j) = (1,6)$ there is a 6-cluster $\{\{b_1\}, \{a_4 b_2\}, P^* \cup \{b_6, a_5, b_4\}, \{a_1\}, \{a_2, b_3\}, \{a_3, b_5\}\}$.
- If $(i,j) = (4,5)$ there is a 6-cluster $\{\{b_1, a_4, b_6, a_5\}, \{b_4\}, P^* \cup \{b_5\}, \{a_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$.
- If $(i,j) = (4,6)$ there is a 6-cluster $\{P^* \cup \{b_6\}, \{b_4\}, \{a_5, b_5\}, \{a_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$.

In each case we have a contradiction. This proves 4.3. ∎

4.4 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(4,7,4)$-subgraph.*

Proof. Suppose that $a_1, \dots, a_4 \in A$ are all adjacent to each of $b_1, \dots, b_7 \in B$, except the pairs $a_1b_1, a_2b_2, a_3b_3, a_4b_4$. Let H be the cover graph with respect to $a_1, \dots, a_4, b_1, \dots, b_7$. We claim that every partition of $V(H)$ into at most three sets is feasible. To see this, let $\{Y_1, \dots, Y_k\}$ be a partition of $V(H)$ with $k \leq 3$. We must show there are disjoint subsets $X_1, \dots, X_k$ of $\{a_1, \dots, a_4\}$ such that $X_i \cup Y_i$ is connected for $1 \leq i \leq k$. If some Y_i, say Y_1, contains all of $b_1, \dots, b_4$ we may set $X_1 = \{a_1, a_2\}$ and $X_2, \dots, X_k$ each to contain one of a_3, a_4; so we may assume that for $1 \leq i \leq k$, there exists $j \in \{1, \dots, 4\}$ such that $b_j \notin Y_i$. But then (from Hall's "marriage" theorem, for instance), there is an injection $\phi : \{1, \dots, k\} \to \{1, \dots, 4\}$ such that $b_{\phi(i)} \notin Y_i$ for $1 \leq i \leq k$; so we may set $X_i = \{a_{\phi(i)}\}$ for $1 \leq i \leq k$. This proves that every partition of $V(H)$ into at most three sets is feasible, and so $\chi(H) \geq 4$ by 3.3.

(1) *b_1, b_2, b_3, b_4 are pairwise adjacent in H, and H has no other edges.*

Suppose that say b_5b_6 are adjacent in H, and let P be a path of G between b_5, b_6 with no other vertex in $\{a_1, \dots, a_4, b_1, \dots, b_7\}$. There is a 6-cluster

$$\{a_1, b_2\}, \{a_2, b_3\}, \{a_3, b_4\}, \{a_4, b_1\}, \{b_5\}, P^* \cup \{b_6\}\},$$

a contradiction. So b_5, b_6, b_7 are pairwise nonadjacent in H. Let S be the set of edges of H with both ends in $\{b_1, \dots, b_4\}$, and T the set of edges of H with one end in $\{b_1, \dots, b_4\}$ and the other in

$\{b_5, b_6, b_7\}$. Thus every edge of H belongs to exactly one of S, T. If say b_3b_4 and b_4b_5 are edges of H, let P, Q be the corresponding paths of G; then there is a 6-cluster

$$\{\{a_1, b_2\}, \{a_2, b_1\}, \{a_3, b_6\}, \{a_4, b_7\}, V(P), Q^* \cup \{b_5\}\},$$

a contradiction. Hence no edge in S shares an end with an edge in T, and so no component of H has an edge in S and an edge in T. But some component H' of H has chromatic number at least four, and so not all its edges are in T, since then it would be bipartite. Hence all its edges are in S, and so $V(H') \subseteq \{b_1, \dots, b_4\}$; and since H' has chromatic number four, H' is a complete graph. This proves (1).

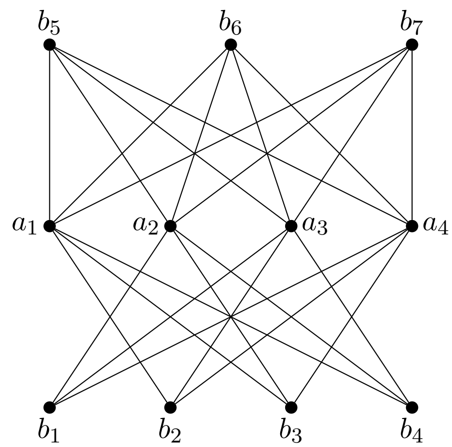

Figure 6: $K(4, 7, 4)$-subgraph.

Choose $c \neq a_1, \dots, a_4$ adjacent to b_1, b_2, and $c' \neq a_1, \dots, a_4$ adjacent to b_3, b_4. Then $c = c'$, since otherwise the connected subgraphs with vertex sets $\{b_1, c, b_2\}, \{b_3, c', b_4\}, \{b_5\}, \{b_6\}, \{b_7\}$ violate 3.2. Consequently c is adjacent to b_1, b_2, b_3, b_4. By the same argument, no other vertex has two neighbours in $\{b_1, \dots, b_4\}$, and hence no other vertex has two neighbours in $\{b_1, \dots, b_7\}$, since b_5, b_6, b_7 have degree zero in H. But then the cover graph with respect to $a_1, \dots, a_4, c, b_1, \dots, b_7$ has no edges and 3.3 is violated. This proves 4.4. ▮

4.5 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(4, 5, 2)$-subgraph.*

Proof. Suppose that $a_1, \dots, a_4 \in A$ are all adjacent to each of $b_1, \dots, b_5 \in B$, except the pairs a_1b_1, a_2b_2.

(1) *Every vertex in $B \setminus \{b_1, \dots, b_5\}$ with a neighbour in $\{a_1, \dots, a_4\}$ has exactly two neighbours in this set.*

Suppose that some vertex $b_6 \in B \setminus \{b_1, \dots, b_5\}$ is adjacent to exactly one of $a_1, \dots, a_4$. By 3.4, for $i = 3, 4, 5$ there is a vertex c_i adjacent to b_i, b_6 and not in $\{a_1, \dots, a_4\}$. But then

$$\{\{a_1, b_2\}, \{a_2, b_3\}, \{a_3, b_1\}, \{b_4\}, \{a_4\}, \{b_5, c_5, b_6, c_4\}\}$$

is a 6-cluster, a contradiction.

So every vertex in $B \setminus \{b_1, \dots, b_5\}$ with a neighbour in $\{a_1, \dots, a_4\}$ has at least two neighbours in this set. No vertex is adjacent to all of a_1, a_3, a_4 or to all of a_2, a_3, a_4, since there is no $K(3, 5, 0)$-subgraph by 4.3. If some vertex b_6 different from $b_1, \dots, b_5$ is adjacent to a_1, a_2, a_3, then each of a_1, a_2, a_4 has exactly one neighbour different from $b_1, \dots, b_6$, and these must all be equal since no vertex has one neighbour in $\{a_1, a_2, a_3, a_4\}$; but then G has a $K(4, 7, 4)$-subgraph, contrary to 4.4. Similarly no vertex different from $b_1, \dots, b_5$ is adjacent to a_1, a_2, a_4; and so every vertex in $B \setminus \{b_1, \dots, b_5\}$ with a neighbour in $\{a_1, \dots, a_4\}$ has exactly two neighbours in this set. This proves (1).

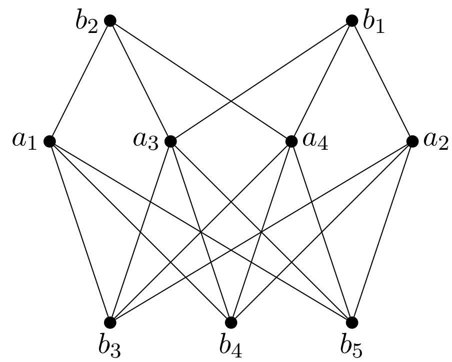

Figure 7: $K(4, 5, 2)$-subgraph.

(2) b_3, b_4, b_5 *each have degree four in* G.

There are four edges between $\{a_1, a_2\}$ and $B \setminus \{b_1, \dots, b_5\}$, and only two between $\{a_3, a_4\}$ and $B \setminus \{b_1, \dots, b_5\}$; and so by (1) there is a vertex $b_6 \in B \setminus \{b_1, \dots, b_5\}$ adjacent to a_1, a_2 and not to a_3, a_4. By 3.4, there is a vertex different from $a_1, \dots, a_4$ adjacent to b_6, b_1 (because b_6, b_1 are adjacent to a_2), and similarly there is a vertex different from $a_1, \dots, a_4$ adjacent to b_6, b_2. Consequently there is a path P between b_1, b_2 with no other vertices in $\{a_1, \dots, a_4, b_1, \dots, b_5\}$ (possibly containing b_6).

We claim that P, b_3, b_4, b_5 all belong to different components of $G \setminus \{a_1, a_2, a_3, a_4\}$. Because suppose not; then, either there is a path Q of G between two of b_3, b_4, b_5 with no other vertex in $\{a_1, \dots, a_4, b_1, \dots, b_5\}$, or there is a path Q between one of b_1, b_2 and one of b_3, b_4, b_5 with no other vertex in $\{a_1, \dots, a_4, b_1, \dots, b_5\}$. In the first case, say Q has ends b_3, b_4; then

$$\{\{b_3\}, Q^* \cup \{b_4\}, \{a_4\}, \{a_1, b_2\}, \{a_2, b_5\}, \{a_3, b_1\}\}$$

is a 6-cluster, a contradiction. In the second case, let Q have ends b_1, b_3 say; then

$$\{\{P^* \cup Q^* \cup \{b_1\}, \{a_1, b_2\}, \{b_3\}, \{a_2, b_4\}, \{a_3, b_5\}, \{a_4\}\}$$

is a 6-cluster, a contradiction.

This proves that P, b_3, b_4, b_5 all belong to different components of $G \setminus \{a_1, a_2, a_3, a_4\}$. By 3.2, the components containing b_3, b_4, b_5 are all singletons. This proves (2).

Now for $i = 3, 4$, a_i has one neighbour not in $\{b_1, \dots, b_5\}$, say c_i. Either $c_3 = c_4$ and c_3 has no neighbour in $\{a_1, a_2\}$, or each of c_3, c_4 has a unique neighbour in $\{a_1, a_2\}$, and not the same one. Thus in either case we may assume that c_3 is not adjacent to a_1 and c_4 is not adjacent to a_2. Consequently, there are at least six vertices in $B \setminus \{b_1, b_2\}$ with a neighbour in $\{a_1, a_3\}$, namely b_3, b_4, b_5, c_3 and the two neighbours of a_1 not in $\{b_1, \ldots, b_5\}$. Similarly there are at least six vertices in $B \setminus \{b_1, b_2\}$ with a neighbour in $\{a_2, a_4\}$. Since b_1, b_2 both have degree four, by contracting $\{a_1, b_1, a_3\}$ and $\{a_2, b_2, a_4\}$ into A we obtain a smaller candidate, a contradiction. This proves 4.5. ∎

4.6 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(4, 4, 1)$-subgraph.*

Proof. Suppose that $a_1, \ldots, a_4 \in A$ are all adjacent to each of $b_1, \ldots, b_4 \in B$, except the pair a_1b_1. No vertex in $B \setminus \{b_1, \ldots, b_4\}$ is adjacent to all of a_2, a_3, a_4 since G has no $K(3, 5, 0)$-subgraph by 4.3. No vertex is adjacent to a_1 and to two of a_2, a_3, a_4 since G has no $K(4, 5, 2)$-subgraph by 4.5. So every vertex in $B \setminus \{b_1, \ldots, b_4\}$ with a neighbour in $\{a_1, \ldots, a_4\}$ has at most two neighbours in this set.

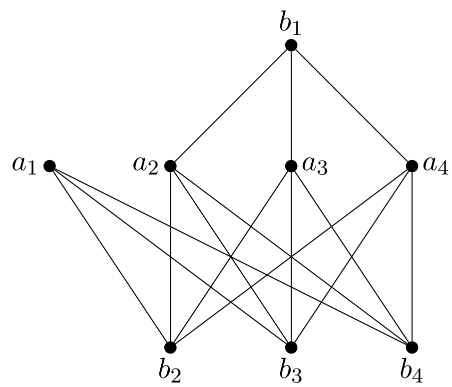

Figure 8: $K(4, 4, 1)$-subgraph.

Suppose that some vertex $b_5 \in B \setminus \{b_1, \ldots, b_4\}$ has only one neighbour in $\{a_1, \ldots, a_4\}$, and that neighbour is different from a_1; let it be a_2 say. By 3.4, for $i = 1, 2, 3, 4$ there is a vertex $c_i \neq a_2$ adjacent to both b_5, b_i. But then

$$\{\{b_5, c_1, c_2, c_3, c_4\}, \{a_2\}, \{a_3, b_1\}, \{a_4, b_2\}, \{a_1, b_3\}, \{b_4\}\}$$

is a 6-cluster. So every vertex in $B \setminus \{b_1, \ldots, b_4\}$ with a neighbour in $\{a_2, \ldots, a_4\}$ has exactly two neighbours in $\{a_1, \ldots, a_4\}$.

But there are an odd number of edges (nine) between $\{a_1, \ldots, a_4\}$ and $B \setminus \{b_1, \ldots, b_4\}$; so some vertex $b_5 \in B \setminus \{b_1, \ldots, b_4\}$ has a unique neighbour in $\{a_1, \ldots, a_4\}$, and consequently this neighbour is a_1. It follows that at most two neighbours of a_1 are different from b_2, b_3, b_4 and have a neighbour in $\{a_2, a_3, a_4\}$. But there are six edges between $\{a_2, a_3, a_4\}$ and $B \setminus \{b_1, \ldots, b_4\}$, and so there is a vertex $b_6 \in B \setminus \{b_1, \ldots, b_4\}$ adjacent to two of a_2, a_3, a_4, say a_2, a_3. By 3.4, for $i = 2, 3, 4$ there is a vertex $c_i \neq a_1$ adjacent to both b_5, b_i. But then

$$\{\{a_2\}, \{a_3, b_6\}, \{a_4, b_1\}, \{a_1, b_3\}, \{b_4\}, \{b_5, c_2, c_3, c_4, b_2\}\}$$

is a 6-cluster, a contradiction. This proves 4.6. ∎

5 Excluding $K(3,4,0)$- and $K(2,6,0)$-subgraphs.

Our main goal in this section is to eliminate $K(3,4,0)$-subgraphs; and to do this, we first eliminate $K(3,7,3)$-subgraphs.

5.1 *Let G be a minimal candidate with bipartition (A,B). Then G has no $K(3,7,3)$-subgraph.*

Proof. Suppose that $a_1,a_2,a_3 \in A$ are all adjacent to each of $b_1,\ldots,b_7 \in B$ except for the pairs a_1b_1, a_2, b_2, a_3b_3.

(1) *No vertex different from a_1,a_2,a_3 has three neighbours in $\{b_4,b_5,b_6,b_7\}$.*

This is immediate since G has no $K(4,4,1)$-subgraph by 4.6.

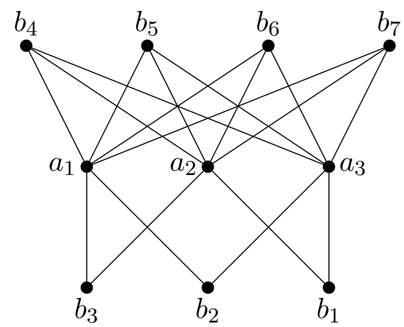

Figure 9: $K(3,7,3)$-subgraph.

(2) *No vertex different from a_1,a_2,a_3 has at least four neighbours in $\{b_1,\ldots,b_7\}$.*

Suppose a_4 has at least four neighbours in $\{b_1,\ldots,b_7\}$. Let I be the set of $i \in \{1,\ldots,7\}$ such that a_4,b_i are adjacent. Thus $|I| \geq 4$. Let b_8 be a neighbour of a_4 not in $\{b_1,\ldots,b_7\}$. Thus b_8 is nonadjacent to a_1,a_2,a_3, since the latter have degree only six. By 3.4, for each $i \in I$ there exists $c_i \neq a_1,\ldots,a_4$ adjacent to b_i,b_8.

Suppose first that $1,2,3 \in I$; and we may assume that $4 \in I$ since $|I| \geq 4$. Then

$$\{\{a_4\},\{b_8,c_1,c_2,c_3,c_4\},\{a_1,b_2\},\{a_2,b_3\},\{a_3,b_1\},\{b_4\}\}$$

is a 6-cluster. So not all $1,2,3$ belong to I; and hence by (1), since $|I| \geq 4$, we may assume that $I = \{1,2,4,5\}$. But then

$$\{\{a_4\},\{b_8,c_1,c_2,c_4,c_5\},\{a_1,b_2\},\{a_2,b_3,b_4\},\{a_3,b_1\},\{b_5\}\}$$

is a 6-cluster. This proves (2).

Let H be the cover graph with respect to $a_1,a_2,a_3,b_1,\ldots,b_7$. By an argument like that in the proof of 4.4, it follows that every partition of $V(H)$ into four sets is feasible, and so $\chi(H) \geq 5$ by 3.3. For each edge b_ib_j of H let $c_{i,j} \in A \setminus \{a_1,a_2,a_3\}$ be adjacent to b_i,b_j.

(3) *The subgraph* $H[\{b_4, b_5, b_6, b_7\}]$ *has no triangle, and so is bipartite; and hence* b_1, b_2, b_3 *are pairwise adjacent in* H.

Suppose that say b_4, b_5, b_6 are pairwise adjacent in H. By (1), $c_{4,5}, c_{5,6}, c_{4,6}$ are all distinct. But then

$$\{\{a_1, b_2\}, \{a_2, b_3\}, \{a_3, b_1\}, \{c_{4,5}, b_4\}, \{c_{5,6}, b_5\}, \{c_{4,6}, b_6\}\}$$

is a 6-cluster. So $H[\{b_4, b_5, b_6, b_7\}]$ is bipartite. Since $\chi(H) \geq 5$ it follows that b_1, b_2, b_3 are pairwise adjacent in H. This proves (3).

(4) *Let* $i \in \{1, 2, 3\}$*, and let* $j, k \in \{4, 5, 6, 7\}$ *be distinct. If* b_i, b_j, b_k *are pairwise adjacent in* H*, then* $c_{i,j}, c_{i,k}, c_{j,k}$ *are all equal.*

Let $i = 3$, $j = 4$ and $k = 5$ say. Suppose first that $c_{3,4}, c_{4,5}, c_{3,5}$ are all different. Since $c_{4,5}$ is different from $c_{2,3}$ by (2) it follows that

$$\{\{a_1\}, \{a_2, b_6\}, \{a_3, b_7\}, \{b_3, c_{2,3}, b_2, c_{3,4}, c_{3,5}\}, \{b_4\}, \{c_{4,5}, b_5\}\}$$

is a 6-cluster. Thus one of $c_{3,4}, c_{4,5}, c_{3,5}$ (say c) is adjacent to all of b_3, b_4, b_5. Suppose that some $d \in \{c_{3,4}, c_{4,5}, c_{3,5}\}$ is different from c. Then d is different from one of $c_{1,3}, c_{2,3}$ by (2), say $d \neq c_{2,3}$, and

$$\{\{a_1\}, \{a_2, b_6\}, \{a_3, b_7\}, \{b_3, c_{2,3}, b_2, c\}, \{b_4\}, \{d, b_5\}\}$$

is a 6-cluster. This proves (4).

(5) *There exists a clique* $X \subseteq V(H)$ *of* H *containing two of* b_1, b_2, b_3 *and two of* $b_4, \ldots, b_7$.

If H is perfect, then it has a clique of cardinality five, which therefore contains all of b_1, b_2, b_3 by (3) and the claim holds. Otherwise, H has an odd hole or antihole as an induced subgraph; and since H has only seven vertices and $\chi(H) \geq 5$, it follows that H has an induced cycle C of length five, and the other two vertices of H are adjacent to each other and to every vertex of C. Since b_1, b_2, b_3 are pairwise adjacent, at least one of them is not in $V(C)$, say b_1; and so at least three vertices of C are not in $\{b_1, b_2, b_3\}$, and consequently an edge of C has both ends in $\{b_4, \ldots, b_7\}$. But this set contains no triangle of C by (3), and so the second vertex of H not in $V(C)$ belongs to $\{b_1, b_2, b_3\}$. This proves (5).

From (5) we may assume that b_2, b_3, b_4, b_5 are pairwise adjacent in H. By (4), $c_{1,4}, c_{1,5}, c_{4,5}$ are all equal, and also $c_{1,5}, c_{1,6}, c_{5,6}$ are all equal. But then $c_{1,4} = c_{5,6}$ contrary to (2). This proves 5.1. ∎

5.2 *Let* G *be a minimal candidate with bipartition* (A, B)*. Then* G *has no* $K(3, 4, 0)$*-subgraph.*

Proof. Suppose that $a_1, a_2, a_3 \in A$ are all adjacent to each of $b_1, \ldots, b_4 \in B$.

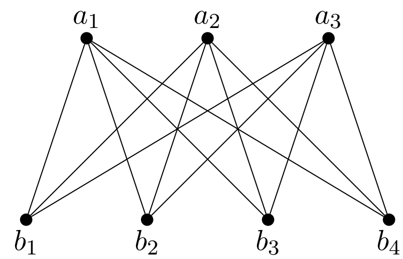

Figure 10: $K(3,4,0)$-subgraph.

No vertex in $A\backslash\{a_1,a_2,a_3\}$ has more than two neighbours in $\{b_1,\ldots,b_4\}$, since G has no $K(4,4,1)$-subgraph or $K(4,4,0)$-subgraph by 4.6 and 4.1. Also no vertex in $B \setminus \{b_1,\ldots,b_4\}$ is adjacent to all three of a_1,a_2,a_3, since G has no $K(3,5,0)$-subgraph by 4.3.

Let B_0 be the set of vertices in B with exactly one neighbour in $\{a_1,a_2,a_3\}$. Thus $|B_0|$ is even.

(1) *$B_0 \neq \emptyset$ and hence $|B_0| \geq 2$.*

Suppose that $B_0 = \emptyset$. Since there are exactly six edges between $\{a_1,a_2,a_3\}$ and $B \setminus \{b_1,\ldots,b_4\}$, it follows that there are exactly three vertices each adjacent to exactly two of a_1,a_2,a_3, and each of a_1,a_2,a_3 is adjacent to exactly two of these three vertices; but then G contains a $K(3,7,3)$-subgraph, contrary to 5.1. This proves (1).

(2) *Every vertex adjacent to exactly two of $b_1,\ldots,b_4$ is adjacent to every vertex in B_0.*

Let a_4 be adjacent to b_1,b_2 say, and let b_5 be adjacent to a_1 and not to a_2,a_3. Suppose that a_4,b_5 are nonadjacent. By 3.4, for $1 \leq i \leq 4$ there is a vertex $c_i \neq a_1$ adjacent to b_5,b_i; and $c_i \neq a_2,a_3,a_4$ since these are not adjacent to b_5. But then

$$\{\{a_4,b_1\},\{b_2\},\{b_5,c_1,c_2,c_3,c_4\},\{a_1\},\{a_2,b_3\},\{a_3,b_4\}\}$$

is a 6-cluster. This proves (2).

Contracting $\{a_1,a_2,a_3,b_1,b_2,b_3,b_4\}$ into B does not yield a smaller candidate, so some vertex a_4 different from a_1,a_2,a_3 has at least two (and hence exactly two) neighbours in $\{b_1,b_2,b_3,b_4\}$. From the symmetry we may assume that a_4 is adjacent to b_1,b_2. Since a_4 has degree six, and is adjacent to every vertex in B_0, it follows that $|B_0| \leq 4$, and so some vertex is adjacent to exactly two of a_1,a_2,a_3.

(3) *There is a vertex different from $a_1,\ldots,a_4$ adjacent to b_3,b_4.*

Suppose not. Choose distinct $b_5,b_6 \in B_0$. By 3.4, for $i \in \{3,4\}$ and $j \in \{5,6\}$ there is a vertex $c_{i,j} \notin \{a_1,\ldots,a_4\}$ adjacent to b_i,b_j; and $c_{i,j} \neq a_4$ since a_4 has only two neighbours in $\{b_1,\ldots,b_4\}$. Moreover $\{c_{3,5},c_{3,6}\}$ is disjoint from $\{c_{4,5},c_{4,6}\}$, since these vertices only have one neighbour in $\{b_3,b_4\}$ by hypothesis. Let b_7 be a vertex adjacent to exactly two of a_1,a_2,a_3, say a_2,a_3; then since a_4 is adjacent to b_5,b_6 by (2),

$$\{\{a_1,b_1\},\{a_2\},\{a_3,b_7\},\{c_{3,5},c_{3,6},b_3,b_5\},\{c_{4,5},c_{4,6},b_4\},\{a_4,b_2,b_6\}\}$$

is a 6-cluster. This proves (3).

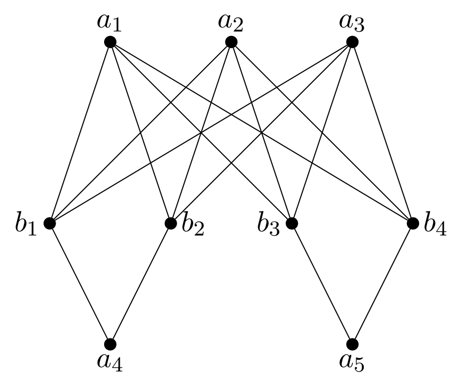

Figure 11: For the proof of 5.2.

Let $a_5 \neq a_1, \dots, a_4$ be adjacent to b_3, b_4. By (2) a_5 is adjacent to every vertex in B_0.

(4) *There is no path P of $G \setminus \{a_1, \dots, a_5\}$ with ends in distinct sets in the list $\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, B_0$.*

Suppose that P is such a path. By choosing P minimal we may assume that no internal vertex of P belongs to $\{b_1, b_2, b_3, b_4\} \cup B_0$; and from the symmetry we may assume that b_1 is an end of P. Let the other end be b_i say. Up to symmetry there are three cases: $i = 2$, $i = 3$, and $i = 5$ for some $b_5 \in B_0$, and in the third case we may assume that a_1 is adjacent to b_5 from the symmetry.

- If $i = 2$ then $\{\{a_1\}, \{a_2, b_3\}, \{a_3, b_4\}, \{b_1\}, P^* \cup \{b_2\}, B_0 \cup \{a_4, a_5\}\}$ is a 6-cluster.
- If $i = 3$ then $\{\{a_1\}, \{a_2, b_2\}, \{a_3, b_4\}, \{b_1\}, P^* \cup \{b_3\}, B_0 \cup \{a_4, a_5\}\}$ is a 6-cluster.
- If $i = 5$ then $\{\{a_1\}, \{a_2, b_3\}, \{a_3, b_4\}, P^* \cup \{b_1\}, \{a_4, b_2\}, \{a_5, b_6\}\}$ is a 6-cluster.

This proves (4).

(5) *$b_1, \dots, b_4$ all have degree four in G.*

From (4), b_1, b_2, b_3, b_4, b_5 all belong to different components of $G \setminus \{a_1, \dots, a_5\}$, and none of these four compoents contains any vertex of B_0. By 3.1, $G \setminus \{a_1, \dots, a_5\}$ has exactly five components, and four of them are singletons; and therefore one contains all of B_0 and so is not a singleton. This proves (5).

As we observed earlier, there is a vertex $c \in B$ that is adjacent to exactly two of a_1, a_2, a_3, say to a_2, a_3. Moreover, since $B_0 \neq \emptyset$, not both neighbours of a_1 in $B \setminus \{b_1, \dots, b_4\}$ have a neighbour in $\{b_2, b_3\}$; and so a_1 has a neighbour in B_0, say b_5. Since we cannot obtain a smaller candidate by contracting $\{a_1, \dots, a_5, b_1, \dots, b_5, c\}$ into B, there exists $a_6 \in A$ different from $a_1, \dots, a_5$ with two neighbours in $\{b_1, \dots, b_5, c\}$. By (5), a_6 is adjacent to b_5, c. Let $b_6 \in B_0 \setminus \{b_5\}$. Then

$$\{\{a_1\}, \{a_2, b_1\}, \{a_3, b_3\}, \{a_4, b_2\}, \{a_5, b_4\}, \{a_6, c, b_5\}\}$$

is a 6-cluster. This proves 5.2. ∎

6 Excluding $K(2,5,0)$-subgraphs

Our next goal is to eliminate $K(2,5,0)$-subgraphs. We begin with:

6.1 *Let G be a minimal candidate with bipartition (A,B). Then G has no $K(2,6,0)$-subgraph.*

Proof. Suppose that $a_1, a_2 \in A$ are both adjacent to each of $b_1, \ldots, b_6 \in B$. The cover graph H with respect to $a_1, a_2, b_1, \ldots, b_6$ has chromatic number at least five, by 3.3 (note that every partition of $\{b_1, \ldots, b_6\}$ into four sets is feasible, since at least two of them will be singletons and therefore already induce connected subgraphs). Consequently H has a clique of size five, and so we may assume that $b_1, \ldots, b_5$ are pairwise adjacent in H. By 5.2, no vertex in $A \setminus \{a_1, a_2\}$ has more than three neighbours in $\{b_1, \ldots, b_6\}$.

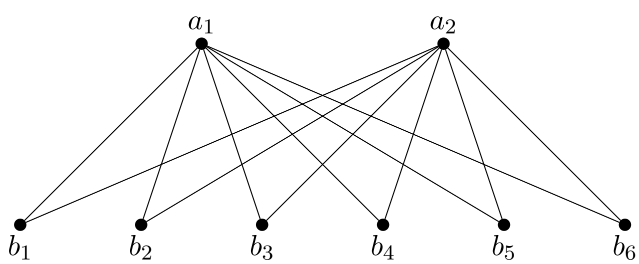

Figure 12: $K(2,6,0)$-subgraph.

For $1 \le i < j \le 5$ let $c_{i,j} \in A \setminus \{a_1, a_2\}$ be adjacent to b_i, b_j. If the six vertices $c_{i,j}$ $(1 \le i < j \le 4)$ are all distinct, there is a 6-cluster

$$\{\{b_1, c_{1,2}, c_{1,3}, c_{1,4}\}, \{b_2, c_{2,3}, c_{2,4}\}, \{b_3, c_{3,4}\}, \{b_4\}, \{a_1\}, \{a_2, b_5\}\},$$

a contradiction. So we may assume that some two are equal, and hence some vertex in $A \setminus \{a_1, a_2\}$ is adjacent to three of b_1, b_2, b_3, b_4; say a_3 is adjacent to b_1, b_2, b_3. If none of the vertices $c_{i,j}$ $(i \in \{1,2,3\}, j \in \{4,5\})$ is adjacent to both b_4, b_5, then

$$\{\{b_1\}, \{b_2, a_3\}, \{b_4, c_{1,4}, c_{2,4}\}, \{b_5, c_{1,5}, c_{2,5}, c_{4,5}\}, \{a_1\}, \{a_2, b_6\}\},$$

is a 6-cluster, a contradiction; so we may assume that some a_4 is adjacent to b_3, b_4, b_5 say. (See figure 13.)

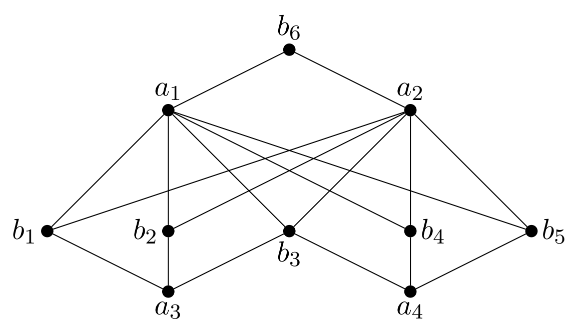

Figure 13: For the proof of 6.1.

If some a_5 different from $a_1, \dots, a_4$ is adjacent to three of b_1, b_2, b_4, b_5, say to b_1, b_2, b_4, then

$$\{\{b_1\}, \{b_2, a_3\}, \{b_4, a_5\}, \{b_5, a_4, c_{1,5}, c_{2,5}\}, \{a_1\}, \{a_2, b_6\}\}$$

is a 6-cluster, a contradiction; so we may assume that no vertex different from $a_1, \dots, a_4$ is adjacent to three of b_1, b_2, b_4, b_5. Consequently $c_{1,4}, c_{2,4}, c_{1,5}, c_{2,5}$ are all different. But then

$$\{\{b_1\}, \{b_2, a_3\}, \{b_4, c_{1,4}, c_{2,4}\}, \{b_5, a_4, c_{1,5}, c_{2,5}\}, \{a_1\}, \{a_2, b_6\}\}$$

is a 6-cluster, a contradiction. This proves 6.1. ∎

6.2 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(2,5,0)$-subgraph.*

Proof. Suppose that $a_1, a_2 \in A$ are both adjacent to each of $b_1, \dots, b_5 \in B$. Let b_6, b_7 be the neighbours of a_1, a_2 respectively that are not in $\{b_1, \dots, b_5\}$. Thus $b_6 \neq b_7$ since there is no $K(2,6,0)$-subgraph by 6.1. No vertex different from a_1, a_2 has four neighbours in $\{b_1, \dots, b_5\}$ since there is no $K(3,4,0)$-subgraph by 5.2.

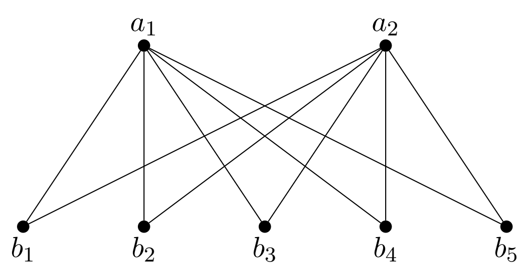

Figure 14: $K(2,5,0)$-subgraph.

The cover graph H with respect to $a_1, a_2, b_1, \dots, b_5$ has chromatic number at least four, by 3.3, and so has a clique of cardinality four, say b_1, b_2, b_3, b_4. For all distinct $i, j \in \{1,2,3,4,6,7\}$ let $c_{i,j} \in A \setminus \{a_1, a_2\}$ be adjacent to b_i, b_j, if there is such a vertex. Thus $c_{i,j}$ exists for $1 \leq i < j \leq 4$, and also for all $i \in \{1, \dots, 5\}$ and $j \in \{6, 7\}$, by 3.4.

(1) *Some vertex in $A \setminus \{a_1, a_2\}$ has three neighbours in $\{b_1, \dots, b_4\}$.*

Suppose not. Then each of the vertices $c_{i,j}$ $(1 \leq i < j \leq 4)$ has only two neighbours in $\{b_1, \dots, b_4\}$, and in particular they are all different. But then

$$\{\{b_1\}, \{c_{1,2}, b_2\}, \{c_{1,3}, c_{2,3}, b_3\}, \{c_{1,4}, c_{2,4}, c_{3,4}, b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. This proves (1).

Thus we may assume that some a_3 is adjacent to b_1, b_2, b_3.

(2) *Some vertex in $A \setminus \{a_1, a_2, a_3\}$ is adjacent to b_4 and to two of b_1, b_2, b_3.*

Suppose not; so $c_{1,4}, c_{2,4}, c_{3,4}$ are all different. Suppose that some vertex $c \neq a_1, a_2, a_3$ is adjacent to two of b_1, b_2, b_3, say to b_1, b_2. Then

$$\{\{b_1\}, \{c, b_2\}, \{a_3, b_3\}, \{c_{1,4}, c_{2,4}, c_{3,4}, b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. Thus there is no such c. There is a vertex $b_8 \in B \setminus \{b_1, \dots, b_7\}$ adjacent to a_3; and by 3.4, for $i = 1, 2, 3$ there exists $d_i \in A \setminus \{a_1, a_2, a_3\}$ adjacent to b_8, b_i. Since no $c \neq a_1, a_2, a_3$ is adjacent to two of b_1, b_2, b_3, it follows that d_1, d_2, d_3 are all different. Consequently $c_{i,4} \neq d_j$ for all distinct $i, j \in \{1, 2, 3\}$. But then

$$\{\{c_{1,4}, d_1, b_1\}, \{a_3, c_{2,4}, d_2, b_2\}, \{c_{3,4}, d_3, b_3, b_8\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. This proves (2).

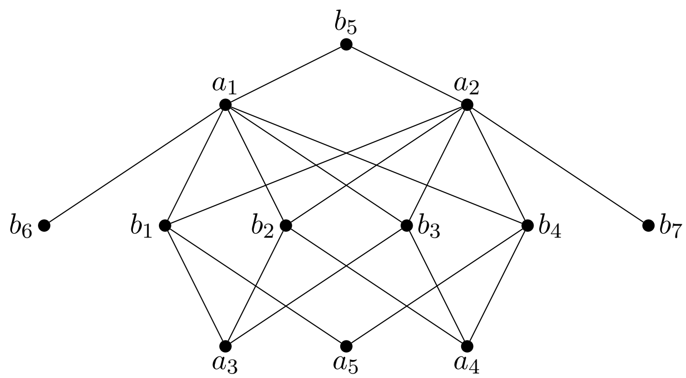

Figure 15: For the proof of 6.2.

Thus we may assume that $a_4 \in \setminus\{a_1, a_2, a_3\}$ is adjacent to b_2, b_3, b_4. Since $\{b_1, b_2, b_3, b_4\}$ is a clique of H, there exists $a_5 \in A \setminus \{a_1, a_2, a_3, a_4\}$ adjacent to b_1, b_4. (See figure 15.)

(3) *No vertex in A different from a_1, a_2, a_3, a_4 has two neighbours in $\{b_1, b_2, b_3\}$, or has two neighbours in $\{b_2, b_3, b_4\}$.*

Suppose that a_6 is such a vertex, adjacent to two of b_1, b_2, b_3 say. (Possibly $a_6 = a_5$.) If a_6 is adjacent to b_2, b_3, then $a_6 \neq a_5$, and

$$\{\{a_3, b_1\}, \{b_2\}, \{a_6, b_3\}, \{a_4, a_5, b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. Thus from the symmetry we may assume that a_6 is adjacent to b_1, b_2, and now possibly $a_6 = a_5$. Then

$$\{\{a_5, a_6, b_1\}, \{b_2\}, \{a_3, b_3\}, \{a_4, b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. This proves (3).

(4) *Every vertex in* $B \setminus \{b_1, \dots, b_7\}$ *adjacent to one of* a_3, a_4 *is adjacent to both* a_3, a_4.

Suppose that $b_8 \in B \setminus \{b_1, \dots, b_7\}$ is adjacent to b_3 and not to b_4. By 3.4, for $i = 2, 3$ there exists $c_{i,8} \in A \setminus \{a_3\}$ adjacent to b_i, b_8; $c_{i,8} \neq a_1, a_2, a_4$ since a_1, a_2, a_4 are not adjacent to b_8, and $c_{i,8} \neq a_5$ since a_5 is not adjacent to b_i by (3). But then

$$\{\{a_3, b_1\}, \{c_{2,8}, c_{3,8}, b_8, b_2\}, \{b_3\}, \{a_4, a_5, b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. This proves (4).

(5) *Each of* b_6, b_7 *is adjacent to at least one of* a_3, a_4.

Suppose that b_6 is nonadjacent to both a_3, a_4. Thus $c_{i,6} \neq a_4, a_5$ for $1 \leq i \leq 4$. But then

$$\{\{c_{1,6}, c_{4,6}, b_1, b_6\}, \{a_3, b_2\}, \{a_4, b_3\}, \{b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. This proves (5).

(6) *Each of* b_6, b_7 *is adjacent to both of* a_3, a_4.

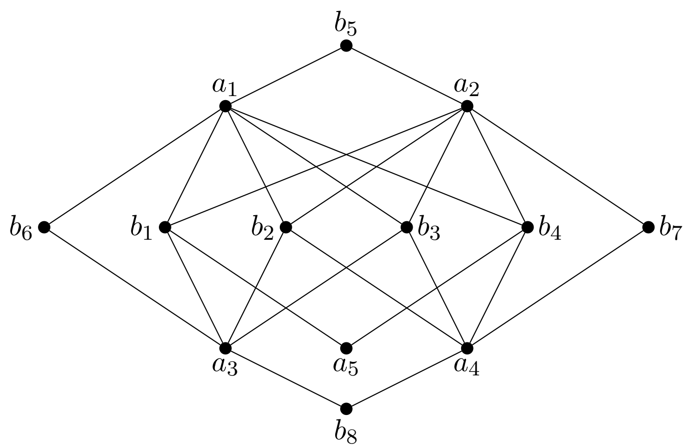

Figure 16: For the proof of 6.2, step (6).

Suppose that a_4, b_6 are nonadjacent. By (5), a_3b_6 is an edge. From (4), since a_3, a_4 both have degree six, and have the same number of neighbours in $B \setminus \{b_1, \dots, b_7\}$, it follows that they have the same number of neighbours in $\{b_6, b_7\}$; and so a_4 has at least one neighbour in $\{b_6, b_7\}$, and therefore a_4b_7 is an edge, and a_3, b_7 are not adjacent. Suppose that $c_{4,6}, c_{1,7}$ are different; then from the symmetry we may assume that $c_{1,7} \neq a_5$. Then

$$\{\{c_{1,7}, b_1\}, \{a_3, b_2, b_6\}, \{a_4, b_3, b_7\}, \{a_5, c_{4,6}, b_4\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. So $c_{1,7} = c_{4,6}$, and we may assume that they both equal a_5. Since a_5 has at most five neighbours in $\{b_1, \dots, b_7\}$, it has a neighbour b_8 different from $b_1, \dots, b_7$. Consequently b_8 is nonadjacent to a_1, a_2. Suppose that b_8 is nonadjacent to both a_3, a_4. By 3.4, for $i = 1, 4, 6, 7$ there exists $c_{i,8}$ adjacent to b_i, b_8, and different from a_5. Since b_8 is nonadjacent to a_1, a_2, a_3, a_4, it follows that $c_{i,8} \neq a_1, \ldots, a_5$. But then

$$\{\{a_5, b_1\}, \{a_3, b_2\}, \{a_4, b_3, b_7\}, \{c_{4,8}, c_{6,8}, b_4, b_6, b_8\}, \{a_1\}, \{a_2, b_5\}\}$$

is a 6-cluster. So b_8 is adjacent to one of a_3, a_4, and hence to both a_3, a_4 by (2). By 3.4, there is a vertex $a_6 \neq a_5$ adjacent to b_6, b_7; and so $a_6 \neq a_1, a_2, a_3, a_4$ since none of these four vertices is adjacent to both b_6, b_7. But then

$$\{\{a_1, b_1\}, \{a_3, b_2\}, \{a_4, b_3\}, \{a_5, b_4\}, \{a_2\}, \{a_6, b_6, b_7\}\}$$

is a 6-cluster. This proves (6).

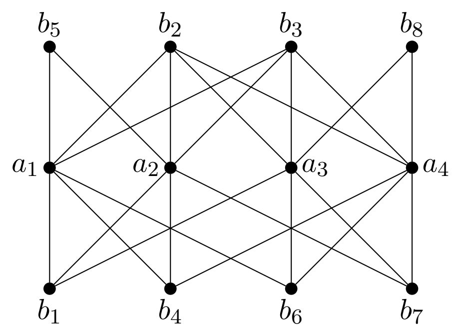

Figure 17: For the last part of the proof of 6.2. (a_5 is not drawn.)

From (6), there is a unique vertex $b_8 \in B \setminus \{b_1, \ldots, b_7\}$ adjacent to a_3, and it is adjacent to both of a_3, a_4 by (4). The subgraph induced on $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}$ (note that a_5 is not included) has some significant symmetry, which will help reduce the case analysis to come. There are symmetries that exchange

- b_2 with b_3;
- a_1 with a_2, and b_6 with b_7;
- a_3 with a_4, and b_1 with b_4;
- a_1 with a_3, a_2 with a_4, b_5 with b_8, b_6 with b_1, and b_7 with b_4.

Let us call these symmetries the first, second, third and fourth symmetries respectively. In the argument to come, we will avoid making use of a_5, in order to maintain these symmetries.

(7) *Let C_2 be the component of $G \setminus \{a_1, a_2, a_3, a_4\}$ that contains b_2; then it contains none of $b_1, \ldots, b_8$ except b_2.*

Suppose not. By 3.4, $c_{5,6}, c_{5,7}, c_{1,8}, c_{4,8}$ exist, and they are different from $a_1, \ldots, a_4$. Let $X =$

$\{c_{5,6}, c_{5,7}, b_5, b_6, b_7\}$ and $Y = \{c_{1,8}, c_{4,8}, b_1, b_4, b_8\}$. ($X, Y$ might not be disjoint.) Since C_1 contains one of $b_1, b_3, \ldots, b_8$, there is a a minimal path P of $G \setminus \{a_1, a_2, a_3, a_4\}$ with one end b_2 and the other end in $\{b_3\} \cup X \cup Y$. From the minimality of P it has no other neighbour in $\{b_3\} \cup X \cup Y$. Let z be the end of P different from b_2. If $z = b_3$ then

$$\{\{b_3\}, P^* \cup \{b_2\}, \{a_1\}, \{a_2, b_5\}, \{a_3, b_1\}, \{a_4, b_4\}\}$$

is a 6-cluster. Thus $z \in X \cup Y$, and from the fourth symmetry we may assume that $z \in X$, and from the second symmetry we may assume that $z \in \{a_1, b_5, b_6\}$. But then

$$\{P^* \cup X, \{a_1, b_4\}, \{a_2\}, \{a_3, b_1\}, \{b_2\}, \{a_4, b_3\}\}$$

is a 6-cluster. This proves (7).

Similarly, let C_3 be the component of $G \setminus \{a_1, a_2, a_3, a_4\}$ that contains b_3; then it contains none of $b_1, \ldots, b_8$ except b_3. Now $a_1, \ldots, a_4$ are the only vertices in $A \setminus V(C_i)$ that have a neighbour in $V(C_i)$, for $i = 2, 3$, and in particular, no vertex in $V(C_i) \cap A$ has a neighbour in $V(G) \setminus V(C_i)$. Since C_i is not a smaller candidate, it follows that $|A \cap V(C_i)| < |B \cap V(C_i)|$, for $i = 2, 3$. But there are at least six vertices in $B \setminus (V(C_2) \cup V(C_3))$ that have a neighbour in $\{a_1, a_3\}$, namely $b_1, b_4, b_5, b_6, b_7, b_8$; and similarly there are six such vertices that have a neighbour in $\{a_2, a_4\}$. Hence contracting $V(C_2) \cup \{a_1, a_3\}$ and $V(C_2) \cup \{a_2, a_4\}$ into A makes a smaller candidate, a contradiction. This proves 6.2. ∎

7 Excluding $K(4, 4, 2)$-subgraphs

7.1 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(4, 4, 2)$-subgraph.*

Proof. Suppose that $a_1, \ldots, a_4 \in A$ are all adjacent to each of $b_1, \ldots, b_4 \in B$ except the pairs a_1b_1, a_2b_2.

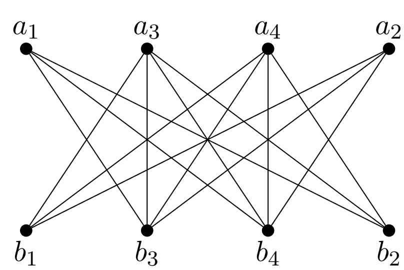

Figure 18: $K(4, 4, 2)$-subgraph.

For $i = 3, 4$, a_i has two neighbours, $d_{1,i}, d_{2,i}$ say, not in $\{b_1, \ldots, b_4\}$. Since there is no $K(2, 5, 0)$-subgraph by 6.2, none of $d_{1,3}, d_{2,3}, d_{1,4}, d_{2,4}$ is adjacent to both a_3, a_4, and so they are all distinct.

(1) *Each of $d_{1,3}, d_{2,3}, d_{1,4}, d_{2,4}$ has a neighbour in $\{a_1, a_2\}$.*

Suppose that $d_{1,3}$ say is nonadjacent to both a_1, a_2. By 3.4, for $1 \le i \le 4$ there is a vertex c_i different from a_3 that is adjacent to both $d_{1,3}, b_i$. But then

$$\{\{a_3\}, \{c_1, c_2, c_3, c_4, d_{1,3}\}, \{a_1, b_2\}, \{a_2, b_3\}, \{a_3, b_1\}, \{b_4\}\}$$

is a 6-cluster. This proves (1).

Not both $d_{1,3}, d_{2,3}$ are adjacent to a_1, since there is no $K(2,5,0)$-subgraph by 6.2, and similarly they are not both adjacent to a_2, so we may assume that $d_{1,3}$ is nonadjacent to a_2, and $d_{2,3}$ is nonadjacent to a_1. The same applies for $d_{1,4}, d_{2,4}$; so for each $i \in \{1,2\}$ and each $j \in \{3,4\}$, $d_{i,j}$ is adjacent to a_i, a_j and nonadjacent to the other two vertices in $\{a_1, \ldots, a_4\}$.

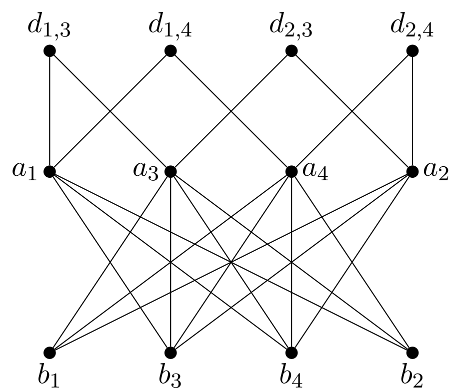

Figure 19: For the proof of 7.1, step (2).

(2) *There is a vertex b_5 not in $\{b_1, \ldots, b_4\}$ adjacent to a_1, a_2*

Suppose not. Let d be the neighbour of a_1 different from $d_{1,3}, d_{1,4}, b_2, b_3, b_4$. Thus d is adjacent to a_1 and to none of a_2, a_3, a_4. By 3.4, for $2 \le i \le 4$ there exists c_i different from a_1 that is adjacent to both d, b_i. Also by 3.4, there is a vertex f different from a_3 that is adjacent to both $b_2, d_{2,3}$. So c_2, c_3, c_4, f are all in $A \setminus \{a_1, \ldots, a_4\}$. But then

$$\{\{c_2, c_4, f, b_2\}, \{b_3\}, \{a_2, d_{2,3}\}, \{a_3, b_1\}, \{a_4, d_{2,4}\}, \{b_4\}\}$$

is a 6-cluster. This proves (2).

By 3.4, there is a vertex c_1 different from a_2 and adjacent to b_5, b_1; and so $c_1 \neq a_1, \ldots, a_4$. Similar there exists $c_2 \neq a_1, \ldots, a_4$ adjacent to b_5, b_2. By 3.4, there is a vertex $f_{1,3}$ different from a_3 that is adjacent to both $b_1, d_{1,3}$; and similarly there exists $f_{1,4}$ different from a_4 that is adjacent to both $b_1, d_{1,4}$; there exists $f_{2,3}$ different from a_3 that is adjacent to both $b_2, d_{2,3}$; and there exists $f_{2,4}$ different from a_4 that is adjacent to both $b_2, d_{1,4}$. Consequently none of $f_{1,3}, f_{1,4}, f_{2,3}, f_{2,4}$ belongs to $\{a_1, \ldots, a_4\}$. If they are all equal, equal to a_5 say, then a_5 is adjacent to $b_1, b_2, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}$; but then

$$\{\{c_1, c_2, b_1, b_2, b_5\}, \{a_5\}, \{a_1, d_{1,4}\}, \{a_2, d_{2,3}, b_3\}, \{a_3, d_{1,3}\}, \{a_4, d_{2,4}, b_4\}\}$$

is a 6-cluster. Thus they are not all equal, and so there exist $i, j \in \{3, 4\}$ such that $f_{1,i}$ is different from $f_{2,j}$.

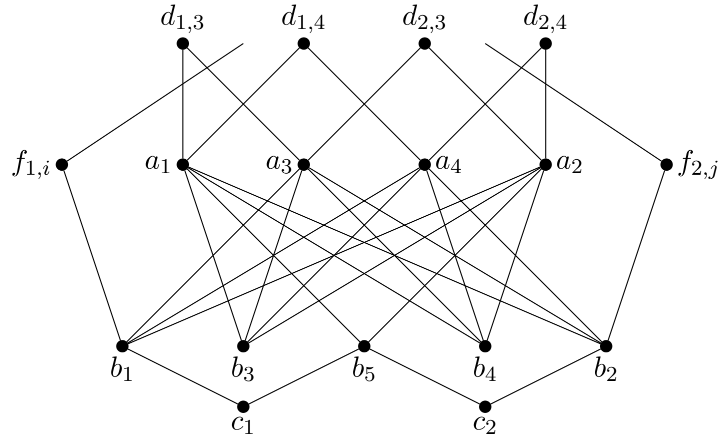

Figure 20: For the last part of the proof of 7.1. $f_{1,i}$ is adjacent to $d_{1,i}$ for some $i \in \{3, 4\}$, and similarly for $f_{2,j}$. The vertices c_1, c_2 might be equal, but $f_{1,i}$ is different from $f_{2,j}$.

(3) *There are two disjoint subsets X, Y of $\{b_1, f_{1,i}, b_2, f_{2,j}, c_1, c_2, b_5\}$, both inducing connected subgraphs, with $b_1, f_{1,3} \in X$ and $b_2, f_{2,j} \in Y$, such that there is an edge between X, Y.*

If $f_{1,i}$ is adjacent to b_2 we may take $X = \{b_1, f_{1,i}\}$ and $Y = \{b_2, f_{2,j}\}$, so we assume that $f_{1,i}$ is nonadjacent to b_2, and similarly $f_{2,j}$ is nonadjacent to b_1. It follows that $f_{1,i} \neq c_2$ and $f_{2,j} \neq c_1$. So the only possible equalities between two of $f_{1,i}, f_{2,j}, c_1, c_2$ are $f_{1,i} = c_1$, $f_{2,j} = c_2$, and $c_1 = c_2$. If $f_{1,i} = c_1$, then c_1 is different from $f_{2,j}, c_2$ and we may set $X = \{b_1, c_1, b_5\}$ and $Y = \{b_2, f_{2,j}, c_2\}$, so we assume $f_{1,i} \neq c_1$, and similarly $f_{2,j} \neq c_2$. But then we may set $X = \{b_1, f_{1,i}\}$ and $Y = \{b_5, c_1, c_2, b_2, f_{2,j}\}$. This proves (3).

But then

$$\{X, Y, \{a_1, d_{1,3}, d_{1,4}\}, \{a_2, b_4, d_{2,3}, d_{2,4}, b_4\}, \{a_3, b_3\}, \{a_4\}\}$$

is a 6-cluster. This proves 7.1. ∎

8 Excluding $K(2, 4, 0)$-subgraphs

We begin with:

8.1 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(3, 4, 1)$-subgraph.*

Proof. Suppose that $a_1, a_2, a_3 \in A$ are all adjacent to each of $b_1, \ldots, b_4 \in B$ except $a_1 b_1$. We observe:

(1) *No vertex in $A \setminus \{a_1, a_2, a_3\}$ is adjacent to b_1 and to two of b_2, b_3, b_4. No vertex in $B \setminus \{b_1, \ldots, b_4\}$ is adjacent to both a_2, a_3. At most one vertex in $B \setminus \{b_1, \ldots, b_4\}$ is adjacent to both a_1, a_2, and at*

most one is adjacent to a_1, a_3,

The first is because there is no $K(4,4,2)$-subgraph by 7.1, and the other three because there is no $K(2,5,0)$-subgraph by 6.2. This proves (1).

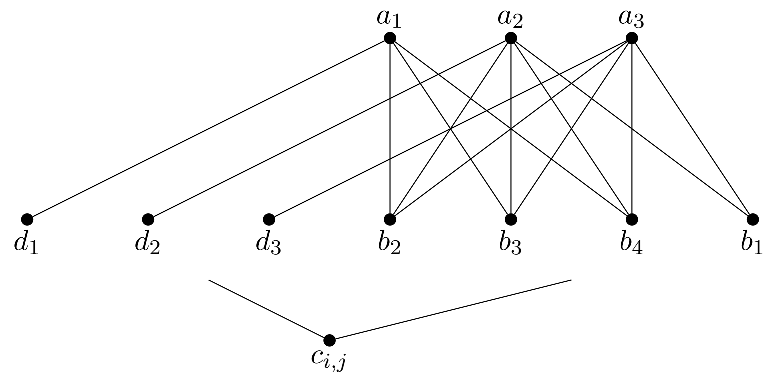

Figure 21: The start of the proof of 8.1. $c_{i,j}$ is adjacent to d_i and b_j.

Let B_0 be the set of vertices that have exactly one neighbour in $\{a_1, a_2, a_3\}$. Thus $|B_0|$ is odd. From (1) it follows that each of a_1, a_2, a_3 has a neighbour in B_0; let us call these neighbours d_1, d_2, d_3 respectively. For $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ (except when $(i, j) = (1, 1)$), by 3.4 there is a vertex $c_{i,j}$ different from a_i and adjacent to d_i, b_j. Hence $c_{i,j} \neq a_1, a_2, a_3$. For $i = 2, 3$ choose the set $\{c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}\}$ minimal, and choose the set $\{c_{1,2}, c_{1,3}, c_{1,4}\}$ minimal.

(2) *Some vertex* $a_4 \in A \setminus \{a_1, a_2, a_3\}$ *has more than one neighbour in* $\{b_2, b_3, b_4\}$.

Suppose not. Thus $c_{2,2}, c_{2,3}, c_{2,4}$ are all different. We chose $\{c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}\}$ minimal; so we may assume that either

- $c_{2,1}, c_{2,2}, c_{2,3}, c_{4,4}$ are all distinct, and each has only one neighbour in $\{b_1, \dots, b_4\}$; or
- $c_{2,1} = c_{2,2}$. In this case $c_{2,1}$ is nonadjacent to b_3, b_4 by (1).

In both cases neither of $c_{2,1}, c_{2,2}$ have a neighbour in $\{b_3, b_4\}$, and so $\{c_{1,3}, c_{1,4}\}$ is disjoint from $\{c_{2,1}, c_{2,2}\}$. But then

$$\{\{c_{2,1}, c_{2,2}, d_2\}, \{a_2\}, \{c_{1,3}, c_{2,3}, d_1, b_3\}, \{c_{1,4}, c_{2,4}, b_4\}, \{a_3, b_1\}, \{a_1, b_2\}\}$$

is a 6-cluster. This proves (2).

We assume that a_4 is adjacent to b_2, b_3 (and possibly to b_4, but not to b_1, by (1)).

(3) a_4 *is adjacent to* d_2, d_3.

Suppose that a_4, d_2 are nonadjacent, say. Then

$$\{\{c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}, d_2\}, \{a_2\}, \{b_2\}, \{a_4, b_4\}, \{a_3, b_1\}, \{a_1, b_4\}\}$$

is a 6-cluster. This proves (3).

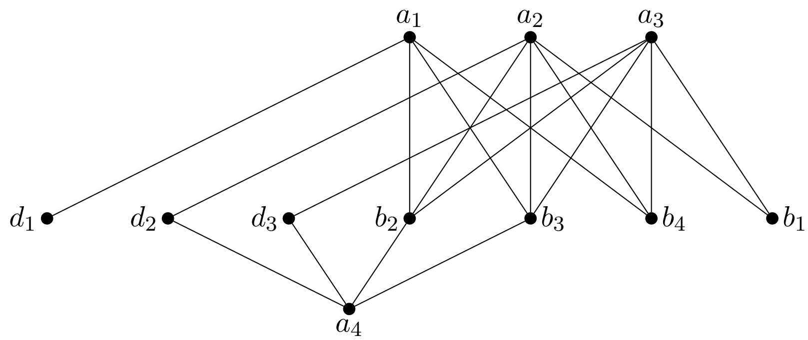

Figure 22: For the proof of 8.1, step (4).

(4) *If a_4 is nonadjacent to b_4, then no vertex in $A \setminus \{a_1, \ldots, a_4\}$ has a neighbour in $\{b_2, b_3\}$ and a neighbour in $\{d_2, d_3\}$.*

Suppose that some $a_5 \in A \setminus \{a_1, \ldots, a_4\}$ is adjacent to b_2, d_2 say. By (1), $c_{2,1} \neq a_4$, and since a_4, b_4 are nonadjacent, $c_{2,4} \neq a_4$. Hence

$$\{\{a_2\}, \{c_{2,1}, c_{2,4}, a_5, d_2\}, \{a_1, b_4\}, \{a_3, b_1\}, \{b_2\}, \{a_4, b_3\}\}$$

is a 6-cluster. This proves (4).

(5) *a_4 is adjacent to d_1 and hence to every vertex in B_0.*

Suppose that a_4 is nonadjacent to d_1. Thus $c_{1,2}, c_{1,3}, c_{1,4} \neq a_4$. If a_4 is adjacent to b_4, then $c_{2,4} = a_4$ by (3) (with b_3, b_4 exchanged), and at most one of $c_{1,2}, c_{1,3}, c_{1,4} = c_{2,1}$ by (1), and so we may assume that $\{c_{2,1}, c_{2,4}\}$ is disjoint from $\{c_{1,2}, c_{1,3}\}$. If a_4 is nonadjacent to b_4, then by (3), neither of $c_{2,1}, c_{2,4}$ has a neighbour in $\{b_2, b_3\}$, and so again $c_{2,1}, c_{2,4} \neq c_{1,2}, c_{1,3}$. In either case

$$\{\{b_2\}, \{b_3, c_{1,2}, c_{1,3}, d_1\}, \{d_2, c_{2,1}, c_{2,4}, a_4\}, \{a_1, b_4\}, \{a_2\}, \{a_3, b_1\}\}$$

is a 6-cluster. Hence a_4 is adjacent to each of d_1, d_2, d_3, and hence to every vertex in B_0. This proves (5).

(6) *a_4 is adjacent to b_4.*

Suppose not. Since a_4 has degree six and is nonadjacent to b_1 by (1), it has a neighbour $b_5 \neq b_1, \ldots, b_4, d_1, d_2, d_3$. By 3.4, for each $v \in \{b_2, b_3, d_1, d_2, d_3\}$ there is a vertex $c(v)$ different from a_4 adjacent to b_5, v; and hence $c(v) \neq a_1, \ldots, a_4$. But then

$$\{\{a_4\}, \{b_5, c(b_2), c(b_3), c(d_1), c(d_2), c(d_3)\}, \{a_1, d_1, b_4\}, \{a_2, b_1, d_2\}, \{a_3, d_3\}, \{b_2\}\}$$

is a 6-cluster. This proves (6).

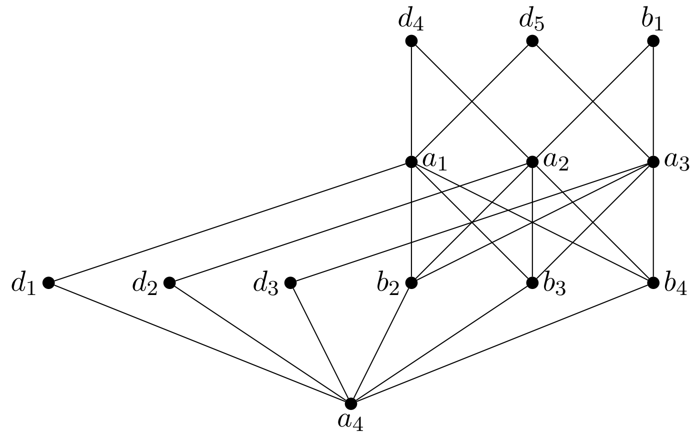

Figure 23: For the last part of the proof of 8.1.

Since a_4 has degree only six, it follows from (5) that $|B_0| \leq 4$. Since $|B_0|$ is odd, it follows that $B_0 = \{d_1, d_2, d_3\}$, and there are two vertices $d_4, d_5 \in B \setminus \{b_1, \ldots, b_4\}$ that have two neighbours in $\{a_1, a_2, a_3\}$. By (1) we may assume that d_4, d_5 are adjacent to a_1, a_2 and to a_1, a_3 respectively.

We do not obtain a smaller candidate by contracting $\{d_4, a_1, b_2\}, \{d_5, a_3, b_3\}, \{b_1, a_2, b_4\}$ into B; and so there is a vertex $a_5 \neq a_1, \ldots, a_4$, adjacent either to both b_2, d_4, or to both b_3, b_5, or to both b_1, b_4. There is symmetry between b_1, d_4, d_5 (see figure 23), so we may assume that a_5 is adjacent to b_1, b_4. By 3.4, there is a vertex $c(d_2)$ different from a_2 adjacent to both b_1, d_2; a vertex $c(d_3)$ different from a_3 adjacent to both b_1, d_3; and a vertex $c(d_5)$ different from a_3 adjacent to both b_1, d_5. It follows that $c(d_2), c(d_3) \neq a_1, \ldots, a_4$. But then

$$\{\{b_1, a_5, c(d_2), c(d_3), c(d_5), d_2\}, \{a_2, b_3\}, \{a_1, d_5\}, \{a_3, d_3\}, \{a_4, b_2\}, \{b_4\}\}$$

is a 6-cluster. This proves 8.1. ∎

8.2 *Let G be a minimal candidate with bipartition (A, B). Then G has no $K(2, 4, 0)$-subgraph.*

Proof. Suppose that $a_1, a_2 \in A$ are both adjacent to each of $b_1, \ldots, b_4 \in B$. No other vertex is adjacent to both a_1, a_2, since there is no $K(2, 5, 0)$-subgraph by 6.2. Let b_5, b_6, b_7, b_8 be the vertices in B that have exactly one neighbour in $\{a_1, a_2\}$, where b_5, b_6 are adjacent to a_1, and b_7, b_8 to a_2.

No other vertex is adjacent to more than two of of $b_1, \ldots, b_4$, since there is no $K(3, 4, 0)$- or $K(3, 4, 1)$-subgraph by 5.2 and 8.1. Let H be the cover graph with respect to $a_1, a_2, b_1, b_2, b_3, b_4$; then $\chi(H) \geq 3$ by 3.3, and so H has a triangle, say with vertices b_1, b_2, b_3. Consequently there are three vertices c_1, c_2, c_3, such that c_i is adjacent to the two vertices in $\{b_1, b_2, b_3\} \setminus \{b_i\}$, and c_i is nonadjacent to b_i.

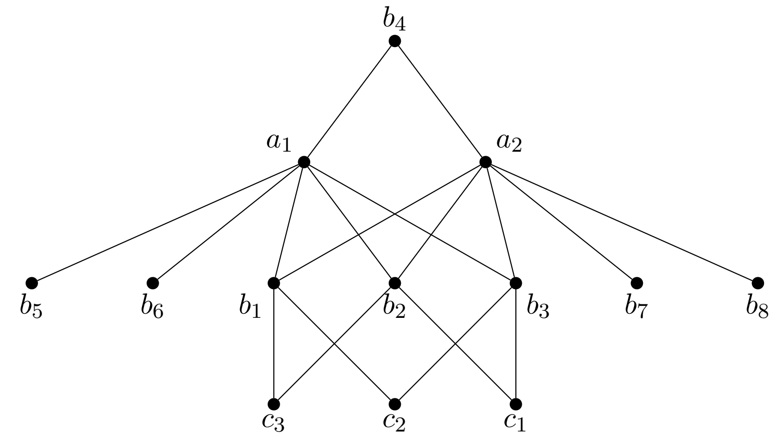

Figure 24: For the proof of 8.2.

Let $Z = \{a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, c_3\}$, and let $\mathcal{C}$ be the set of components of $G \setminus Z$ that contain at least one of b_5, b_6, b_7, b_8. For each $C \in \mathcal{C}$, let $N(C)$ be the set of vertices in $\{b_1, b_2, b_3, c_1, c_2, c_3\}$ that have a neighbour in $V(C)$.

(1) *If $C \in \mathcal{C}$, then b_4 has a neighbour in $V(C)$, and either*

- *$N(C) = \{b_i, c_i\}$ for some $i \in \{1, 2, 3\}$; or*
- *$N(C) = \{c_i, c_j\}$ for some two distinct $i, j \in \{1, 2, 3\}$.*

Let $b_5 \in V(C)$ say. By 3.4, for $i = 1, 2, 3, 4$ there is a vertex different from a_1 adjacent to both b_5, b_i. Since none of c_1, c_2, c_3 is adjacent to b_4, it follows that b_4 has a neighbour in $V(C)$. Also, for $1 \leq i \leq 3$, either $b_i \in N(C)$ or $N(C)$ contains a vertex in $\{c_1, c_2, c_3\} \setminus \{c_i\}$. In summary, $N(C)$ contains a member of each of the sets $\{b_1, c_2, c_3\}$, $\{b_2, c_3, c_1\}$, $\{b_3, c_1, c_2\}$. Consequently, if $|N(C)| = 2$ then the claim holds, so we assume that $|N(C)| \geq 3$. Since

$$\{\{c_3, b_1\}, \{c_2, b_3\}, \{c_1, b_2\}, \{a_1\}, \{a_2, b_4\}, V(C)\}$$

is not a 6-cluster, it follows that $N(C)$ is disjoint from one of the sets $\{c_3, b_1\}, \{c_2, b_3\}, \{c_1, b_2\}$, and we may assume from the symmetry that $b_1, c_3 \notin N(C)$. Since $N(C)$ contains a member of $\{b_1, c_2, c_3\}$, it follows that $c_2 \in N(C)$. By a similar argument, $N(C)$ is disjoint from one of $\{c_3, b_2\}, \{c_2, b_1\}, \{c_1, b_3\}$, and hence from one of $\{c_3, b_2\}, \{c_1, b_3\}$. Since $|N(C)| \geq 3$, $N(C) \cap \{c_1, b_3\} \neq \emptyset$; and hence $b_2 \notin N(C)$. Since $|N(C)| \geq 3$ it follows that $c_1, b_3 \in N(C)$. But then

$$\{\{c_2, c_3, b_1\}, \{c_1, b_2\}, \{b_3\}, \{a_1\}, \{a_2, b_4\}, V(C)\}$$

is a 6-cluster. This proves (1).

(2) *If $C \in \mathcal{C}$, then every vertex of $V(C) \cap \{b_5, b_6, b_7, b_8\}$ is adjacent to every vertex of $N(C) \cap \{c_1, c_2, c_3\}$.*

We may assume that $b_5 \in V(C)$ and $c_3 \in N(C)$, and we must show that b_5, c_3 are adjacent. It follows from (1) that $b_1, b_2 \notin N(C)$. But for $i = 1, 2$ there is a vertex different from a_1 adjacent to both b_5, b_i, and this vertex is one of c_1, c_2, c_3 since $b_1, b_2 \notin N(C)$. Thus b_5 is adjacent to one of c_1, c_3, and also to one of c_2, c_3. But b_5 is not adjacent to both c_1, c_2 since not all $c_1, c_2, c_3 \in N(C)$; and so b_5 is adjacent to c_3. This proves (2).

(3) *Let $C \in \mathcal{C}$ contain one of b_5, b_6, b_7, b_8; then $N(C) \cap \{b_1, b_2, b_3\} = \emptyset$.*

Suppose that $b_1, b_5 \in V(C)$ say. Thus $N(C)$ contains c_1 and none of b_2, b_3, c_2, c_3. Let $C' \in \mathcal{C}$ contain b_6. We claim that b_6 is also adjacent to c_1; because suppose not. Then $b_6 \notin V(C)$, by (2), and so $C' \neq C$. Since b_6 is nonadjacent to c_1, (2) implies that $c_1 \notin N(C')$. Since b_5, b_6 are both adjacent to a_1, they have a second common neighbour c say. But $c \notin \{c_1, c_2, c_3\}$, since c_1 is not adjacent to b_6, and $c_2, c_3 \notin N(C)$ and so are not adjacent to b_5. Also $c \neq a_1, a_2$; so $c \notin Z$, contradicting that $C \neq C'$ are components of $G \setminus Z$. This proves that b_6 is adjacent to c_1.

We claim also that $c_2, c_3 \notin N(C')$. There is a symmetry exchanging c_2, c_3 and fixing c_1, so we suppose without loss of generality that $c_2 \in N(C')$, and hence $C \neq C'$. But then

$$\{\{V(C') \cup \{c_2\}, V(C) \cup \{c_1, b_3\}, \{b_1\}, \{b_2, c_3\}, \{a_1\}, \{a_2, b_4\}\}$$

is a 6-cluster. This proves that $c_2, c_3 \notin N(C')$, and hence $N(C') = N(C)$.

Now a_1, c_1 have four common neighbours, namely b_2, b_3, b_5, b_6; and so they satisfy the same conditions as a_1, a_2. At the start of this proof, we showed the existence of c_1, c_2, c_3, that with three of b_1, b_2, b_3 induce a 6-cycle. Consequently the same is true for b_2, b_3, b_5, b_6, and in particular there are two vertices c_4, c_5, different from a_1, c_1, that have a neighbour in $\{b_2, b_3\}$. Since $c_2, c_3 \notin N(C) = N(C')$, it follows that $c_4, c_5 \notin Z$, and hence at least one of b_2, b_3 belongs to $N(C) = N(C')$, contrary to (1). This proves (3).

For $i = 5, 6, 7, 8$, let $C_i \in \mathcal{C}$ contain b_i (they are not necessarily all different). From (1), (2) and (3) it follows that each of b_5, b_6, b_7, b_8 is adjacent to two of c_1, c_2, c_3. They are not all adjacent to the same two of c_1, c_2, c_3, since there is no $K(2, 5, 0)$-subgraph by 6.2; so there exists $C \in \{C_5, C_6\}$ and $C' \in \{C_7, C_8\}$ with $N(C) \neq N(C')$. Thus we may assume that $c_2, c_3 \in N(C_5)$ and $c_1, c_2 \in N(C_7)$. In particular $C_5 \neq C_7$; and by (2), b_5, b_7 are adjacent to c_2. By 3.4 they have another common neighbour, say c. Thus $c \notin \{a_1, a_2\}$ because b_5, b_7 each have only one and different neighbours in that set; $c \notin \{c_1, c_2, c_3\}$, since $c \neq c_2$ from its definition, and $c_1 \notin N(C_5)$, and $c_3 \notin N(C_7)$; and $c \notin V(G) \setminus Z$ since $C \neq C'$, a contradiction. This proves 8.2. ∎

9 The end

Next we need the following lemma. Let H be a complete graph with six vertices. If C is a triangle of H, a *C-path* means a path of H with vertex set C. Let $C_1, \ldots, C_k$ be triangles of H, not necessarily all different. We denote by $M(C_1, \ldots, C_k)$ the graph with vertex set $V(H)$ and edge set the set of all edges uv of H such that $\{u, v\}$ is not a subset of any of $C_1, \ldots, C_k$. Let J be the graph obtained from a six-vertex complete graph by deleting four edges, the edges of two disjoint three-vertex paths. We observe that J admits a 5-cluster (one of the five sets contains the two vertices of degree three, the others are singletons).

9.1 *Let $H, C_1, \ldots, C_k$ be as above. Then for $1 \le i \le k$ there is a C_i-path P_i, such that*

$$M(C_1, \ldots, C_k) \cup P_1 \cup \cdots \cup P_k$$

has a subgraph isomorphic to J.

Proof. We proceed by induction on k. Suppose first that $u, v \in V(H)$ and two of $C_1, \ldots, C_k$ contain u, v, say C_1, C_2. Let $C_1 = \{u, v, w\}$ say. Let P_1 be the path u-w-v. From the inductive hypothesis, for $2 \le i \le k$ there is a C_i-path P_i such that $M(C_2, \ldots, C_k) \cup P_2 \cup \cdots \cup P_k$ admits a 5-cluster. But every edge of $M(C_2, \ldots, C_k)$ that is not an edge of $M(C_1, \ldots, C_k)$ is one of uw, vw, and they are edges of P_1; so $M(C_1, \ldots, C_k) \cup P_1 \cup \cdots \cup P_k$ contains a copy of J as required.

So we may assume that no two of $C_1, \ldots, C_k$ share more than one vertex. Every subgraph of H obtained by deleting at most two edges contains a copy of J, so we may assume that $k \ge 3$ and hence no two of $C_1, \ldots, C_k$ are disjoint (because if $C_1 \cap C_2 = \emptyset$ then C_3 shares two vertices with one of them). Let $V(H) = \{h_1, \ldots, h_6\}$; then we may assume that $C_1 = \{h_1, h_2, h_3\}$, $C_2 = \{h_1, h_4, h_5\}$, $C_3 = \{h_2, h_4, h_6\}$ and either $k = 3$, or $k = 4$ and $C_4 = \{h_3, h_5, h_6\}$. Define $P_1 = h_2$-h_1-h_3, $P_2 = h_1$-h_5-h_4, $P_3 = h_4$-h_2-h_6, and if $k = 4$, define $P_4 = h_3$-h_6-h_5. If $k = 4$, $M(C_1, \ldots, C_k) \cup P_1 \cup \cdots \cup P_k$ is isomorphic to J, and if $k = 3$, $M(C_1, \ldots, C_k) \cup P_1 \cup \cdots \cup P_k$ contains a copy of J. This proves 9.1. ∎

We deduce 1.7, which we restate as follows:

9.2 *No graph is a candidate.*

Proof. Assume some graph is a candidate, and let G be a minimal candidate, with bipartition (A, B). Let $a \in A$, and let its neighbours be $b_1, \ldots, b_6$. By 3.3 (or by 3.4), the cover graph H with respect to $a, b_1, \ldots, b_6$ is complete. No vertex different from a has more than three neighbours in $\{b_1, \ldots, b_6\}$, since there is no $K(2,6,0)$-, $K(2,5,0)$- or $K(2,4,0)$-subgraph, by 6.1, 6.2 and 8.2. Consequently there is a set of vertices $a_1, \ldots, a_\ell \in A \setminus \{a\}$, each with two or three neighbours in $\{b_1, \ldots, b_6\}$, such that for all distinct $u, v \in \{b_1, \ldots, b_6\}$, there exists $i \in \{1, \ldots, \ell\}$ such that a_i is adjacent to u, v. We may assume that $a_1, \ldots, a_k$ have three neighbours in $\{b_1, \ldots, b_6\}$ and $a_{k+1}, \ldots, a_\ell$ have two. For $1 \le i \le k$ let C_i be the set of three neighbours of a_i in $\{b_1, \ldots, b_6\}$. By 9.1, for $1 \le i \le k$ there is a C_i-path P_i such that $M(C_1, \ldots, C_k) \cup P_1 \cup \cdots \cup P_k$ contains a copy of J as a subgraph, and hence admits a 5-cluster $\{X_1, \ldots, X_5\}$ say. Let p_i be the middle vertex of P_i for $1 \le i \le k$, and let p_i be one of the two members of C_i for $k+1 \le i \le \ell$.

For each $v \in \{b_1, \ldots, b_6\}$ let $C(v)$ be the union of $\{v\}$ and all the vertices c_i with $1 \le i \le \ell$ such that $p_i = v$. For $1 \le j \le 5$ let Y_j be the union of the sets $C(v)$ over all $v \in X_j$. We claim that $\{\{a\}, Y_1, \ldots, Y_5\}$ is a 6-cluster. To see this, we must check that these six sets are pairwise disjoint subsets of $V(G)$, which is clear; that $Y_1, \ldots, Y_5$ each contain a neighbour of a, which is true since $X_1, \ldots, X_5$ are nonempty subsets of $\{b_1, \ldots, b_6\}$; that each of the sets Y_i induces a connected subgraph of G; and that for $1 \le i < i' \le 5$, some vertex in Y_i has a neighbour in $Y_{i'}$. To see both these final statements, it suffices to show that if $u, v \in \{b_1, \ldots, b_6\}$ are adjacent in $M(C_1, \ldots, C_k) \cup P_1 \cup \cdots \cup P_k$, there is an edge of G between $C(u), C(v)$. To see this, there are two cases: $uv \in E(M(C_1, \ldots, C_k))$, and $uv \in E(P_i)$ for some $i \in \{1, \ldots, k\}$. In the first case, there exists i with $k+1 \le i \le \ell$ such that c_i is adjacent to both u, v, and since p_i is one of u, v, the claim holds. In the second case, let $uv \in E(P_i)$ for some $i \in \{1, \ldots, k\}$; then one of u, v equals p_i, say v; the other, u, is an end of P_i; and in G there is an edge between c_i and u. This proves 9.2. ∎

References

[1] J. Barát, "List Hadwiger conjecture and extremal K_5-minor-free graphs with fixed girth", lecture in 2013, https://kam.mff.cuni.cz/workshops/mcw/work19/barat.pdf.

[2] J. Barát, "On the number of edges in a K_5-minor-free graph of given girth", manuscript dated 22 May 2020.

[3] L. Jørgensen, "Contractions to K_8", *J. Graph Theory* **18** (1994), 431–448.

[4] W. Mader, "Homomorphiesätze für Graphen", *Math. Ann.* **178** (1968), 154–168.

[5] N. Robertson, P. Seymour and R. Thomas, "Hadwiger's conjecture for K_6-free graphs", *Combinatorica* **13** (1993), 279–361.

[6] K. Wagner, "Über eine Eigenschaft der ebenen Komplexe", *Math. Ann.* **114** (1937), 570–590.