Tight bounds for hypercube minor-universality

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Abstract

Benjamini, Kalifa and Tzalik recently proved that there is an absolute constant $c > 0$ such that any graph with at most $c \cdot 2^d/d$ edges and no isolated vertices is a minor of the d-dimensional hypercube Q_d, while there is an absolute constant $K > 0$ such that Q_d is not $(K \cdot 2^d/\sqrt{d})$-minor-universal. We show that Q_d does not contain 3-uniform expander graphs with $C \cdot 2^d/d$ edges as minors. This matches the lower bound up to a constant factor and answers one of their questions.

1 Introduction

A (finite or infinite) graph is universal for a class $\mathcal{G}$ of graphs if it contains every graph from $\mathcal{G}$. Different instances of graph universality arise by specifying the notion of containment and class of graphs. In the infinite case, the question is often whether a universal graph exists; in the finite case, there is considerable work on finding small universal graphs, and such results are significant in extremal graph theory, structural graph theory, and various algorithmic problems.

Results on universal graphs date back to the early 1960s, when Rado [14] proved that there exists a countable graph that contains every countable graph as an induced subgraph. There is now substantial literature with results on many graph classes with respect to subgraph or induced subgraph universality [1, 2, 3, 4, 7, 8, 11, 12]. There are also interesting negative results: answering a question of Ulam, a classical theorem of Pach [13] from 1981 states that there does not exist a countable planar graph that contains every countable planar graph as a subgraph.

A different picture emerges for minor containment. For graphs H and G, we say that H is a *minor* of G if H can be obtained from G by a finite sequence of edge and vertex

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deletions, and edge contractions. In this case, the analogue of Pach's theorem does not hold: in 1999, Diestel and Kühn [9] constructed a countable planar graph that contains every countable planar graph as a minor. The investigation of minor-universality has been fruitful (see [10] for some recent developments). A notable example is the theorem of Robertson, Seymour and Thomas [15] that the $2n \times 2n$ grid is minor-universal for planar graphs on n vertices, which has important consequences for bounding treewidth.

In this paper, we are concerned with minors of hypercubes. Hypercubes are a natural candidate graph in which to embed other graphs for various computational reasons (see, for example, [16], which studies universality of hypercubes containing k-ary trees as subgraphs). Very recently, Benjamini, Kalifa and Tzalik [5] investigated the minor-universality of hypercubes. They define a graph G to be m-minor-universal if every graph H with at most m edges and no isolated vertices is contained as a minor in G. Let Q_d denote the d-dimensional hypercube whose vertices are binary strings of length d where two vertices are adjacent whenever their Hamming distance is one. Benjamini, Kalifa and Tzalik's main result is the following.

Theorem 1.1 (Theorem A in [5]). *The hypercube Q_d is $\Omega(\frac{2^d}{d})$-minor-universal. Moreover, there is an absolute constant $K > 0$ such that Q_d is not $\frac{K \cdot 2^d}{\sqrt{d}}$-minor-universal.*

The upper and lower bounds differ by a factor of $\sqrt{d}$. Benjamini, Kalifa and Tzalik went on to ask which (if either) of these two bounds is tight [5, Question 1]. We resolve this question by exhibiting a graph with $C \cdot 2^d/d$ edges which is not a minor of Q_d, which shows that the lower bound is in fact tight. For completeness, we also give a short version of the lower-bound proof from [5].

2   Lower bound

We present a substantially shorter version of the lower-bound argument from [5]. Given a permutation σ of $X = [n_1] \times \cdots \times [n_d]$, we say that σ is *one-dimensional in direction* i if, for all $x \in X$, we have that $\sigma(x)$ and x may differ only in coordinate i. We begin with a short proof of a key lemma, stating that permutations of a d-dimensional grid can be decomposed into a linear number of one-dimensional permutations.

Lemma 2.1. *Let $X = [n_1] \times \cdots \times [n_d]$ and let σ be a permutation of X. Then σ can be written as*

$$\sigma = \sigma_{2d-1} \circ \cdots \circ \sigma_1,$$

where each σ_i is a one-dimensional permutation of X in direction $j_i = |d - i| + 1$.

Proof. We first sketch the idea. Thinking of the coordinate d direction as 'vertical', and all other directions as 'horizontal', we can view X as either $n_1 \cdots n_{d-1}$ one-dimensional columns, or n_d horizontal layers. We construct the one-dimensional permutations in three steps: first, choose σ_1 to permute the entries in each column so that each layer

consists of vectors whose images under σ are in different columns of X; then, inductively find permutations $\sigma_2, \ldots, \sigma_{2d-2}$ that rearrange the vectors in each layer so that every vector lies in the correct column; finally, choose σ_{2d-1} to permute each column so that every vector is in the correct layer.

We proceed by induction on d. Clearly, the statement holds for $d = 1$, so suppose that $d \geqslant 2$ and that the statement holds for $d - 1$. Let $Y = [n_1] \times \cdots \times [n_{d-1}]$, and denote by $\pi_d(x) = (x_1, \ldots, x_{d-1})$ the projection that forgets coordinate d. We construct a bipartite multigraph G with bipartition $V(G) = U \cup V$ where U and V are both copies of Y, and for each $x \in X$ we add an edge e_x from $\pi_d(x) \in U$ to $\pi_d(\sigma(x)) \in V$. Note that G is n_d-regular. Therefore, by repeatedly applying Hall's theorem we can decompose G into n_d perfect matchings $M_1, \ldots, M_{n_d}$. For $x \in X$, let $\ell_x \in [n_d]$ be such that $e_x \in M_{\ell_x}$.

Define $\sigma_1 \colon X \to X$ by $\sigma_1(x) = (\pi_d(x), \ell_x)$. Observe that σ_1 is one-dimensional in direction d. Also, this is a permutation because if $x, y \in X$ are distinct and satisfy $\pi_d(x) = \pi_d(y)$, then e_x and e_y are incident in G, so they cannot both belong to the same matching. This implies that $\ell_x \neq \ell_y$ and so $\sigma_1(x) \neq \sigma_1(y)$. Moreover, if $x, y \in X$ are distinct and satisfy $\pi_d(\sigma(x)) = \pi_d(\sigma(y))$, then e_x and e_y are again incident in G and so $\ell_x \neq \ell_y$. Thus, for any $\ell \in [n_d]$ and any distinct $x, y \in X_\ell = \{x \in X : \ell_x = \ell\}$ we have $\pi_d(x) \neq \pi_d(y)$ and $\pi_d(\sigma(x)) \neq \pi_d(\sigma(y))$. This implies that $\tau_\ell \colon Y \to Y$ defined by $\tau_\ell(\pi_d(x)) = \pi_d(\sigma(x))$ for $x \in X_\ell$ is a permutation of Y. By induction, it follows that τ_ℓ can be written as $\tau_{\ell,2d-2} \circ \cdots \circ \tau_{\ell,2}$ where each $\tau_{\ell,i}$ is one-dimensional in direction j_i.

For any $2 \leqslant i \leqslant 2d - 2$, let $\sigma_i(y, \ell) = (\tau_{\ell,i}(y), \ell)$ which is one-dimensional in direction j_i, and let $\sigma' = \sigma_{2d-2} \circ \cdots \circ \sigma_2$. Note that $\sigma'(y, \ell) = (\tau_\ell(y), \ell)$ which implies that $\sigma'(\sigma_1(x)) = \sigma'(\pi_d(x), \ell_x) = (\tau_{\ell_x}(\pi_d(x)), \ell_x) = (\pi_d(\sigma(x)), \ell_x)$ for all $x \in X$. Now define $\sigma_{2d-1} \colon X \to X$ so that $\sigma_{2d-1}(\pi_d(\sigma(x)), \ell_x) = \sigma(x)$. Then $\sigma = \sigma_{2d-1} \circ \sigma' \circ \sigma_1$ and σ_{2d-1} is one-dimensional in direction d, as required. $\qquad\square$

Given two graphs F_1 and F_2, their *Cartesian product* $F_1 \mathbin{\square} F_2$ is the graph with vertices $V(F_1) \times V(F_2)$ and edges between (u, v) and (u', v') whenever either $u = u'$ and $vv' \in E(F_2)$ or $v = v'$ and $uu' \in E(F_1)$. We will focus on the cube for simplicity, but the argument generalises to products of other graphs as shown in [5].

Theorem 2.2. *There is an absolute constant $c > 0$ such that Q_d is $\frac{c \cdot 2^d}{d}$-minor-universal.*

Proof. We follow the argument of [5] with some small changes for efficiency. Let G be a graph with at most $c \cdot 2^d/d$ edges and no isolated vertices. We need to show that G is a minor of Q_d. Let $a = \lceil \log_2(2|E(G)|) \rceil$, and consider a copy of $Q_a \mathbin{\square} C_{2d}$ in Q_d where Q_a uses only the first a coordinates of Q_d and C_{2d} the next $\lceil \log_2(2d) \rceil$ coordinates. All vertices of this copy have the same value on all remaining coordinates. Label the vertices of C_{2d} with elements of $\{0, 1, \ldots, 2d-1\}$ so that consecutive integers are adjacent. We will refer to this cycle as the *temporal* dimension.

To each vertex $v \in V(G)$, assign a path $P(v)$ in $Q_a \square \{0\}$ with $|N(v)|$ vertices such that $\{P(v) : v \in V(G)\}$ is a collection of vertex-disjoint paths. This is possible since Q_a contains a Hamiltonian cycle with $2^a \geqslant 2|E(G)|$ vertices which can be split into these paths. Index the vertices in $P(v)$ arbitrarily by vertices of $N(v)$, so we can write $V(P(v)) = \{v_x : x \in N(v)\}$.

It now suffices to find a collection of vertex-disjoint paths joining the vertices x_y and y_x for each edge $xy \in E(G)$ with internal vertices outside of $Q_a \square \{0\}$. Let σ be a permutation of $Q_a \square \{0\}$ which satisfies $\sigma(x_y) = y_x$ for each $xy \in E(G)$. By Lemma 2.1, we may write $\sigma = \sigma_{2d-1} \circ \cdots \circ \sigma_1$ where each σ_i is one-dimensional.

For an edge $xy \in E(G)$, we define a path between $x_0 = x_y$ and $x_{2d-1} = y_x$ piecewise as follows. Consider the sequence of vertices $(x_0, 0), (x_1, 1), \ldots, (x_{2d-1}, 2d-1), (x_{2d-1}, 0)$ where $x_i = \sigma_i(x_{i-1})$ for each $i > 0$. Fix an $i > 0$. We can go from $(x_{i-1}, i-1)$ to (x_i, i) by first moving in the temporal dimension, and then in the unique coordinate in which x_{i-1} and x_i differ (if they differ); call this path P_x. Consider an analogously defined path P_z between $(z_{i-1}, i-1)$ and (z_i, i). If P_x and P_z are not already disjoint, it must be that $z_i = \sigma_i(z_{i-1}) = x_{i-1}$, and hence also that $x_i = \sigma_i(x_{i-1}) = z_{i-1}$. In this case, from the last four coordinates of Q_d choose any two coordinates which were not used in time step $i-1$, say coordinates $d-1$ and d. Starting from the first vertices of the paths, flip the bits $d-1$ and d respectively, then follow P_x and P_z respectively, and finally flip back the bits $d-1$ and d (see Figure 1). This ensures that the paths P_x and P_z are traversed in distinct copies of $Q_a \square C_{2d}$ and therefore do not intersect. Moreover, the moves in coordinates $d-1$ and d do not intersect the existing paths at time step $i-1$ as we chose two coordinates which were not used in that time step.

Since the paths defined above are vertex-disjoint for each time step, they are also vertex-disjoint in Q_d. Thus, stitching them together produces a path between x_y and y_x such that these paths for all $xy \in E(G)$ are pairwise vertex-disjoint as required. $\square$
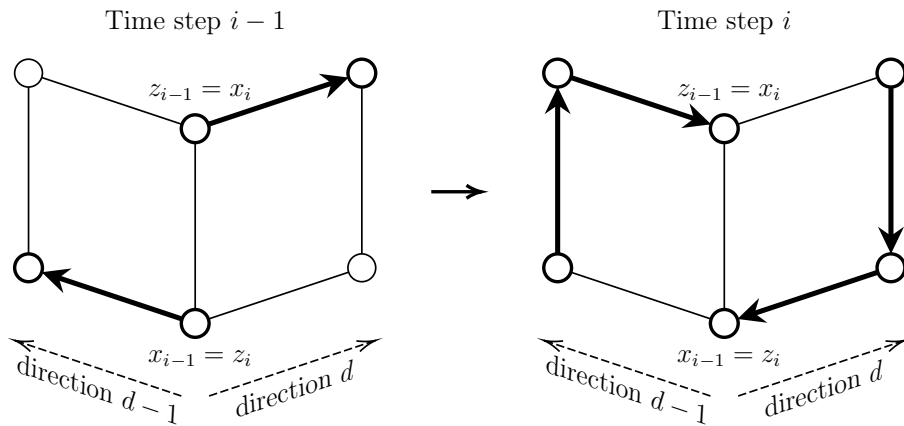

Figure 1: Constructing paths $(x_{i-1}, i-1) \rightarrow (x_i, i)$ and $(z_{i-1}, i-1) \rightarrow (z_i, i)$ when $z_i = x_{i-1}$.

3 Tight upper bound

Benjamini, Kalifa and Tzalik [5] showed that there is an absolute constant $K > 0$ such that Q_d is not $\frac{K \cdot 2^d}{\sqrt{d}}$-minor-universal. In this section we improve this by a $\sqrt{d}$ factor, to obtain an upper bound that matches the lower bound of Theorem 1.1 to within a constant factor. As in [5], we work with expander graphs, which are a natural candidate for graphs that should be 'difficult' to embed.

Theorem 3.1. *There is an absolute constant $C > 0$ such that Q_d is not $\frac{C \cdot 2^d}{d}$-minor-universal.*

Proof. Let G be a 3-regular graph with $2n$ vertices, where $2n \in [45 \cdot 2^d/d, 50 \cdot 2^d/d]$, that satisfies the following expansion property: for any set $S \subseteq V(G)$ of at most n vertices, $|N(S)| \geqslant 0.18|S|$, where $N(S)$ denotes the neighbours of S in $V(G) \setminus S$. It is well-known that such a graph exists (in fact, almost every 3-regular graph satisfies this property; see [6, Theorem 1], and observe that 0.18 satisfies the condition in the theorem when $r = 3$). Note that G has $3n \leqslant 100 \cdot 2^d/d$ edges, and so it suffices to show that G is not a minor of Q_d.

Suppose for a contradiction that Q_d contains G as a minor. Since G is 3-regular, it is easy to see that this implies that Q_d contains a subgraph G' which is a subdivision of G. We will show that there are not enough vertices in Q_d to support such a subdivision.

For every vertex $v \in V(G)$, let $s_v \in V(G') \subseteq \{0, 1\}^d$ be the corresponding vertex in the subdivision G', and let $S = \{s_v : v \in V(G)\}$ (so $|S| = 2n$). For any edge $uv \in E(G)$, the length of the corresponding (s_u, s_v)-path in G' is at least the Hamming distance $\|s_u - s_v\|_1$ between s_u and s_v. Since these paths are edge-disjoint, the sum of the corresponding Hamming distances over all edges in G counts every edge of G' at most once. That is,

$$|V(G')| = |E(G')| - |S|/2 \geqslant \sum_{uv \in E(G)} \|s_u - s_v\|_1 - |S|/2 = \sum_{uv \in E(G)} \|s_u - s_v\|_1 - n.$$

For each $i \in [d]$, let S_i be the smaller set out of $\{s \in S : s_i = 0\}$ and $\{s \in S : s_i = 1\}$, and let E_i denote the set of edges $uv \in E(G)$ such that $s_u \in S_i$ and $s_v \in S \setminus S_i$ (i.e., such that s_u and s_v differ in bit i). Since $|S_i| \leqslant |V(G)|/2$, the expansion property of G implies that $|E_i| \geqslant 0.18|S_i|$. Note that for any edge $uv \in E(G)$, its contribution to $\sum_i |E_i|$ is precisely the Hamming distance between s_u and s_v, and so

$$\sum_{uv \in E(G)} \|s_u - s_v\|_1 = \sum_{i=1}^d |E_i| \geqslant 0.18 \sum_{i=1}^d |S_i|.$$

Without loss of generality, we may assume that $S_i = \{s \in S : s_i = 1\}$ for all $i \in [d]$. Then $\sum_i |S_i| = \sum_{s \in S} \|s\|_1$. When d is sufficiently large, the number of binary strings

with at most $d/4$ ones is much less than $2^d/d$, which is less than $|S|/2$. Thus S contains at least $|S|/2$ elements with more than $d/4$ ones, and so we have $\sum_{s \in S} \|s\|_1 > d|S|/8$.

Therefore, when d is sufficiently large, we have

$$|V(G')| \geqslant 0.18 \sum_{i=1}^{d} |S_i| - n > \frac{0.18 \cdot d \cdot 45 \cdot 2^d}{8d} - \frac{50 \cdot 2^d}{2d} \geqslant 2^d = |V(Q^d)|.$$

This contradicts the assumption that Q^d contains G' as a subgraph. □

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