FINITE SUBSETS OF THE PLANE ARE 18-RECONSTRUCTIBLE.

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Abstract. We prove that every finite subset of the plane is reconstructible from the multiset of its subsets of at most 18 points, each given up to rigid motion. We also give some results concerning the reconstructibility of infinite subsets of the plane.

1. Introduction

Combinatorial reconstruction problems arise when we are given the subobjects of a certain size of some combinatorial object, up to isomorphism, and are asked whether this is sufficient information to reconstruct the original object. For instance the Reconstruction Conjecture, made sixty years ago by Ulam [36] and Kelly [12], asserts that all finite graphs on at least 3 vertices can be reconstructed from the collection of all their (non-trivial) induced subgraphs. Similarly the Edge Reconstruction Conjecture (Harary [10]) asserts that every graph with at least 4 edges can be reconstructed from the collection of all its (non-trivial) subgraphs. There is a substantial literature on graph reconstruction (see, for instance, [3, 2, 4, 26]). Reconstruction problems have been considered for a variety of other combinatorial objects, including directed graphs [34, 35], hypergraphs [15], infinite graphs [27], codes [19], sets of real numbers [30], sequences [33, 17], and combinatorial geometries [5, 6].

The necessary ingredients for a combinatorial reconstruction problem are a notion of isomorphism and a notion of subobject. Some progress has been made in recent years in the general case where we have a group action $G \rightarrowtail X$ providing the notion of isomorphism, and we wish to reconstruct a subset S of X from the multiset of isomorphism classes of its k-element subsets, known as the *k-deck* (see Alon, Caro, Krasikov, and Roditty [1], Babai [2], Cameron [7, 9, 8], Maynard and Siemons [20], Mnukhin [21, 22, 23], and [31]). Several authors [1, 7, 22] have noted that we can reconstruct S provided $k > \log_2|G| + 1$; the $n \log_2 n$ bound for edge reconstruction (Müller [25]; Lovász [18]) also follows from this. In general, however, much smaller decks may suffice (see [28, 31]).

In this paper we focus on the case of the plane, $\mathbb{R}^2$, with the group R of rigid motions acting on it. Thus the k-deck of a set S of points in the plane is the multiset of its k-subsets given up to rigid motion. (For instance, the 2-deck is essentially the multiset of distances between pairs of points in S.) We want to know how large k must be so that S is determined up to rigid motion by its k-deck. Alon, Caro, Krasikov, and Roditty [1] proved that subsets of n points in the plane can be

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reconstructed from their $(\log_2 n + 1)$-decks. Our first aim in this paper is to prove that every finite subset of the plane can be reconstructed from its 18-deck.

We begin by considering sets of points in the plane together with an "orientation", which leads naturally to the problem of reconstructing finite subsets of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is crucial to our approach that finite subsets of $\mathbb{T}$ are reconstructible from bounded decks, under the action of $\mathbb{T}$ on itself by translation. This in turn is proved by considering the circle as a limit (in an appropriate sense) of the groups $\mathbb{Z}_n$ for n large. Alon, Caro, Krasikov and Roditty [1] proved that if $\mathbb{Z}_n$ acts on itself then arbitrary subsets S are reconstructible from their $(\log_2 n + 1)$-decks (see also Mnukhin [22, 24]). Radcliffe and Scott [29] improved their bound substantially in the case of $\mathbb{Z}_n$ acting on itself. Using a Fourier analytic approach, they showed (among other results) that if S is a finite multiset in Z_p and p is prime then S is reconstructible from its 3-deck. Using more refined Fourier analytic arguments, Pebody [28] proved the following result."

Theorem 1. *If S is a finite multiset of elements of $\mathbb{Z}_n$ then S can be reconstructed from its 6-deck.*

In fact Pebody proved rather more, computing for every abelian group A the minimum k (as a function of A) for which all multisets in A are k-reconstructible.

In this paper we prove first that finite subsets of $\mathbb{T}$ are reconstructible from their 6-decks and then that finite subsets of the plane $\mathbb{R}^2$, under the action of the group R of rigid motions, are reconstructible from their 18-decks. Our proof for the plane works by reducing the problem of reconstructing a set up to the action of the group of rigid motions to that of reconstructing it up to the action of the group of translations. This requires us to reconstruct the orientations of the sets in an appropriately sized deck. The technique which allows us to do this we the method of "features", and we present it in Section 2, in a quite general form, before proving our results on finite subsets of $\mathbb{T}$ and $\mathbb{R}^2$ in Section 3. It turns out that we can use this approach in another, slightly different situation, and in Section 4 prove some results concerning the reconstructibility of infinite subsets of the plane.

1.1. Definitions.

In the following we suppose that a group action $G \rightarrowtail X$ has been specified. We write the group action generically as $(g, x) \mapsto g.x$. We shall most often be dealing with the group R of rigid motions of the plane acting on $\mathbb{R}^2$, in which case we shall usually think of the elements of R as functions mapping the plane to itself, and write the action as function application. A rigid motion of the plane is an affine isometry preserving orientation. For notation and terminology, see ***. We will always assume that $G \rightarrowtail X$ is transitive.

An essential part of our approach to reconstructing *subsets* of the plane is to consider the more general problem of reconstructing *multisets* of points in the plane, where each point is allowed to have finite multiplicity. This should not be too surprising since [29] and [28] both proceed by proving results concerning the action of $\mathbb{Z}_n$ on the group ring $\mathbb{Q}\,\mathbb{Z}_n$.

Definition 1. Formally, a *multiset S in X with finite multiplicities* is a function $m_S : X \to \{0, 1, 2, \ldots\}$. We say that $m_S(x)$ is the *multiplicity* of x in S, and define the *support* of S to be the set $\operatorname{supp}(S) = \{x \in X : m_S(x) > 0\}$. The *size* of S is $|S| = \sum_{x\in X} m_S(x)$. We shall often refer to a multiset in X of finite size as a *configuration*. We write $\mathcal{M}(X)$ for the collection of all finite multisets in X.

A multiset K is *contained* in a multiset S if $m_K(x) \leq m_S(x)$ for all $x \in X$. The *power set* $\mathcal{P}(S)$ of S is the multiset in which each $K \subset S$ has multiplicity $\prod_{x\in\mathrm{supp}(K)} \binom{m_S(x)}{m_K(x)}$; we write $\mathcal{P}_r(S) = \{A \in \mathcal{P}(S) : |A| = r\}$. With this convention the size of $\mathcal{P}(S)$ is $2^{|S|}$, and $|\mathcal{P}_r(S)| = \binom{|S|}{r}$.

We shall have to consider two different notions of union. The *multiset union* of a collection $\mathcal{S}$ of multisets (or sets) is the multiset $\bigoplus_{S\in\mathcal{S}} S$ in which each $x \in X$ has multiplicity $\sum_{S\in\mathcal{S}} m_S(x)$. The *set union* $\bigcup_{S\in\mathcal{S}} S$ gives to each $x \in X$ the multiplicity $\max_{S\in\mathcal{S}} m_S(x)$.

Definition 2. Given two multisets S, T in X we say that they are *isomorphic*, and write $S \simeq T$, if there exists $g \in G$ such that $g.S = T$. The collection of all multisets in X isomorphic to S is the *isomorphism class* of S, written $[S]_G$ (or simply $[S]$ if the group action is sufficiently clear).

Definition 3. If S is a multiset in X then the *k-deck* of S is the multiset

$$D_k(S) = \{[K]_G : K \in \mathcal{P}(S), |K| \leq k\}.$$

Note that $K \subset S$ might well arise multiple times as a subset of S: to be precise, K arises $\prod_{x\in\mathrm{supp}(K)} \binom{m_S(x)}{m_K(x)}$ times. Thus, for $|K| \leq k$, the multiset $D_k(S)$ gives the cardinality of the collection of multisets in $\mathcal{P}(S)$ belonging to a fixed isomorphism class $[K]$. We write $m_S([K])$ for the multiplicity $m_{D_k(S)}([K])$. In some cases we will want to emphasize the particular group action, in which case we will write $D_k(G \rightarrowtail S)$. The entire collection of isomorphism classes of finite subsets of S we will call the *$(<\omega)$-deck of S*, written $D(S) = \{[K] : K \in \mathcal{P}(S), |K| < \infty\}$.

We remark that the k-deck is often defined in terms of the subsets of S of size *exactly k*. However the two definitions are easily seen to be equivalent here for $\infty|S| \geq k$, by a variant of Kelly's Lemma [13]. (Further discussion can be found in [32].)

Definition 4. We say that a multiset S in X is *reconstructible from its k-deck* (or *k-reconstructible*) if every T in X with the same k-deck as S is, in fact, isomorphic to S. Similarly, if $f : \mathcal{M}(X) \to Y$ is an arbitrary function then we say $f(S)$ is *k-reconstructible* if $D_k(T) = D_k(S) \Rightarrow f(T) = f(S)$. More generally we say that $f : \mathcal{M}(X) \to Y$ is *k-reconstructible* if $f(S)$ is k-reconstructible for all finite multisets S in X. This is equivalent to saying that f is reconstructible iff it factors through the map $S \mapsto D_k(S)$. Note that if f is k-reconstructible it must depend only on $[S]$, since $D_k(S)$ does. We will say that *(finite) multisets in X are reconstructible from their k-decks* if the function $S \mapsto [S]_G$ on $\mathcal{M}(X)$ is k-reconstructible (in other words, finite multisets can be identified up to isomorphism from their k-decks).

2. The method of features

In this section we present a method central to our results in this paper; the method of features. We show that from an appropriately sized deck of $G \rightarrowtail S$ we can reconstruct the k-deck of any collection of features naturally associated with configurations lying in S. To make this clearer let us give an example that we will use later.

Example 5. We would like to associate to a configuration C in $\mathbb{R}^2$ a direction. This requires us to distinguish two points of C to use to define a reference line, whose direction we will call the *direction of C*. Thus we are led naturally to the notion of an oriented configuration: an *oriented configuration* is a triple $\langle C, x, y\rangle$ consisting of a finite multiset C in $\mathbb{R}^2$ together with points $x, y \in \operatorname{supp}(C)$ with $x \neq y$.

With the example of oriented configurations in mind we describe the general formalism we will use.

Definition 6. A *configuration style* is a finite sequence $a = (a_1, a_2, \ldots, a_r)$ of positive integers. A *coloured configuration in style* a is a pair $\langle C, c\rangle$ consisting of a finite multiset C in X and a colouring $c : \operatorname{supp}(C) \to \{0, 1, \ldots, r\}$ such that $|c^{-1}(i)| = a_i$ for $i = 1, 2, \ldots, r$. There is a natural action of G on coloured configurations, where $g.\langle C, c\rangle = \langle g.C, c \circ g^{-1}\rangle$. Two coloured configurations $\langle C, c\rangle$ and $\langle C', c'\rangle$ are therefore *isomorphic* if there exists $g \in G$ such that $g.C = C'$ and $c'(g.x) = c(x)$ for all $x \in C$. As usual we write $[\langle C, c\rangle]_G$ for the isomorphism class of $\langle C, c\rangle$ under the action of G. The *size* of a coloured configuration $\langle C, c\rangle$ is simply the size of C. We write $\mathcal{C}_a$ for the collection of all coloured configurations in style a. We say that $\langle C, c\rangle$ is an *a-coloured configuration in S* if c is an a-colouring of C and $C \subset S$.

Example 7. We define a *pointed configuration* $\langle C, x\rangle$ to be a coloured configuration in style (1), that is, a finite multiset C together with one distinguished element $x \in \operatorname{supp}(C)$, which has colour 1. An oriented configuration is, similarly, a coloured configuration in style $(1, 1)$. The colouring picks out two distinguished elements of $\operatorname{supp}(C)$, the first, x, having colour 1 and the second, y, having colour 2.

Now we turn to the central reason for discussing coloured configurations. We want to talk about a "feature" of a coloured configuration, and, eventually, to be able to reconstruct the set of all such features associated with particular classes of configurations. (Recall the example of the direction of an oriented configuration.) Since these features are also the object of a reconstruction problem we insist that there be a group H acting on the features and that isomorphic coloured configurations have isomorphic features.

Definition 8. Given group actions $G \rightarrowtail X$ and $H \rightarrowtail Y$ we define an *H-feature* of a-coloured configurations in X to be a function $f : \mathcal{C}_a \to Y$ on coloured configurations together with a homomorphism $\phi : G \to H$ such that $f(g.\langle C, c\rangle) = \phi(g).f(\langle C, c\rangle)$ for all $\langle C, c\rangle$ and g. In other words isomorphic configurations have isomorphic features, and moreover the isomorphism is chosen in a uniform way.

Definition 9. Let $\mathcal{C}$ be a set of isomorphism classes of a-coloured configurations. The *$\mathcal{C}$-list* of S is

$$L_{\mathcal{C}}(S) = \{\langle C, c\rangle \,:\, C \in \mathcal{P}(S),\, c \text{ an } a\text{-colouring of } C,\, [\langle C, c\rangle]_G \in \mathcal{C}\}.$$

If f is an H-feature of such configurations then the *$\mathcal{C}$-feature set of S* is the multiset

$$F_{f,\mathcal{C}}(S) = \{f(\langle C, c\rangle) \,:\, \langle C, c\rangle \in L_{\mathcal{C}}(S)\}.$$

Example 10. Given an oriented configuration $\langle C, x, y\rangle$ we can associate with it the direction of the directed line segment from x to y. We consider this direction as an element of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This is a $\mathbb{T}$-feature, with associated homomorphism mapping $g \in R$, the group of rigid motions, to $\theta + \mathbb{Z}$ where $2\pi\theta$ is the common angle through which all line segments rotate under the action of g. So if we let $\mathcal{C}$ consist only of the equivalence class of oriented configurations containing two points at distance 1 apart, then the $\mathcal{C}$-list of S is the collection of all ordered pairs of points in S at distance 1 apart, and the feature set of S is the multiset of all directions of these line segments.

Remark 11. Note that the $\mathcal{C}$-list of S, and hence the $\mathcal{C}$-feature set F of S are *not* isomorphism invariants, so there is no hope that we will literally be able to reconstruct them. What we hope is that the isomorphism class $[F]_H$ of the feature set will be reconstructible.

Now are now ready for the first theorem of the section. Where unambiguous we shall suppress the qualifiers in H-feature, a-coloured configuration, and $\mathcal{C}$-feature set.

Theorem 2 (Feature Theorem)**.** *Let f be a feature of coloured configurations (with associated homomorphism ϕ), $\mathcal{C}$ a set of isomorphism classes of coloured configurations, each of size at most m, and S a multiset in X. Set $F = F_{f,\mathcal{C}}(S)$, the feature set of S. Then the k-deck of $H \rightarrowtail F$ is reconstructible from the mk-deck of $G \rightarrowtail S$. In particular if multisets in Y are reconstructible from their k-decks then $[F]_H$ is reconstructible from the mk-deck of S.*

Proof. Note first that there is a natural bijection between the feature set F and the $\mathcal{C}$-list $L = L_{\mathcal{C}}(S)$. Thus there is also a natural bijection between $\mathcal{P}_r(F)$ and the collections $\{\langle C_i, c_i\rangle : i = 1, 2, \dots, r\} \in \mathcal{P}_r(L)$. We will partition $\mathcal{P}_r(L)$ according to the set union (of multisets) $C = \bigcup_1^r C_i$: note that a given C may arise in many different ways. For a configuration C in X we say that a $\mathcal{C}$*-splitting* of C is a representation of C as a set union $C = \bigcup_1^r C_i$ together with a-colourings c_i for the C_i such that $[\langle C_i, c_i\rangle]_G \in \mathcal{C}$ for $i = 1, 2, \dots, r$.We can then write

$$f(C) = \{\{f(\langle C_i, c_i\rangle)\}_1^r : \{\langle C_i, c_i\rangle\}_1^r \text{ is a } \mathcal{C}\text{-splitting of } C\}.$$

We obtain the multiset identity

$$\bigoplus_{i\le k} \mathcal{P}_i(F) = \bigoplus_{\substack{C\in\mathcal{P}(S)\\ |C|\le mk}} f(C),$$

and hence

$$D_k(H \rightarrowtail F) = \left\{ [K]_H : K \in \bigoplus_{i\le k} \mathcal{P}_i(F) \right\} = \bigoplus_{\substack{C\in\mathcal{P}(S\\ |C|\le mk}} \{[L]_H : L \in f(C)\}. \tag{1}$$

The last, crucial, observation is that the multiset of isomorphism classes

$$\{[L]_H : L \in f(C)\}$$

is reconstructible from $[C]_G$. To see this note that if $D \simeq C$, with say $g.C = D$, then the $\mathcal{C}$-splittings of C are isomorphic to the $\mathcal{C}$-splittings of D: if $C = \bigcup_1^k C_i$ and c_i are appropriate colourings then we set $D_i = g.C_i$ with colourings $d_i(x) = c_i(g^{-1}.x)$ for all $x \in D_i$. The set of features arising from $\{\langle D_i, d_i\rangle\}_1^k$ is isomorphic to that arising from $\{\langle C_i, c_i\rangle\}_1^k$ because we have

$$\begin{aligned}\{f(\langle D_i, d_i\rangle)\} &= \{f(g.\langle C_i, c_i\rangle)\}\\ &= \{\phi(g).f(\langle C_i, c_i\rangle)\}\\ &= \phi(g).\{f(\langle C_i, c_i\rangle)\}.\end{aligned}$$

Thus, by (1), $D_k(F)$ depends only on the collection of all isomorphism classes of elements of $\mathcal{P}(S)$ of size at most mk, which is the mk-deck of $G \rightarrowtail S$. ∎

We will use the method of features both directly and via the "certification lemma" below. The certification lemma applies to the situation in which S might be infinite, and shows that if some subset P of S can be picked out by a property which can be determined from examining small configurations then we can reconstruct the decks of P from (larger) decks of S.

Definition 12. Recall that if C is a finite multiset of points in X and $x \in \operatorname{supp}(C)$ then we call the pair $\langle C, x\rangle$ a pointed configuration. Let S be a multiset in X and let P be a subset of S. We say that P has a *certificate of size* m if there exists a set $\mathcal{C}$ of isomorphism classes of pointed configurations, each of size at most m, such that P is exactly the set of points in S "pointed at" by elements of $\mathcal{C}$. To be precise, we require

$$P = \{y \in S \,:\, \exists C \subset S, y \in \operatorname{supp}(C) \text{ such that } [\langle C, y\rangle] \in \mathcal{C}\}$$

Definition 13. If S is a multiset in X and $\mathcal{C}$ is a collection of pointed configurations then we write

$$\mathcal{C}(x) = \{\langle C, y\rangle \,:\, C \in \mathcal{P}(S), y \in \operatorname{supp}(C) \text{ such that } [\langle C, y\rangle] \in \mathcal{C}\}.$$

We also define $\lambda_{\mathcal{C}}(x) = |\mathcal{C}(x)|$.

Lemma 3 (Certification Lemma)**.** *Let S be a subset of X and P be a subset of S having a certificate of size m, $\mathcal{C}$ say. We can reconstruct the k-deck of the multiset P^λ, consisting of $\lambda_{\mathcal{C}}(x)$ copies of x for each $x \in P$, from the mk-deck of S. In particular, if $[P^\lambda]_G$ is reconstructible from its k-deck then it is reconstructible from the mk-deck of S, as is $[P]$.*

Proof. The map taking $p : \langle C, x\rangle \mapsto x$ is trivially a G-feature of pointed multisets (with associated homomorphism the identity map $G \to G$), and moreover P^λ is exactly the feature set $F_{p,\mathcal{C}}(S)$. Thus by Theorem 2 the claims of the lemma hold. ∎

3. The Circle and the Plane

In this section we prove that finite multisets in the circle are 6-reconstructible and that finite multisets of $\mathbb{R}^2$ are 18-reconstructible.

We deal first with the reconstructibility of multisets of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ acting on itself by translation. It turns out that we are able to relate this problem to that of reconstructing multisets in the cyclic group $\mathbb{Z}_n$. Because of this it is helpful to identify $\mathbb{Z}_n$ with the specific subgroup $\{i/n + \mathbb{Z} \,:\, i = 0, 1, \ldots, n-1\} < \mathbb{T}$. We will also make use of the fact that $\mathbb{T}$ is a topological group with metric

$$d(r + \mathbb{Z}, s + \mathbb{Z}) = \min\{|r' - s'| \,:\, r' \in r + \mathbb{Z}, s' \in s + \mathbb{Z}\}.$$

We shall often identify elements of $\mathbb{T}$ with elements of $[0, 1) \subset \mathbb{R}$.

Theorem 4. *All finite multisets in $\mathbb{T}$ are 6-reconstructible.*

We give two proofs of this result.The first proof considers the subgroup of $\mathbb{T}$ generated by the multiset S that we wish to reconstruct; the second proof works by approximating S by a "nearby" copy of $\mathbb{Z}_n$ (standard results on Diophantine approximation imply that such a copy exists).

First proof. Given finite multisets S_1, S_2 in $\mathbb{T}$ with the same 6-deck, we will show that S_1 is a translate of S_2. Consider the subgroup G of $\mathbb{T}$ generated by $S_1 \bigcup S_2$. It is a finitely generated subgroup of $\mathbb{T}$, and therefore there exist integers k, n such that $G \simeq \mathbb{Z}_k \oplus \mathbb{Z}^n$. Let $\theta : G \to \mathbb{Z}_k \oplus \mathbb{Z}^n$ be an isomorphism, and let $T_i = \theta(S_i)$. Then T_1, T_2 are multisets of $\mathbb{Z}_k \oplus \mathbb{Z}^n$ with the same 6-deck.

Represent the elements of $\mathbb{Z}_k \oplus \mathbb{Z}^n$ by sequences of $n+1$ integers. The sequences $(a_1, a_2, \dots, a_{n+1})$ and $(b_1, b_2, \dots, b_{n+1})$ represent the same element if $k|(a_1 - b_1)$ and $a_i = b_i$ for all $i > 1$. For $2 \le i \le n+1$, say that a_i is *the ith co-ordinate* of $(a_1, a_2, \dots, a_{n+1})$. Let x_i be the smallest ith coordinate of elements of $T_1 \bigcup T_2$, and let y_i be the largest such. Finally, let $(p_2, \dots, p_{n+1})$ be a sequence of distinct primes such that p_i is not a factor of k, and $p_i > 2(y_i - x_i)$.

Let H be the subgroup of $\mathbb{Z}_k \oplus \mathbb{Z}^n$ generated by the elements $(0, p_2, 0, \dots, 0)$, $(0, 0, p_3, 0, \dots, 0)$, …, $(0, 0, \dots, p_{n+1})$, and let

$$\theta' : \mathbb{Z}_k \oplus \mathbb{Z}^n \to (\mathbb{Z}_k \oplus \mathbb{Z}^n)/H \simeq \mathbb{Z}_{kp_2p_3\dots p_{n+1}}$$

be the quotient map. Then if $T_i' = \theta'(T_i)$, T_1' and T_2' have the same 6-deck. Since these multisets are multisets of a cyclic group, Theorem 1 implies that they are translates.

Therefore there exists a translate T of T_1 and a bijection $\gamma : T \to T_2$ such that for all $t \in T$, $t - \gamma(t) \in H$. Furthermore, by picking T wisely, we may assume that there exists t such that $t = \gamma(t)$ for some t. Then the ith co-ordinate of t is between x_i and y_i. Therefore for any $u \in T$, the ith co-ordinate of u is between $x_i - (y_i - x_i) = 2x_i - y_i$ and $y_i + (y_i - x_i) = 2y_i - x_i$. Furthermore, the ith co-ordinate of $\gamma(u)$ is between x_i and y_i. Therefore the ith co-ordinate of $u - \gamma(u)$ is between $2(x_i - y_i)$ and $2(y_i - x_i)$ and is definitely less in magnitude than p_i. Since $u - \gamma(u) \in H$, $u = \gamma(u)$. Since this holds for all u. $T = T_2$ (as multisets), and hence T_1 and T_2 are translates. Since θ was an isomorphism, it follows that S_1 and S_2 are translates, and hence multisets in $\mathbb{T}$ are 6-reconstructible. ■

Second proof. Given a finite multiset S in $\mathbb{T}$, we will show that it is reconstructible from its 6-deck. First note that we may assume, by translating S if necessary, that $0 \in S$. For $T \in \mathcal{M}(\mathbb{T})$ we will write $\Delta(T) = \{t - t' : t, t' \in T\}$ for the multiset of differences of elements of T. Let $\Delta_1 = \Delta(S)$ and $\Delta_2 = \Delta(\Delta(S))$. It is clear that Δ_1, and hence Δ_2, can be reconstructed from the 2-deck of S; note that $S \subset \Delta_1 \subset \Delta_2$.

By standard results concerning Diophantine approximation (see, for instance, [11], Chapter 1, Proposition 2) there exists $\rho > 0$ and a sequence $n_i \to \infty$ such that

$$\epsilon_i := \max\{d(\delta, \mathbb{Z}_{n_i}) : \delta \in \Delta_2\} < 1/n^{1+\rho}.$$

(This approximation is used in a similar context in [1].) In particular we may assume

$$\epsilon_i < \frac{1}{4n_i} < \frac{1}{4} \min\{d(\delta, 0) : \delta \in \Delta_2\} \tag{2}$$

We shall say that n_i is *good for S* if it satisfies (2). Notice that for any particular n, the property that n is good for S is reconstructible from the 2-deck of S. For each of the n_i we define a "projection" $\pi : \Delta_2 \to \mathbb{Z}_{n_i}$ by letting $\pi(\delta)$ be the point in $\mathbb{Z}_{n_i}$ closest to δ. There is no possible ambiguity since by (2) the nearest element of $\mathbb{Z}_{n_i}$ is within distance $\epsilon_i < 1/4n_i$. Moreover π is injective on Δ_1: if $\delta, \delta' \in \Delta_1$ have $\pi(\delta) = \pi(\delta')$ then $d(\delta, \delta') \le 2\epsilon_i < 1/n_i$ while $\delta - \delta' \in \Delta_2$. By (2) this implies that $\delta = \delta'$.

Now we define $S_{n_i} = \pi(S) = \{\pi(x) : x \in S\}$. It is easily checked that the 6-deck of S_{n_i} is reconstructible from the 6-deck of S, and hence that $[S_{n_i}]$ is reconstructible. Now take an arbitrary orientation of each S_{n_i}: dropping to a convergent subsequence yields an orientation of S. ■

We turn now to the proof of our central result, that finite multisets in the plane are reconstructible from their 18-decks.

Theorem 5. *Any finite multiset S in $\mathbb{R}^2$ is reconstructible, up to the action of the group R of rigid motion acting on the plane, from its 18-deck.*

Proof. We begin by defining a $\mathbb{T}$-feature of configurations contained in S. We identify, in the natural way, the collection of unit vectors in $\mathbb{R}^2$ with the group $\mathbb{T}$. To be precise let $\psi : \{u \in \mathbb{R}^2 : |u| = 1\} \to \mathbb{T}$ be defined by $\psi((x_1, x_2)) = \alpha/(2\pi)$ if $(x_1, x_2) = (\sin\alpha, \cos\alpha)$. As in the discussion in Section 2 recall that an oriented configuration $\langle C, x, y\rangle$ is a finite multiset C in $\mathbb{R}^2$ together with points $x, y \in \operatorname{supp}(C)$ with $x \neq y$. The *direction* of $\langle C, x, y\rangle$ is the element $u(\langle C, x, y\rangle) = \psi((x-y)/|x-y|)$ of $\mathbb{T}$.

We claim that u is a $\mathbb{T}$-feature of oriented configurations. To see this note that there is a homomorphism ρ from R to $\mathbb{T}$ which takes g to $\alpha/2\pi$ if g rotates all lines segments through α radians. Moreover $u(g.\langle C, x, y\rangle) = \rho(g).u(\langle C, c\rangle)$. If $\mathcal{C}$ is any collection of isomorphism classes of oriented configurations we define the *orientation set* of $\mathcal{C}$ (in S) to be the multiset in $\mathbb{T}$ given by

$$\begin{aligned} O(\mathcal{C}) &= F_{u,\mathcal{C}}(S) \\ &= \{u(\langle C, x, y\rangle) : C \in \mathcal{P}(S), x, y \in \operatorname{supp}(C), x \neq y, [\langle C, x, y\rangle] \in \mathcal{C}\}. \end{aligned}$$

By Theorem 2, if all the configurations in $\mathcal{C}$ have size at most m then we can reconstruct $[O(\mathcal{C})]_{\mathbb{T}}$ from the $6m$-deck of S.

Similarly, if $\epsilon : \mathcal{C} \to \mathbb{T}$ is an arbitrary function then the map $\langle C, x, y\rangle \mapsto u(\langle C, x, y\rangle) + \epsilon([\langle C, x, y\rangle])$ is also a $\mathbb{T}$-feature, with the same associated homomorphism. Thus, by the same result, we can also reconstruct $[O(\mathcal{C}, \epsilon)]_{\mathbb{T}}$ from the $6m$-deck of S, where

$$\begin{aligned} O(\mathcal{C}, \epsilon) = \{u(\langle C, x, y\rangle) + \epsilon([\langle C, x, y\rangle]) :& \\ C \in \mathcal{P}(S), x, y \in \operatorname{supp}(C), x \neq y, [\langle C, x, y\rangle] \in \mathcal{C}\}.& \end{aligned}$$

Suppose now that $\mathcal{C} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_t\}$. We will show that we can reconstruct $[(O(\Gamma_i))_{i=1}^t]_{\mathbb{T}}$ from the $6m$-deck of S. (Note that the relevant $\mathbb{T}$ action is that on $\mathcal{M}(\mathbb{T})^t$ given by $s.(A_i)_{i=1}^t = (s.A_i)_{i=1}^t$.) To see this let $\Delta = \{t - t' : t, t' \in O(\mathcal{C})\}$ and let $W \subset \mathbb{R}$ be the subspace of $\mathbb{R}$ (considered as a vector space over $\mathbb{Q}$) generated by $\Delta \cup \{1\}$. This is clearly independent of the choice of representatives for elements of Δ. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_t$ be elements of $\mathbb{R}$ linearly independent of each other and W, and define $\epsilon : \mathcal{C} \to \mathbb{T}$ by $\epsilon(\Gamma_i) = \epsilon_i$. As above we can reconstruct $[O(\mathcal{C}, \epsilon)]_{\mathbb{T}}$ from the $6m$-deck of S. Now pick $O \in [O(\mathcal{C}, \epsilon)]$ and consider $x, y \in O$. We have $O = O(\mathcal{C}, \epsilon) + s$ for some unknown s. If $x \in O(\Gamma_i) + \epsilon_i + s$ and $y \in O(\Gamma_j) + \epsilon_j + s$ then $x - y \in W + \epsilon_i - \epsilon_j$. Thus we can recognize, from the difference $x - y$, that $x \in O(\mathcal{C}_i) + \epsilon_i + s$ and that $y \in \mathcal{C}_j + \epsilon_j + s$ and we are therefore able to label every element of O with the Γ_i from which it came. From this we deduce $(O(\Gamma_i) + s)_{i=1}^t$ for some fixed unknown $s \in \mathbb{T}$, by subtracting ϵ_i from every direction labelled with Γ_i. Hence we can reconstruct $[(O(\Gamma_i))_{i=1}^t]_{\mathbb{T}}$ from the $6m$-deck of S.

We are now ready to finish the proof. The group R of rigid motions contains a normal subgroup $\ker(\rho)$ isomorphic to $\mathbb{R}^2$ and consisting of the translations. We refer to this subgroup as $\mathbb{R}^2$ in what follows. The quotient $R/\mathbb{R}^2$ is isomorphic to $\mathbb{T}$.

Let $(\Gamma_i)_{i=1}^t$ be a list of all equivalence classes of oriented configurations of size 3 in S (deducible from the 3-deck of S), and let $(O_i)_{i=1}^t$ be a representative of $[(O(\Gamma_i))_{i=1}^t]_{\mathbb{T}}$. Note that we can determine a suitable $(O_i)_{i=1}^t$ from the 18-deck of S; we'll show that from this information we can reconstruct $[D_3(\mathbb{R}^2 \rightarrowtail S)]_{\mathbb{R}^2}$. Here it is crucial to understand what we are reconstructing. $\mathbb{R}^2$ acts on itself by translation. In turn there is an action of $\mathbb{T}$ on $\mathbb{R}^2$-isomorphism classes by $s.[C]_{\mathbb{R}^2} = [g.C]_{\mathbb{R}^2}$, where $g \in R$ is any rigid motion with $\rho(g) = s$, since if $\rho(g_1) = \rho(g_2)$ then $g_1 g_2^{-1}$ is a translation. Hence $\mathbb{T}$ acts on multisets of $\mathbb{R}^2$-isomorphism classes, and in particular on the deck $D_3(\mathbb{R}^2 \rightarrowtail S)$.

Starting from $(O_i)_{i=1}^t \in [(O(\Gamma_i))_{i=1}^t]_{\mathbb{T}}$ we build an element D of $[D_3(\mathbb{R}^2 \rightarrowtail S)]_{\mathbb{R}^2}$, i.e., we reconstruct $D_3(\mathbb{R}^2 \rightarrowtail S)$ up to a global rotation. For any $[C]_R \in D_3(R \rightarrowtail S)$ one can work out which Γ_i arise from orientations of C and for each one the sequence $(O_i)_{i=1}^t$ tells us which directions to pick for the corresponding elements of D. Clearly we have $D = D_3(\mathbb{R}^2 \rightarrowtail r.S)$ for some unknown $r \in R$. Now pick some unit vector $u \in \mathbb{R}^2$ such that no two points $x, y \in r.S$ have $\langle u, x\rangle = \langle u, y\rangle$; this property can be easily checked from D (indeed, from the 2-deck of $\mathbb{R}^2 \rightarrowtail r.S$) since it is invariant under translations of S. Then let $\lambda = \max\{\langle u, x\rangle - \langle u, y\rangle \,:\, x, y \in r.S\}$. Again, λ can be computed from the 2-deck of $\mathbb{R}^2 \rightarrowtail r.S$. Now $r.S$ can be recovered up to translation – it is a translate of

$$T = \big\{x \,:\, \{0, x, \lambda u\} \in D_3(\mathbb{R}^2 \rightarrowtail r.S)\big\}.$$

Thus, from some unknown $r \in R, x \in \mathbb{R}^2$ we have $r.S = x + T$. In particular $[S]_R$ is determined by the 18-deck of S. ■

4. Infinite subsets of $\mathbb{R}^2$

In this section we discuss the reconstructibility of some infinite subsets of the plane. We shall no longer be concerned with multisets. We immediately run into several examples of non-reconstructible sets.

Example 14. Let $S = (0,1)$ and let $S' = (0,1) \setminus \{\frac{1}{2}\}$. Clearly these sets are not isomorphic. On the other hand their decks both consist of an (uncountably) infinite number of copies of every finite configuration which is linear and has diameter strictly less than 1. Moreover these examples have the same k-deck (for every k) as any set of the form $(0,1) \setminus C$ where C is any countable subset of $(0,1)$. Since these are all mutually non- isomorphic this gives quite a large range of examples of non-reconstructible sets. (These examples are all reconstructible from their $\aleph_0$-decks.)

Example 15. Similarly if we take the disc $\{x \in \mathbb{R}^2 : |x| \leq 1\}$, it has the same k-deck, for every k, as the disc with a countable number of points (none of which is the origin) removed. Every configuration for which a copy appears in the disc can be rotated (in uncountably many ways) so as to avoid the missing points. In fact even the $\aleph_0$-deck does not distinguish these examples from one another. Thus the disc is not even $\aleph_0$-reconstructible.

Example 16. Let $\mathbb{P}$ be the standard symmetric probability distribution on the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N} \subset \mathbb{R}^2$. Pick two subsets $S, S' \subset \mathbb{N}$ at random according to $\mathbb{P}$. With probability 1 they will each contain infinitely many copies of every finite subset of $\mathbb{N}$ (and of course no copies of any other configuration) and will not be isomorphic. Thus we can find countable subsets of the plane which are not reconstructible.

We have given examples showing that if S is not compact, or has infinite automorphism group then S may not be reconstructible. The next result proves that otherwise there exists N_S depending only on S such that given an arbitrary set $S' \subset \mathbb{R}^2$ either $S \simeq S'$ or the N_S-decks of S and S' are different. This property of S we call being *finitely reconstructible.*

Theorem 6. *Every compact subset of the plane with finite automorphism group is finitely reconstructible.*

Our proof of this theorem will use the certification lemma, Lemma 3, to show that the existence of even one configuration C which appears in S but does not appear infinitely often in S is enough to ensure that that S is finitely reconstructible.

Definition 17. If $S \subset \mathbb{R}^2$ and $C \subset S$ is a finite subset with the property that the deck of S contains only finitely many copies of $[C]_R$ (or, equivalently, that S contains only finitely many copies of C) then C is called a *characteristic configuration in* S.

Lemma 7. *If* $S \subset \mathbb{R}^2$ *contains a characteristic configuration* C *of size* k *then* S *is* $18(2k+1)$*-reconstructible.*

Proof. Let S_0 be the subset of S consisting of points belonging at least one copy of C. For each $D \subset \mathbb{R}_+$ let S_D be the subset of S containing all points whose distances to at least two points of S_0 belong to D. Note that S_0 is finite and thus S_D is finite for all finite D. Also S_D is an increasing function of D, and $S = \bigcup_{|D|<\infty} S_D$.

We claim that for any D we can reconstruct S_D from the $18(2k+1)$-deck of S. Certainly S_D has a certificate of size $2k+1$ since $y \in \mathcal{S}_D$ if and only if it belongs to a pointed configuration $\langle C_1 \cup C_2 \cup \{y\}, y\rangle$ where $C_1, C_2 \simeq C$ and at least two of the distances from y to points in $C_1 \cup C_2$ belong to D. We therefore let $\mathcal{C}$ be the set of isomorphism classes of pointed configurations of this sort. By Lemma 3 and Theorem 5, S_D is reconstructible from the $18(2k+1)$-deck of S.

Now let H be the automorphism group of S. Clearly, since S has a characteristic configuration, H must be finite. For finite subsets D of $\mathbb{R}_+$ let H_D be the automorphism group of S_D. Clearly $H \leq H_D$ for all finite D and if $E \supset D$ then $H_E \leq H_D$. We claim that there is some finite $D_0 \subset \mathbb{R}_+$ such that $H = H_D$. To see this pick D_0 with $|H_{D_0}|$ minimal. Now suppose that $H < H_{D_0}$. Pick $h \in H_{D_0} \setminus H$. There must be some $x \in S$ with $hx \notin S$. Now pick $E \supset D_0$ with $x \in S_E$. Then $H_E \leq H_{D_0}$ and $h \in H_{D_0} \setminus H_E$, contradicting the minimality of $|H_{D_0}|$.

Now since we can reconstruct S_D for all finite D we build

$$\{[S_D]_R : |D| < \infty, D_0 \subset D\}.$$

We fix a copy T_0 of S_{D_0} and choose $T_D \in [S_D]_R$ such that the copy of S_{D_0} in T_D is equal to T_0. We claim that $\bigcup_{|D|<\infty, D_0 \subset D} T_D \simeq S$. If $D, E \supset D_0$ and we have chosen T_D and T_E to agree on T_0, then $T_D = g_D S_D$ and $T_E = g_E S_E$ for some $g_D, g_E \in R$ such that $g_D^{-1} g_E T_0 = T_0$. Thus, by the minimality of H_{D_0}, we have $g_D^{-1} g_E T_D = T_D$ and $g_D g_E^{-1} T_E = T_E$. Thus T_D and T_E are consistent, and a similar argument shows that both agree with $T_{D \cup E}$. The union of $\{T_D : D_0 \subset D, |D| < \infty\}$ is therefore a set isomorphic to S. ■

Before completing the proof of Theorem 6 we note some simple facts concerning subgroups of R. We write $\mathbb{T}_x$ for the subgroup of R consisting of all rotations about x, and $\mathbb{Z}_{n,x}$ for the subgroup of all rotations about x through an integer multiple of $2\pi/n$ radians. We will need some elementary topological properties of R. We note that any element $g \in R$ rotates all line segments through some fixed angle $\alpha(g)$ and we define a metric on R by $d(g, g') = |g((0,0)) - g'((0,0))| + d_{\mathbb{T}}(\alpha(g), \alpha(g'))$. This metric makes R into a topological group.

Proposition 8. *If K is any compact subgroup of R then there exists x in $\mathbb{R}^2$ such that K is either $\mathbb{T}_x$ or $\mathbb{Z}_{n,x}$ for some n.*

Proof. Clearly the set of iterates of a (non-trivial) translation form an infinite discrete set, thus K cannot contain a translation. Since the commutator of two rotations about different centres is a translation, K cannot contain such a pair. Therefore K consists purely of rotations about some fixed center x. The set of allowed rotations is either discrete, in which case K is easily seen to be $\mathbb{Z}_{n,x}$ for some n, or dense in $\mathbb{T}_x$, in which case (since K is closed) $K = \mathbb{T}_x$. ■

Lemma 9. *If S is a compact subset of $\mathbb{R}^2$ with $\mathrm{Aut}(S)$ finite and $C \subset S$ is finite then for every $\epsilon > 0$ there exists a finite superset $E_\epsilon \supset C$ such that whenever $E_1, E_2 \subset S$ have $E_1, E_2 \simeq E_\epsilon$ and $g \in R$ maps D_1 to E_2 then g is within ϵ of some automorphism of R.*

Proof. For any finite subset $E \subset S$ we set

$$K_E = \{g \in R \,:\, g(E) \subset S\} \setminus \{g \in R \,:\, d(g, \mathrm{Aut}(S)) < \epsilon\}.$$

This is clearly a compact subset of R. Suppose that no finite subset E as described in the lemma exists. Then the collection $\{K_E \,:\, E \text{ finite}, C \subset E\}$ has the finite intersection property and thus $\bigcap_{|E|<\infty, E\supset C} K_E$ is non- empty. This intersection consists however only of rigid motions which map S to S and are at least ϵ away from any automorphism of S which is a contradiction. ■

We will use Lemma 9 to restrict our search for a characteristic configuration in S to subsets which only have "nearby" copies. To be precise, if E_1, $E_2 \subset S$ are both copies of one another we will write $d(E_1, E_2)$ for $\min\{d(g, \mathrm{id}) \,:\, g(E_1) = E_2\}$.

Proof of Theorem 6. Note first that $\mathrm{Aut}(S)$, being finite, must be $\mathbb{Z}_{n,x}$ for some n, x, by Proposition 8. Put $\epsilon = 1/2n$. Let M be the diameter of S and let C consist of two points in S at distance M apart. By Lemma 9 there exists E containing C such that any two copies of E in S are related by a rigid motion which is within ϵ of an automorphism of S. Pick a copy E' of E with $E' \subset S$ and distinguish a copy C' of C in E'. Among all images $g(E')$ in S with $d(g, \mathrm{id}) \le \epsilon$ pick a pair E_1, E_2 with the angle between their distinguished copies of C' being maximal. This is possible by compactness. Note that it is an elementary geometric fact that, since $M = \mathrm{diam}(S)$, there is at most one copy of C with any given orientation. Now it is clear that $E'' = E_1 \cup E_2$ is a characteristic configuration for S, indeed $[E'']$ occurs with multiplicity at most $|\,\mathrm{Aut}(S)|$ in the k-deck of S. For, if $F'' \subset S$ is a copy of E'' then by hypothesis $F'' = g(E'')$ for some $g \in R$ with $d(g, \mathrm{Aut}(S)) \le \epsilon$. Suppose that $h \in \mathrm{Aut}(S)$ has $d(h, g) < \epsilon$. Thus $h^{-1}(F'')$ is the image of E'' under a rigid motion at most ϵ from the identity. This however, by the construction of E'' ensures that $h^{-1}(F'') = E$ and so $F'' = h(E'')$. In summary, the only copies of E'' in S are the images of E'' under $\mathrm{Aut}(S)$. Now by Lemma 7 we are done. ■

Example 18. Consider the "notched disc"

$$N_\epsilon = \{x : |x| \leq 1, |x - (1,0)| \geq \epsilon\}.$$

Any finite configuration C for which the multiplicity of $[C]$ in $D(N_\epsilon)$ is different than in the deck of the unnotched disc must have $|C| \geq \pi / \sin^{-1} \epsilon$ (since otherwise either C wouldn't turn up in the disc or uncountably many rotations of C would fit in the notched disc). Thus there is no uniform bound N such that all compact subsets of $\mathbb{R}^2$ with finite automorphism group are reconstructible from their N-decks.

Remark 19. It is worth remarking that if S, T are compact subsets with $\mathrm{Aut}(S) = \mathbb{T}_x$ and $\mathrm{Aut}(T) = \mathbb{T}_y$ and $D_3(S) = D_3(T)$ then $S \simeq T$. To see this note that for such S with diameter $2R$ if we pick an arbitrary unit vector u we have $S \simeq \mathbb{T}_{(0,0)}.\{\lambda u : \{-Ru, \lambda u, Ru\} \in D_3(S)\}$.

We have seen that if S is bounded but not closed then it may not be reconstructible even from its $\aleph_0$-deck. However, we *can* reconstruct the closure of S.

Theorem 10. *If $S \subset \mathbb{R}^2$ is bounded then $[\bar{S}]_R$ can be reconstructed from the $(< \omega)$-deck of S.*

Proof. Let $K = \bar{S}$. Given two finite subsets $C, C' \subset \mathbb{R}^2$ we say that they are ϵ-copies of one another if there exists a map $\phi : C \to C'$ and a rigid motion $g \in R$ such that $|\phi(x) - g(x)| \leq \epsilon$ for all $x \in C$. By compactness, for any finite subset $C \subset \mathbb{R}^2$, the deck of K contains $[C]$ if and only if for all $\epsilon > 0$ there exists an ϵ-copy C_ϵ of C such that $[C_\epsilon] \in D(S)$. However it may be hard to compute the multiplicity of $[C]$ in $D(K)$. It turns out that we can get away with only using the "reduced deck" of K: the set of isomorphism classes of finite subsets of K. Let $\tilde{D} = \tilde{D}(K)$ be this set. By the observation above, $\tilde{D}$ is reconstructible from $D(S)$.

We now show that the automorphism group of K is reconstructible (up to isomorphism) from $\tilde{D}$. Note first that by Proposition 8 the automorphism group of K, which is certainly compact, is either $\mathbb{Z}_{n,x}$ or $\mathbb{T}_x$ for some $x \in \mathbb{R}^2$. If H is a group of rigid motions, we say that K is *H-full* if every finite subset $C \subset K$ can be extended to a configuration $C_G \subset K$ with $H \leq \mathrm{Aut}(C_H)$. Clearly if $H \leq \mathrm{Aut}(K)$ is finite then K is H-full, since for $C \subset K$ we can take C_H to be the union $\bigcup_{h \in H} h(C)$. In particular, if $\mathrm{Aut}(K)$ is infinite then K is $\mathbb{Z}_n$-full for all n. On the other hand, if $\mathrm{Aut}(K)$ is finite then we know from the proof of Lemma 7 that there is a (finite) subset $C \subset K$ such that $\mathrm{Aut}(C) = \mathrm{Aut}(K)$ and every extension D with $C \subset D \subset K$ has $\mathrm{Aut}(D) \leq \mathrm{Aut}(K)$. Thus if $\mathrm{Aut}(K)$ is finite then K is H'-full if and only if $H' \leq \mathrm{Aut}(K)$. By this observation we see that the isomorphism type of $\mathrm{Aut}(K)$, that is $\mathbb{Z}_n$ or $\mathbb{T}$, can be reconstructed from $\tilde{D}$.

Now that we know $\mathrm{Aut}(K)$ we can reconstruct as follows. If $\mathrm{Aut}(K)$ is finite then

$$K = \bigcup_{D \supset C, [D] \in \tilde{D}} D.$$

where C is as above; moreover the right hand side can be reconstructed up to rigid motion from $\tilde{D}$. On the other hand if $\mathrm{Aut}(K)$ is infinite then we can reconstruct K as in Theorem 19, from the reduced 3-deck of K, which can be determined from $\tilde{D}$. ∎

We can also attempt to weaken the boundedness hypothesis. However, as the following example shows, we cannot remove it altogether.

Example 20. There are closed subsets of the plane that cannot be reconstructed even from the set of isomorphism classes of *all* their subsets. For instance $S = \{(x, y) : x, y \geq 0\}$ and $T = \{(x, y) : x, y \geq 0, x + y \geq 1\}$ each contain a copy of the other and both sets contain any configuration (of arbitrary cardinality) either uncountably often or not at all.

In Theorem 6 the compactness of S serves to limit the complexity of S. However some unbounded sets are finitely- reconstructible. We impose a different condition to ensure that the complexity is not too high, namely that S can be covered by a finite number of lines. This is clearly not enough to prove even finite reconstructibility, as Example 16 shows. However the counterexamples are all contained in finite collections of parallel lines. This last property is of course equivalent to that of $P_u(S)$ being finite for some unit vector u, where P_u is the orthogonal projection from $\mathbb{R}^2$ onto the line through the origin perpendicular to u.

Theorem 11. *If $S \subset \mathbb{R}^2$ is contained in the union of the finite set of lines $\mathcal{L}$ and the projection $P_u(S)$ is infinite for all unit vectors u then S is 162-reconstructible.*

We first prove a lemma showing that certain configurations appear only finitely many times on a given collection of lines.

Lemma 12. *If L_1, L_2, L_3 are three pairwise non-parallel lines in the plane and C is a configuration consisting of three points x_1, x_2, x_3 in a straight line with $|x_1 - x_2| = d_1$ and $|x_2 - x_3| = d_2$ then there are only finitely many images $g(C)$ of C with $g(x_i) \in L_i$, $i = 1, 2, 3$.*

Proof. Parameterize the lines L_1, and L_2 using parameters s and t respectively: $z_1(s) = a_1 + sv_1$ and $z_2(t) = a_2 + tv_2$. Pick $w_3 \in \mathbb{R}^2 \setminus \{0\}$, $\lambda \in \mathbb{R}$ such that $L_3 = \{z : \langle z, w_3\rangle = \lambda\}$. The condition $|z_1(s) - z_2(t)|^2 = d_1^2$ is a quadratic equation for s, t. Let $P(s, t) = z_2(t) + \frac{d_2}{d_1}(z_2(t) - z_1(s))$. This is the third point of the copy of C having $g(x_1) = z_1(s)$ and $g(x_2) = z_2(t)$. Values of the parameters s, t describe a copy of C if and only if (s, t) lies on the conic $|z_1(s) - z_2(t)|^2 - d_1^2 = 0$ and the straight line $\langle P(s, t), w_3\rangle - \lambda = 0$, so there are at most two solutions. ∎

Proof of Theorem 11. Let $\mathcal{L}$ be partitioned into parallel classes of lines $\mathcal{L}_1$, $\mathcal{L}_2$,..., $\mathcal{L}_k$, parallel to directions $u_1, u_2, \ldots, u_k$. Let the *ratios appearing in the i^{th} parallel class* be the set of ratios $|x_2 - x_1|/|x_3 - x_1|$ where $x_1, x_2, x_3 \in \bigcup \mathcal{L}_i$ are collinear points belonging to distinct lines in $\mathcal{L}_i$. Note that this set is finite, and is the same as if one required that the line on which x_1, x_2, x_3 lie were perpendicular to those in $\mathcal{L}_i$. Let us write R_i for this set of ratios and let $R = \bigcup_1^k R_i$. Pick a line $L \in \mathcal{L}$ containing infinitely many points; we may assume that $L \in \mathcal{L}_1$. Pick 3 points $x_1, x_2, x_3 \in L \in \mathcal{L}_1$ such that the ratio $|x_2 - x_1|/|x_3 - x_1|$ does not belong to R. This is possible simply by picking x_1 and x_2 arbitrarily on L and then avoiding a finite number of possibilities for x_3. Now consider $P_{u_1}(S)$. It is, by hypothesis, infinite, and therefore there exists $y \in S$ such that $P_{u_1}(y) \notin P_{u_1}(\bigcup \mathcal{L}_1)$. We claim that $\{x_1, x_2, x_3, y\}$ is a characteristic configuration in S. Note first that by Lemma 12 the configuration $\{x_1, x_2, x_3\}$ only occurs a finite number of times with the images of x_1, x_2, x_3 not all on one line from L_i. On the other hand given a line $L \in \mathcal{L}_i$ there exist only finitely many copies of $\{x_1, x_2, x_3, y\}$ with the images of x_1, x_2, x_3 on L since there are at most two such copies with the image of y on L' for each $L' \in \mathcal{L} \setminus \{L\}$. By Lemma 7, it follows that S is (18×9)-reconstructible. ∎

5. Further questions

There are several extremely interesting questions still open. In this paper we have shown that finite subsets of the plane can be reconstructed from their 18-decks. However, we know very little in higher dimensions.

Conjecture 1. For all $n \geq 1$ there exists $k = k(n)$ such that every finite multiset in $\mathbb{R}^n$ can be reconstructed from its k-deck.

The main difficulty here seems to be reconstructing finite subsets of S^{n-1} under the action of $SO(n)$. In Section 3 we showed that finite subsets of S^1 are 6-reconstructible under the action of $SO(1)$. In [32] we show that a similar result for S^{n-1} would prove Conjecture 1. Note that, for $n \geq 3$, $SO(n)$ presents some difficulties absent in the planar case — $SO(n)$ is nonabelian, and there is no "approximating sequence" of finite subgroups analogous to $\mathbb{Z}_n < \mathbb{T}$.

A seemingly more general question is that of reconstructing finite multisets in $\mathbb{R}^n$ up to isometry from the k-deck (given up to isometry). In fact, it is shown in [32] that if finite multisets in $\mathbb{R}^n$ are reconstructible up to rigid motion from their k-decks then they can be reconstructed up to isometry from their $2k$-decks (given up to isometry).

Returning to two dimensions, we can ask about the reconstructibility of the hyperbolic plane under the action of its isometry group. Very much in this line also is the problem of reconstructing subsets of the extended complex plane $\mathcal{C}_\infty$ under the action of the group of Möbius transformations. We conjecture that in both cases there is a constant k such that all finite multisets are k-reconstructible (under the appropriate group action).

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