ON INDUCED SUBGRAPHS WITH ALL DEGREES ODD

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ABSTRACT. Gallai proved that the vertex set of any graph can be partitioned into two sets, each inducing a subgraph with all degrees even. We prove that every connected graph of even order has a vertex partition into sets inducing subgraphs with all degrees odd, and give bounds for the number of sets of this type required for vertex partitions and vertex covers. We also give results on the partitioning and covering problems for random graphs.

Note: For the final version of this paper, see the journal publication.

1. INTRODUCTION

Gallai (see [10], Exercise 5.19) proved that every graph G has a vertex partition into two sets, each of which induces a subgraph with all degrees even. Thus every graph G contains an induced subgraph with all degrees even on at least $\lceil |G|/2 \rceil$ vertices. For induced subgraphs with all degrees odd, however, the situation is less clear. Clearly no subgraph with odd degrees can have an isolated vertex, so we must restrict our attention to graphs without isolated vertices. Let us write $f(G)$ for the maximum order of an induced subgraph of G with all degrees odd and $f(n)$ for the minimum of $f(G)$ over the set of graphs of order n without isolated vertices. Caro [5] proved that $f(n) \geq c\sqrt{n}$ for $n \geq 2$. It was proved in [12] that

$$f(n) \geq cn/\log n. \tag{1}$$

Best possible bounds for $f(T)$ where T is a tree have been proved by Radcliffe and Scott [11], while Berman, Wang and Wargo [3] have proved optimal bounds for graphs with maximum degree 3. It has been conjectured (see [5]) that there is $c > 0$ such that $f(n) \geq cn$ for $n \geq 2$.

In this paper we prove results about partitioning and covering by induced subgraphs with all degrees odd. We begin by asking under what conditions a graph G has a vertex partition into sets that induce subgraphs with all degrees odd. A graph with all degrees odd must have even order, so clearly every component of a graph with such a vertex partition must have even order. Surprisingly, this trivial necessary condition is in fact sufficient, and we prove this in §2. It would be interesting to determine the number of sets required for a vertex partition of a graph of order n into sets inducing subgraphs with all degrees odd. It is known that $o(n)$ sets suffice and we give an example showing that $c\sqrt{n}$ sets may be required.

In §3, we look at vertex *coverings* by sets that induce subgraphs with all degrees odd. It turns out that we can in general achieve this with far fewer sets than for a vertex partition: we prove that every graph has a vertex covering of this type with $c\log^2 n$ sets and give an example showing that we may need $c\log n$ sets.

We would expect that for most graphs we could do rather better than this, and in §4 we prove that this is the case. In particular, we prove that almost every graph $G \in \mathcal{G}(n, \frac{1}{2})$ has a vertex partition (and hence a vertex covering) with three sets inducing subgraphs with all degrees odd.

In §5, we turn to residue classes modulo k. In particular, following Caro [5] and Caro, Krasikov and Roditty [6], for a graph G without isolated vertices, we define $f_k(G)$ to be the maximum order of an induced subgraph of G with all degrees congruent to 1 modulo k. We give a lower bound on $f_k(G)$ in terms of the chromatic number of G, and prove that every connected claw-free graph G of order n satisfies $f_k(G) \geq (1 + o(1))\sqrt{n/12}$. This improves upon a result of Caro [5], who gave a lower bound of $c(n\log n)^{1/3}$. Best possible bounds for $f_k(T)$, where T is a tree, are given by Berman, Radcliffe, Scott, Wang and Wargo [2].

Finally, in §6, we consider some open questions.

2. Partitions into induced subgraphs

Under what conditions does a graph G have a vertex partition into sets inducing subgraphs with all degrees odd? Given a graph G, we say that a partition $V_1, \dots, V_k$ of $V(G)$ is a *good partition of* G if $G[V_i]$ has all degrees odd for $i = 1, \dots, k$. Any subgraph with all degrees odd must have an even number of vertices, so if G has a good partition then it must have an even number of vertices in each component. Surprisingly, this trivial condition is also sufficient.

Theorem 1. *Let G be a graph. Then G has a good partition if and only if every component of G has even order.*

Proof. If a graph G has a good partition then it is clearly necessary that every component of G have even order. We prove by induction on $|G|$ that if every component of G has even order then G has a good partition.

If $|G| = 2$ then the statement is trivial. We suppose that $|G| > 2$, that every component of G has even order, and that the inductive statement is true for graphs of smaller order. We may assume that G is connected (and thus has even order), or else we could partition every component of G separately.

If, for every edge xy in G, we have $d(x) = 1$ or $d(y) = 1$, then G is a star: since $|G|$ is even, G must itself have all degrees odd, so we can take the trivial partition. Otherwise, we may assume that we can find an edge xy with $d(x) > 1$ and $d(y) > 1$. Let $G' = G \setminus \{x, y\}$.

If G' is connected, then by the inductive hypothesis we can find some good partition $V_1, \ldots, V_k$ of G', in which case $V_1, \ldots, V_k, \{x, y\}$ is a good partition of G. Also, if any component S of G' has even order then we can find a good partition of S, and since $G \setminus S$ is connected we can find a good partition of $G \setminus S$ and combine these partitions to give a good partition of G. Thus we may assume that every component of G' has odd order.

Let $X_1, \ldots, X_i$ be the components of G' adjacent to x but not to y; let $Y_1, \ldots, Y_j$ be the components of G' adjacent to y but not to x; and let $Z_1, \ldots, Z_k$ be the components of G' adjacent to both x and y. Since G has even order, $i + j + k$ must be even. If $k > 0$ then let

$$X = \{x\} \cup \bigcup_{m=1}^{i} V(X_m) \cup \bigcup_{m=2}^{k} V(Z_m),$$

and

$$Y = \{y\} \cup \bigcup_{m=1}^{j} V(Y_m).$$

Exactly one of $|X|$ and $|Y|$ is odd: adding $V(Z_1)$ to that set gives a partition of $V(G)$ into two sets inducing non-empty connected graphs with even order. Both graphs have a good partition by the inductive hypothesis, and these can be combined to obtain a good partition for G. We may therefore assume that $k = 0$, so that i and j are both even or both odd.

Now if i and j are both odd, then let

$$X = \{x\} \cup \bigcup_{m=1}^{i} V(X_m),$$

and

$$Y = \{y\} \cup \bigcup_{m=1}^{j} V(Y_m).$$

Both $G[X]$ and $G[Y]$ are connected graphs with even order smaller than $|G|$, so both have good partitions, and hence G does as well.

Otherwise, both i and j are even (and $i + j \geq 2$). Let

$$X = \{x, y\} \cup \bigcup_{m=1}^{i} V(X_m),$$

and

$$Y = \{x, y\} \cup \bigcup_{m=1}^{i} V(Y_m).$$

Let $V_1, \ldots, V_p$ be a good partition of $G[X]$ and $W_1, \ldots, W_q$ be a good partition of $G[Y]$. We may assume that $x \in V_1$ and $y \in W_1$, and so $y \in V_1$ and $x \in W_1$. Then $V_1 \cup W_1, V_2, \ldots, V_p, W_2, \ldots, W_q$ is a good partition of G. □

Can we say anything further about the subgraphs induced by the vertex sets in a good partition? We know from Theorem 1 that if G is a connected graph of even order, then G has a good partition. We say that G is a *basic odd graph* if G is minimal with respect to these partitions: in other words, G has all degrees odd, and the only good partition of G is the trivial partition $\{V(G)\}$.

It is easy to characterize the basic odd graphs.

Lemma 2. *A graph G is a basic odd graph iff it is a tree with all degrees odd.*

Proof. Suppose first that G is a tree with all degrees odd and it has a good partition into sets $V_1, \ldots, V_k$, with $k \geq 2$. Let xy be any edge in G between V_1 and $V(G) \setminus V_1$, and let G_x and G_y be the components of G with the edge xy removed, where $x \in V(G_x)$ and $y \in V(G_y)$. Since every vertex of G_x has odd degree (in G and hence in G_x), except for x, which has even degree in G_x, we must have $|G_x|$ odd, and similarly $|G_y|$ odd. However, there are clearly no edges other than xy between $V_i \cap V(G_x)$ and $V_i \cap V(G_y)$, for $i = 1, \ldots, k$, so $V_1 \cap V(G_x), \ldots, V_k \cap V(G_x)$ must be a good partition of G_x, which is impossible, since $|G_x|$ is odd.

Now suppose that G is not a tree with all degrees odd. It is enough to find a partition of $V(G)$ into at least two non-empty sets, each of which has even order and induces a connected graph, since such a partition can always be refined to a good partition. If G is not connected then we are done. If G does not have all degrees odd then Theorem 1 gives the required partition. Thus we may assume that G has all degrees odd and is not a tree. Let H be a spanning tree of G with one edge (from $E(G)$) added. If H does not have all degrees odd then, by Theorem 1, we can partition $V(H)$ into sets inducing connected subgraphs of H with all degrees odd (which thus induce connected subgraphs of G), each of which must have even order. Otherwise H has all degrees odd. Let C be the unique cycle in H. Then consider $H' = (V(H), E(H) \setminus E(C))$, which also has all degrees odd. This has $|C|$ components, and since all degrees of H' are odd each component has even order. Thus the vertex sets of the components of H' give the required non-trivial partition of G. □

The following corollary follows immediately from Theorem 1 and Lemma 2.

Corollary 3. *Let G be a graph, and suppose that every component of G has even order. Then G has a vertex partition such that every vertex class induces a tree with all degrees odd.*

How many sets do we need for a good partition? Let G be a connected graph such that every component has even order. We know from Theorem 1 that G has a good partition: we ask for bounds in terms of the order of G. Define

$$p(G) = \min\{k : G \text{ has a good partition with } k \text{ sets}\},$$

and, for even n,

$$p(n) = \max\{p(G) : |G| = n, \text{ every component of } G \text{ has even order}\}.$$

We begin with a lower bound on $p(n)$, and prove that

$$p(n) \geq (1 + o(1))\sqrt{2n}.$$

Indeed, consider the bipartite graph B_n, with vertex classes $V_0 = [n]^{(1)}$ and $V_1 = [n]^{(2)}$, and edges from $\{i\}$ to $\{j, k\}$ iff $i \in \{j, k\}$. Note that $|B_n| = (1 + o(1))n^2/2$. If $n \equiv 0, 3 \bmod (4)$ then $|B_n|$ is even, so by Theorem 1, we can partition $V(B_n)$ into sets inducing subgraphs with all degrees odd. However, let $S \subset V(B_n)$ be one set in the partition. If $\{i\} \in S$ and $\{j\} \in S$, where $i \neq j$, then we cannnot place $\{i, j\}$ into any set in the partition. Thus $\{1\}, \ldots, \{n\}$ are all in separate sets, and $p(B_n) \geq n$. Since $p(n)$ is clearly increasing in n, we see $p(n) \geq (1 + o(1))\sqrt{2n}$.

It does not seem to be obvious how large $p(n)$ is: it is not even immediately clear that $p(n) = o(n)$, and proving this takes a little work. In fact the best upper bound we have proved is only

$$p(n) \leq cn(\log\log n)^{-1/2}.$$

In order to prove this, it is enough to prove that, for sufficiently large even n, every graph G of order n with every component of even order has a set $W \subset V(G)$ with $|W| \geq (\log\log n)^{1/2}$ such that $G[W]$ has all degrees odd and $G \setminus W$ has a good partition. Successively removing such sets, we obtain the required partition. A subset W of this form is found by taking a partition of G into basic odd graphs: if any has order at least $(\log\log n)^{1/2}$ we are done. Otherwise there is some odd subgraph T that occurs many times. A careful examination of the edges between the copies of T enables us to find the required set W. Details were given in [13].

This upper bound is rather weak. We conjecture that the lower bound is essentially correct, and that $p(n) = (1 + o(1))c\sqrt{n}$ for some constant c.

Let us remark that the bound $p(n) \geq c\sqrt{n}$ contrasts sharply with Gallai's result that, for induced subgraphs with all degrees even, two sets suffice to partition any graph. It is therefore interesting to ask for conditions under which a graph G has a good partition into fairly few sets. For instance, what if G is k-connected? For k even, we consider a generalisation of the graph B_n defined above. Let $B_n^{(k)}$ be the bipartite graph with vertex sets $V_0 = [n]^{(1)}$ and $V_1 = [n]^{(k)}$, and an edge from $i \in V_0$ to $S \in V_1$ iff $i \in S$. Thus $B_n = B_n^{(2)}$. Now $|B_n^{(k)}|$ is even for infinitely many n (for instance, any large power of 2), and $B_n^{(k)}$ is k-connected. However, when k is even, as in the case $k = 2$, any set S in a good partition of $B_n^{(k)}$ must have $|S \cap V_0| \leq k - 1$, since no vertex in V_1 can have all its neighbours in one set of the partition. We deduce that $p(B_n^{(k)}) \geq n/(k-1)$. Since $|B_n^{(k)}| \leq n^k$, we get a sequence G_i of k-connected graphs with $|G_i| \to \infty$ and $p(G_i) \geq c_k|G_i|^{1/k}$, where c_k depends only on k. Perhaps these give an essentially best possible lower bound.

We can also consider the effects of demanding high minimal degrees: for any odd integer m, let $G_{m,n}^{(k)}$ be the graph obtained by modifying the construction of $B_n^{(k)}$ as follows. We replace each vertex $v \in V_1 = [n]^{(k)}$ by a copy $K^{(v)}$ of K_m, and join every vertex of $K^{(v)}$ to all the neighbours (in $B_n^{(k)}$) of v. It is easily seen that the graph $G_{m,n}^{(k)}$ satisfies $\delta(G_{m,n}^{(k)}) \geq m$, $\kappa(G_{m,n}^{(k)}) = k$, $p(G_{m,n}^{(k)}) \geq n/(k-1)$ and $|G_{m,n}^{(k)}| < mn^k$. This gives us some information about the relationship between minimal degree, connectivity and the number of sets required for a good partition. For instance, given $\omega(i) \to \infty$, we can find a sequence of graphs G_i with $|G_i| \to \infty$, $\kappa(G_i) \to \infty$, $\delta(G_i) \geq |G_i|/\omega(i)$ and $p(G_i) \to \infty$.

We have not however dealt with the case when $\delta(G) \geq c|G|$, for some fixed constant c. In fact, it seems quite possible that if $\delta(G) \geq c|G|$ then $p(G) \leq k(c)$, where $k(c)$ depends only on c. Of course, 'most' graphs satisfy this condition, so if this were true then we would expect 'most' graphs to have a partition into a very few sets inducing subgraphs with all degrees odd. We shall return to this in §3.

Finally, we might also try to find good partitions into fairly few sets by relaxing our conditions on the subgraphs that we want to be induced. For instance, we might allow them to have just a few degrees even. Let k be an integer and G be a graph. We define $p_k(G)$ to be the minimum number of sets in a partition of $V(G)$ into sets that induce subgraphs of G with at most k degrees odd. For n even, we define $p_k(n)$ to be the maximal value of $p_k(G)$ for connected graphs G of order n. Is $p_k(n)$ much smaller than $p(n)$? Consider the graph B_n defined above, and suppose that we have a partition of $V(B_n)$ into p sets, say $W_1, \dots, W_p$. Let $s_i = |W_i \cap V_0|$, for $i = 1, \dots, p$. Now if $\{a, b\} \subset W_i$, for some i, then the vertex $\{a, b\} \in V_1$ cannot have odd degree in our partition (it must either belong to W_i and have degree 2, or else to a different set, in which case it has degree 0). Since we have a total of at most kp vertices of even degree in all the subgraphs induced by our partition, we deduce that

$$\begin{aligned} kp &\geq \sum_{i=1}^{p} \binom{s_i}{2} \\ &= \sum_{i=1}^{p} \frac{s_i^2}{2} - \frac{n}{2} \\ &\geq p\left(\frac{n/p}{2}\right)^2 - \frac{n}{2}, \end{aligned}$$

and so $p \geq c(k)n$, for some constant $c(k)$. Thus $p_k(B_n) \geq c(k)n$ and so $p_k(n) \geq c(k)\sqrt{n}$.

3. Covers by induced subgraphs

We now turn from vertex partitions to vertex coverings. For a graph G without isolated vertices, let $t(G)$ be the smallest integer k such that G has a vertex cover with k sets inducing subgraphs with all degrees odd. We define

$$t(n) = \max\{t(G) : |G| = n,\ \delta(G) \geq 1\}.$$

How large is $t(n)$? Clearly $t(n) \leq n$, as we can always find a vertex covering by taking for each vertex the subgraph induced by that vertex and an adjacent vertex. With a little thought it is easy to prove that $t(n) = o(n)$. However, we can give a more accurate answer.

Theorem 4. *There exist a, $b > 0$ such that*

$$a \log n < t(n) < b \log^2 n \tag{4}$$

for all $n > 1$.

Proof. We begin with the upper bound on $t(n)$. Given a graph G, we define inductively sets $V_1, \ldots, V_k$, each inducing a subgraph with all degrees odd. Let $V_1 \subset V(G)$ be a set of maximal size inducing a subgraph of G with all degrees odd. Suppose now we have defined $V_1, \ldots, V_i$. If $W_i = V(G) \setminus \bigcup_{j=1}^{i} V_j$ is an independent set then put $k = i$ and finish. Otherwise, choose in each component of W_i a largest set of vertices that induces a subgraph with all degrees odd, and let the union be V_{i+1}. By considering the order of the largest component of W_i, it is clear from (1) that we finish in at most $c \log^2 n$ steps, for some constant c.

We now consider W_k, which is an independent set. If we can show that for any independent set $W \subset V(G)$ there is a set $V' \subset V(G)$ with odd degrees in G such that $|V' \cap W| > \frac{1}{4}|W|$ then we will be done, since W_k can then be covered with $c' \log n$ sets inducing subgraphs with all degrees odd, and hence the whole graph can be covered by $c'' \log^2 n$ such subsets.

Let $X = \{x_1, \ldots, x_m\}$ be a minimal set in $G \setminus W$ such that $\Gamma(x_1) \cup \cdots \cup \Gamma(x_m) \supset W$. For $i = 1, \ldots, m$, we can find $y_i \in W$ such that $\Gamma(y_i) \cap \{x_1, \ldots, x_m\} = \{x_i\}$, or else we could replace X by $X \setminus \{x_i\}$.

Suppose $m \geq \frac{1}{2}|W|$. By Gallai's theorem there is $S' \subset \{x_1, \ldots, x_m\}$ with $|S'| \geq m/2$ such that $G[S']$ has all degrees even. Set $V' = S' \cup \{y_i : x_i \in S'\}$. Then $G[V']$ has all degrees odd and $|V' \cap W| = |S'| \geq |W|/4$.

Otherwise, $m < \frac{1}{2}|W|$. Let $W' = W \setminus \{y_1, \ldots, y_m\}$, so $|W'| \geq \frac{1}{2}|W|$. Let $S \subset \{x_1, \ldots, x_m\}$ be a random subset, where each x_i is chosen independently with probability $1/2$, and let

$$W_S = \{v \in W' : |\Gamma(v) \cap S| \text{ is odd}\}.$$

Then $\mathbb{E}(|W_S|) = |W'|/2 \geq |W|/4$. Thus, for some $S_0 \subset X$, we have $|W_{S_0}| \geq |W|/4$. Set

$$V' = S_0 \cup W_{S_0} \cup \{y_i : x_i \in S_0 \text{ and } |\Gamma(x_i) \cap (S_0 \cup W_{S_0})| \text{ is even}\}.$$

Then $G[V']$ has all degrees odd and $|V' \cap W| \geq |W_{S_0}| \geq \frac{1}{4}|W|$.

To prove the lower bound, consider the bipartite graph B_n defined after Theorem 1. If $W_1, \ldots, W_m$ is a vertex covering then $\{j, k\} \in W_i$

implies that exactly one of $\{j\}$ and $\{k\}$ are in W_i. Thus for every $\{j,k\}$ there must be some i with exactly one of $\{j\}$ and $\{k\}$ in W_i. Therefore $m \geq c_1 \log n \geq c_2 \log |B_n|$, for some c_1 and c_2. □

We remark that the average size of the sets used in the vertex cover of G in Theorem 4 is at least $cn/\log^2 n$.

We obtain an alternative proof of Theorem 5 by using Theorem 13 from [12], which asserts that there is $c > 0$ such that for every graph G without isolated vertices and every $S \subset V(G)$ there is an induced subgraph of G with all degrees odd that contains at least $c|S|/\log|S|$ vertices from S. To prove Theorem 5, we cover $V(G)$ with sets $S_0, S_1, \ldots$ defined inductively as follows. Let $V_0 = V(G)$ and apply the theorem to get S_0. At the ith step, let $V_i = V(G) \setminus \bigcup_{j=1}^{i-1} S_i$, and apply the theorem to get S_i. Continue until $V_i = \emptyset$. This gives a covering of $V(G)$ with at most $c \log^2 n$ sets. Note that the longer proof is more general; furthermore, if the conjecture that $f(n) \geq \epsilon n$ for all $n > 1$ were true, then the same proof would give an upper bound of $b \log n$ for $t(n)$.

4. Induced subgraphs of random graphs

In this section, we give results on the partitioning and covering problems for random graphs. We prove that, for almost every graph, both problems can be solved with very few sets. It was proved in [12] that almost every graph $G \in \mathcal{G}(n, \frac{1}{2})$ satisfies $f(G) \sim cn$, where $c = 0.7729\cdots$. The following result about partitions uses a similar method.

Theorem 5. *Almost every $G \in \mathcal{G}(n, \frac{1}{2})$, for n even, has a good partition into three sets.*

Proof. Let $G \in \mathcal{G}(n, \frac{1}{2})$ be a random graph with vertex set $V = [n]$. For $A \subset V$, let X_A be the indicator variable of the event

$$\{G[A] \text{ has all degrees odd}\},$$

and for $x, y \in V$ let E_{xy} be the indicator variable of the event $\{xy \in E(G)\}$. We define

$$X = \sum{}' X_A X_B X_C,$$

where the sum is taken over partitions $V = A \cup B \cup C$ into three nonempty sets A, B, C which have even size and satisfy $\min(A) < \min(B) < \min(C)$. Thus $X > 0$ implies $p(G) \leq 3$.

We first calculate $\mathbb{E}(X)$. Let $V = A \cup B \cup C$ with $|A|$, $|B|$, $|C|$ even and non-zero, and pick $a \in A$, $b \in B$ and $c \in C$. We condition on $G[V \setminus \{a,b,c\}]$. Then, for $x \in A \setminus \{a\}$, the parity of $d_{G[A]}(x)$ depends only on E_{ax}, and the events $\{E_{ax} : x \in A\setminus\{a\}\}$ are independent. Since $|A|$ is even, we get

$$\mathbb{P}\{G[A] \text{ has all degrees odd}\} = 2^{1-|A|}.$$

Similar statements hold for B and C; thus the probability that $G[A]$, $G[B]$ and $G[C]$ have all degrees odd is

$$2^{1-|A|}2^{1-|B|}2^{1-|C|} = 2^{3-n}$$

Now let $M(n)$ denote the number of partitions of $[n]$ into three nonempty sets with even size. We note that $m(n) \sim 3^{n-1}/8$, so

$$\mathbb{E}X \sim 2^{3-n}3^{n-1}/8 = \left(\frac{3}{2}\right)^n/3.$$

We now estimate $\mathrm{var}(X)$. We claim that if $V = V_1 \cup V_2 \cup V_3 = W_1 \cup W_2 \cup W_3$ are two partitions of V into nonempty sets with even size such that $|V_i \bigtriangleup W_j| > 0$ for $1 \le i, j \le 3$, then $X_{V_1}X_{V_2}X_{V_3}$ and $X_{W_1}X_{W_2}X_{W_3}$ are independent.

Indeed, for each i, if $|V_i \cap W_j| > 0$ for more than one value of j then let E_i be the edge set of a spanning tree for V_i, every edge of which joins vertices from different W_j; otherwise, let E_i be the edge set of any spanning tree. Similarly, for each j, let E'_j be the edge set of a spanning tree for W_j, with all edges going between different V_i if possible. We set $E_0 = \bigcup_{i=1}^3 E_i \cup \bigcup_{i=1}^3 E'_i$, and let H be the spanning subgraph of G with edge set $V^{(2)} \setminus E_0$. Conditioning on H, we see that there is exactly one subset E_H of E_0 such that $G[V_i]$ and $G[W_i]$ have all degrees odd, for $i = 1, 2, 3$. Indeed, there are $2^{|V_i|-1}$ possibilities for the degree sequence of $H[V_i]$ modulo 2 (ie the sequence of parities), and similarly for $H[W_i]$; thus there are at most $\prod_i 2^{|V_i|-1}2^{|W_i|-1} = 2^{2n-6}$ possibilites for the the parities of the vertices in all the subgraphs $H[V_i]$ and $H[W_j]$. Now $|E_0| = 2n - 6$, so if every subset of E_0 gives a different set of parities then there is exactly one subset of E_0 giving the required sequence of parities. If there are two such subsets then their symmetric difference, say E^*, is a nonempty subset of E_0 such that restricting to any V_i or W_j gives a graph with all degrees even; this is clearly not possible, by definition of E_0 (indeed, if E_i has only edges in a single W_j, then no edges in E'_j lie inside $V_i \cap W_j$, so since E^* restricted to V_i is the edge set of a graph with all degrees even we deduce that E^* contains no edges from E_i; arguing similarly for E'_j, we may assume that E^* contains no edges that lie inside $V_i \cap W_j$ for any i, j; considering each V_i and W_j separately, since E^* restricted to each of these sets is a forest, we deduce that E^* must be empty).

Thus $\mathbb{P}(X_{V_1}X_{V_2}X_{V_3}X_{W_1}X_{W_2}X_{W_3} = 1) = 2^{6-2n}$, and so $X_{V_1}X_{V_2}X_{V_3}$ and $X_{W_1}X_{W_2}X_{W_3}$ are independent.

Therefore, writing $X_{\mathcal{V}}$ for $X_{V_1}X_{V_2}X_{V_3}$ and $X_{\mathcal{W}}$ for $X_{W_1}X_{W_2}X_{W_3}$,

$$\begin{aligned} \mathrm{var}(X) &= \mathbb{E}\left(\sum_{\mathcal{V}} X_{\mathcal{V}}\right)^2 - \left(\mathbb{E}\sum_{\mathcal{V}} X_{\mathcal{V}}\right)^2 \\ &\leq \mathbb{E}X + \sum{}'' X_{\mathcal{V}}X_{\mathcal{W}}, \end{aligned}$$

where the sum is over pairs of partitions with one set in common, and the inequality follows since $X_{\mathcal{V}}^2 = X_{\mathcal{V}}$ and $X_{\mathcal{V}}$ and $X_{\mathcal{W}}$ are independent if $\mathcal{V}$ and $\mathcal{W}$ do not share a set. Now the number of pairs of partitions with a set of size k in common is at most

$$\binom{n}{k}(2^{n-k})^2,$$

and, for such a pair $(\mathcal{V}, \mathcal{W})$,

$$\mathbb{P}(X_{\mathcal{V}}X_{\mathcal{W}} = 1) = 2^{1-k}(2^{1-(n-k)})^2 = 4 \cdot 2^{-k}2^{2(k-n)},$$

since, if $V_1 = W_1$, but $V_i \neq W_j$ otherwise, then $X_{V_2}X_{V_3}$ and $X_{W_2}X_{W_3}$ are independent (consider appropriate spanning trees, as above). Thus

$$\begin{aligned} \mathrm{var}(X) &\leq \sum_{k \text{ even}} \binom{n}{k}(2^{n-k})^2 \cdot 4 \cdot 2^{-k}2^{2(k-n)} + \mathbb{E}(X) \\ &= 4\sum_{k \text{ even}} \binom{n}{k} 2^{-k} + \mathbb{E}(X) \\ &\leq 4\left(\frac{3}{2}\right)^n + \mathbb{E}(X) \\ &= o(\mathbb{E}(X)^2), \end{aligned}$$

and so, by Chebyshev's inequality, $\mathbb{P}(X = 0) \to 0$ as $n \to \infty$. $\square$

It follows immediately from Theorem 6 that almost every graph $G \in \mathcal{G}(n, \frac{1}{2})$, for n even, satisfies $p(G) = 2$ or $p(G) = 3$. Let $X(G)$ be the number of good partitions of G into two sets. An easy calculation gives $\mathbb{E}X = 1$ and $\mathbb{E}(X^2) = 2 - 2^{2-n}$. The Chebyshev inequality does not tell us anything here, so we use a slightly more specific sieve inequality: Theorem I.16 from [4] asserts that a non-negative integer-valued random variable Y with $\mathbb{E}(Y) \leq \mathbb{E}(Y^2) \leq 2\mathbb{E}(Y) \leq 2$ satisfies

$$\mathbb{P}(Y = 0) \leq 1 - (3\mathbb{E}(Y) - \mathbb{E}(Y^2))/2.$$

Since X satisfies the required inequalities, it follows that $\mathbb{P}(X = 0) \leq \frac{1}{2} + o(1)$. It would be interesting to know the probability that a graph $G \in \mathcal{G}(n, \frac{1}{2})$ satisfies $p(G) = 2$.

Problem 6. *For a random graph $G \in \mathcal{G}(n, \frac{1}{2})$, where n is even, what is $\mathbb{P}(p(G) = 2)$?*

Let us remark that if we impose only a slightly weaker condition on our graphs, then we can almost always find a partition into two sets.

Theorem 7. *For odd n, almost every $G \in \mathcal{G}(n, \frac{1}{2})$ has a vertex partition $V(G) = V_1 \cup V_2$ such that $G[V_1]$ has all degrees odd and $G[V_2]$ has exactly one vertex with even degree.*

Proof. (Sketch.) Let $X(G)$ denote the number of vertex partitions of this type. Then an easy calculation shows that $\mathbb{E}(X) \sim \frac{n}{2}$, and $\mathrm{var}(X) \sim \mathbb{E}(X) = o(\mathbb{E}(X)^2)$, since the events that given partitions work are pairwise independent. The result follows by Chebyshev's inequality. □

Finally, we remark that for the covering problem, Theorem 6 gives $t(G) \leq 3$ for almost every $G \in \mathcal{G}(n, \frac{1}{2})$. We conjecture that $t(G) = 2$ for almost every graph.

5. Residues modulo k

We now turn to the consideration of residues modulo k rather than modulo 2. This problem was considered by Caro Krasikov and Roditty [6], Caro [5] and Berman, Radcliffe, Scott, Wang and Wargo [2]. We define $f_k(G)$ to be the maximum order of an induced subgraph of G with all degrees congruent to 1 modulo k and

$$f_k(n) = \min\{f_k(G) : \delta(G) \geq 1 \text{ and } |G| = n\}.$$

Thus the conjecture concerning odd subgraphs asserts that there is some constant $c > 0$ such that $f_2(n) \geq cn$ for all $n \geq 2$. Perhaps $f_k(n) \geq c_k n$ for all n, where c_k is a constant dependent only on k. If this were so, then we would have $c_k \leq 1/k$ for all k, as can be seen by considering $K_{k,k}$.

We begin by giving a bound on $f_k(G)$ when G is bipartite.

Lemma 8. *Let $k \geq 2$ be an integer. There exists $c(k) > 0$ such that for every bipartite graph G with $\delta(G) \geq 1$ there is a set $W \subset V(G)$ such that $|W| \geq c(k)|G|$ and $G[W]$ has all degrees congruent to 1 modulo k.*

Proof. Let $c(k) = 1/(2^k+k+1)$. Let G be a bipartite graph on n vertices with vertex classes V_1 and V_2, where $|V_1| \geq |V_2|$, and $\delta(G) \geq 1$. Let

$W_1 = \{w_1, \ldots, w_p\} \subset V_2$ be a minimal set such that $|\Gamma(v) \cap W_1| > 0$ for all $v \in V_1$. For each $w_i \in W_1$ we can find $v_i \in V_1$ such that $\Gamma(v_i) \cap W_1 = \{w_i\}$. Let $S_1 = \{v_1, \ldots, v_p\}$. Then $|S_1| = |W_1|$, and $G[S_1 \cup W_1]$ is a matching and so has all degrees equal to 1. Therefore, if $|W_1| > c(k)n/2$ we are done.

Otherwise, we define inductively sets $S_2, \ldots, S_{k-1}$ and $W_2, \ldots, W_{k-1}$. Suppose we have defined $S_1, \ldots, S_{i-1}$ and $W_1, \ldots, W_{i-1}$. Let W_i be a minimal subset of W_{i-1} such that $|\Gamma(v) \cap W_i| > 0$ for all $v \in V_1 \setminus \bigcup_{j=1}^{i-1} S_j$. We can, as before, find $S_i \subset V_1 \setminus \bigcup_{j=1}^{i-1} S_j$ such that $|S_i| = |W_i|$ and $G[S_i \cup W_i]$ is a matching.

Let $T = V_1 \setminus \bigcup_{i=1}^{k-1} S_i$. Then

$$\begin{aligned} |T| &= |V_2| - \sum_{i=1}^{k-1} |S_i| \\ &= |V_2| - \sum_{i=1}^{k-1} |W_i| \\ &\geq |V_2| - (k-1)|W_1| \\ &\geq \left(\frac{1-(k-1)c(k)}{2} \right) n. \end{aligned}$$

Now let U be a random subset of W_{k-1}, where each $w \in W_{k-1}$ is chosen independently with probability $1/2$. Let

$$T_U = \{t \in T : |\Gamma(t) \cap U| \equiv 1 \bmod k\}.$$

For $t \in T$, we have $\mathbb{P}(t \in T_U) \geq 2^{1-k}$ (if $|W_{k-1}| \leq k$ this is clear; otherwise, run through the vertices one at a time: there is some assignment of the last k that puts $t \in U$), so $\mathbb{E}(|T_U|) \geq 2^{1-k}|T|$. Thus, for some $U_0 \subset W_{k-1}$, we get $|T_{U_0}| \geq 2^{1-k}|T|$. Now consider the graph H induced by $U_0 \cup T_{U_0} \cup S$, where S is chosen from $\bigcup_{i=1}^{k-1} S_i$ so that each $w \in U_0$ has $d_H(w) \equiv 1 \bmod k$. Then

$$|H| \geq 2^{1-k}|T| \geq 2^{1-k} \left(\frac{1-(k-1)c(k)}{2} \right) n = c(k)n.$$

□

It would be interesting to know the best possible constant $c(k)$ in Lemma 8. Our best upper bound for $c(k)$ is $1/k$, as noted above. A more careful version of the argument above yields $c(k)$ with $1/c(k) = O(k^2 \log k)$; an important step here is bounding $\mathbb{P}(t \in T_U)$ from below, which can be done using arguments from [9]. However, it seems more likely that $1/c(k)$ is $O(k)$.

As a consequence of Lemma 8, we get a bound on $f_k(G)$ in terms of $\chi(G)$.

Theorem 9. *Let $k \geq 2$ be an integer. There exists a constant $c(k) > 0$ such that, for every graph G without isolated vertices,*

$$f_k(G) \geq 2c(k)|G|/\chi(G).$$

Proof. By Corollary 4 from [12], there exists an induced bipartite subgraph H of G with $|H| \geq 2|G|/\chi(G)$ and $\delta(H) \geq 1$. The result then follows from Lemma 8. □

What conditions (other than low chromatic number) can we put on G to ensure that $f_k(G)$ is large? We say that a graph G is *claw-free* if it contains no induced copy of $K_{1,3}$. Caro [5] proved that if G is a claw-free, connected graph on n vertices then

$$f_k(G) \geq c_k(n \log n)^{1/3},$$

where c_k depends only on k. We give an improvement on this.

Theorem 10. *Let $k \geq 2$ be a fixed integer. Let G be a claw-free graph without isolated vertices with order n. Then*

$$f_k(G) \geq (1 + o(1))\sqrt{n/12}.$$

Proof. Let G be a claw-free of order n without isolated vertices. We shall prove that, for any $\epsilon > 0$, we have $f_k(G) \geq (1 - \epsilon)\sqrt{n/12}$, for large enough n.

Let $c = \sqrt{4/27} - (\epsilon\sqrt{3})$. Let I be an independent set of maximal size in G. Since G has no isolated vertices, $|\Gamma(x) \cap I| \leq 2$ for $x \notin I$ (because G is claw-free) and so $|I| \leq 2n/3$. Let $W = V(G) \setminus I$, so $|W| \geq n/3$.

If $\mathrm{ind}(G[W]) \geq c\sqrt{n}$ then let I' be an independent set in $G[W]$ and consider the bipartite graph induced by $S = I' \cup \{v \in I : |\Gamma(v) \cap I'| > 0\}$. Since $G[S]$ is claw-free, bipartite and has no isolated vertices, it must consist of paths and even cycles, and $|S| \geq 3|I'|/2 \geq 3c\sqrt{n}/2$. It is easily verified that $f_k(P_m) \geq m/2$ and $f_k(C_{2m}) \geq m$ for all m, so $f_k(G) \geq f_k(G[S]) \geq |S|/2 \geq 3c\sqrt{n}/4 > (1 - \epsilon)\sqrt{n/12}$.

Thus we may assume that $\mathrm{ind}(G[W]) < c\sqrt{n}$. For $S \subset V(G)$ and $v \in S$, we write $d_S(v) = d_{G[S]}(v) = |\Gamma(v) \cap S|$. Suppose first that there is some $v \in W$ with $d_W(v) < \sqrt{3n/4}$. We define vertices $v_1, \ldots, v_k$ as follows. Let $v_1 = v$, where $d_W(v) < \sqrt{3n/4}$. Now suppose we have defined $v_1, \ldots, v_i$. Let $W_i = W \setminus (\bigcup_{j=1}^{i} \Gamma(v_j) \cup \{v_j\})$. If there exists $v \in W_i$ such that $d_{W_i}(v) < \sqrt{3n/4}$, then set $v_{i+1} = v$. Otherwise, let $k = i$ and we are finished.

Now $v_1, \dots, v_k$ is an independent set in $G[W]$, so $k < c\sqrt{n}$. Therefore

$$|W_k| > \frac{n}{3} - c\sqrt{n}(\sqrt{\frac{3n}{4}} + 1) = \frac{\epsilon\sqrt{3}}{2}n + O(\sqrt{n}),$$

and $\delta(G[W_k]) \geq \sqrt{3n/4}$. Let $H = G[W_k]$, so $|H| > (1+o(1))(\epsilon\sqrt{3}/2)n$ and $\delta(H) \geq \sqrt{3n/4}$. Suppose that there is $xy \in E(H)$ with

$$|\Gamma_H(x) \cap \Gamma_H(y)| < \frac{2}{3}\max\{d_H(x), d_H(y)\},$$

say $|\Gamma_H(x) \cap \Gamma_H(y)| < \frac{2}{3}d_H(x)$. Then let $X = \Gamma(x) \setminus (\Gamma(y) \cup \{y\})$. Since H is claw-free, $H[X \cup \{x\}]$ must be complete, since if $x_1, x_2 \in X$ and $x_1x_2 \notin E(H)$, then $H[\{x, x_1, x_2, y\}] = K_{1,3}$. But $|X \cup \{x\}| \geq d_H(x)/3 \geq \sqrt{n/12}$. Deleting at most $k-1$ vertices from X, we get a complete graph with all degrees congruent to 1 modulo k, with order at least $\sqrt{n/12} - (k-1) > (1-\epsilon)\sqrt{n/12}$, for large n.

Otherwise, we may suppose that $\delta(H) > 2$ and

$$|\Gamma_H(x) \cap \Gamma_H(y)| \geq \frac{2}{3}\max\{d_H(x), d_H(y)\} \tag{2}$$

for every $xy \in E(H)$. Therefore, as in the proof of Lemma 8 in [12], we have $\Delta(H) \geq \frac{1}{2}|H|$ (suppose x has degree $\Delta = \Delta(H)$: then, using (2), we see that $d(y, x) \leq 2$ for every $y \in V(H)$, every $v \in \Gamma(x)$ has at least $2\Delta/3$ neighbours in $\Gamma(x)$, every $y \notin \{x\} \cup \Gamma(x)$ has more than $\Delta/3$ neighbours in $\Gamma(x)$, and hence that if $\Delta < |H|/2$ then some vertex in $\Gamma(x)$ has degree more than Δ). Let $x \in V(H)$ be a vertex of degree $\Delta(H)$ and let $Y = \Gamma_H(x)$. Then, since H is claw-free, Y does not contain three independent vertices. Therefore, by the result of Ajtai, Komlós and Szemerédi [1] that every triangle-free graph of order n has an independent set of size at least $c(n \log n)^{1/2}$, we deduce that

$$\begin{aligned}
\mathrm{cl}(G[Y]) &\geq c(|Y| \log |Y|)^{1/2} \\
&= c\left(\frac{|H|}{2}\log(\frac{|H|}{2})\right)^{1/2} \\
&\geq (1+o(1))c\left(\frac{\epsilon\sqrt{3}}{4}n\log(\frac{\epsilon\sqrt{3}}{4}n)\right)^{1/2} \\
&= \omega(n)\sqrt{n}
\end{aligned}$$

where $\omega(n) \to \infty$ as $n \to \infty$. Thus $f_k(G) \geq \omega(n)\sqrt{n} - (k-1) > (1-\epsilon)\sqrt{n/12}$, for large n. □

It seems likely that this bound is too low, and we conjecture that every claw-free graph G without isolated vertices satisfies $f_k(G) \geq c_k|G|$, where c_k is a constant depending only on k.

6. Open problems

There are many interesting open questions concerning induced subgraphs with all degrees odd. The main problem is the conjecture that $f(n) \geq cn$ for some positive constant c. If this were true then c would be at most $2/7$, which can be seen from an example of Caro: consider the graph with vertex set $\mathbb{Z}_7$, and edges between i and $i \pm 1$, $i \pm 2$ $(\mathrm{mod} 7)$, for $i = 0, \ldots, 6$.

There are various ways of ensuring that $f(G)$ is large. It was conjectured in [12] that $f(G) \geq |G|/\chi(G)$, and proved that $f(G) \geq |G|/2\chi(G)$; it was shown that it would be enough to prove this for bipartite graphs. It would be interesting to find other conditions under which $f(G)$ is large: for instance, can we prove $f(G) \geq c|G|$ when G is triangle-free?

It would also be interesting to narrow the gap between the upper and lower bounds for $p(n)$ and $t(n)$. As remarked in §2, the upper bound for $p(n)$ given in Theorem 4 is fairly weak, and it should be possible to improve it. Indeed, we expect that $p(n) = (1 + o(1))c\sqrt{n}$; perhaps this could be proved for the special case of bipartite graphs. It would also be interesting to know, for $G \in \mathcal{G}(n, 1/2)$, the probability that G satisfies $p(G) = 2$.

All these questions can be asked again modulo k. We proved above (Theorem 10) that $f_k(n) \geq (1 + o(1))\sqrt{n/12}$ for claw-free graphs; it should be possible to improve this result. Perhaps $f_k(n) \geq c_k n$ for some constant c and all $n \geq 2$. However, it would also be interesting to find sufficient conditions for this to hold.

Caro, Krasikov and Roditty [7] asked whether, for every integer k, there is a constant $c(k)$ such that every graph has a vertex partition into $c(k)$ sets, each of which induces a subgraph with all degrees divisible by k. It should be possible to prove a partition result of this type for random graphs: for $k > 1$ and $i \leq k$, there exists $c(k)$ such that almost every $G \in \mathcal{G}(n, p)$ has a vertex partition into $c(k)$ classes such that the degrees in each class are congruent to i mod k.

Finally, it would be interesting to prove results about induced subgraphs with many edges rather than many vertices. Given a graph with m edges, can we find an induced subgraph of with all degrees odd and at least cm edges? How many induced subgraphs with all degrees odd do we need to cover the edges of the graph? These questions are also open for induced subgraphs with all degrees even.

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