# Independent sets and repeated degrees

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**Abstract.** We answer a question of Erdős, Faudree, Reid, Schelp and Staton by showing that for every integer $k \geq 2$ there is a triangle-free graph $G$ of order $n$ such that no degree in $G$ is repeated more than $k$ times and $\mathrm{ind}(G) = (1 + o(1))n/k$.

## §1. Introduction

In [2], Erdős, Fajtlowicz and Staton proved that every triangle-free graph $G$ in which no degree is repeated more than twice is bipartite and thus has independence number at least $|G|/2$. In this paper, we consider triangle-free graphs in which no degree is repeated more than $k$ times. What can one say about the independence number of such graphs? As observed in [3], if $G$ is a triangle-free graph of order $n$, and no degree in $G$ is repeated more than $k$ times, then some vertex $v$ has degree at least $(n/k) - 1$; if $G$ has no isolated vertices then some vertex $v$ has degree at least $n/k$. Then, since $\Gamma(v)$ is an independent set, we must have $\mathrm{ind}(G) \geq n/k$. In fact, Erdős, Faudree, Reid, Schelp and Staton [3] asked whether this inequality is best possible. In other words, are there graphs $G$ of arbitrarily large order $n$ such that $G$ is triangle-free, no degree in $G$ is repeated more than $k$ times, and $\mathrm{ind}(G) = (1 + o(1))n/k$? In [3] it is shown that for $k = 2$ and $k = 4$ this is indeed the case. Our main aim is to prove that the inequality is essentially best possible for all values of $k$.

**Theorem 1.** *For every integer $k \geq 2$, and for every $\epsilon > 0$, there is an $n_0(k, \epsilon)$ such that if $n \geq n_0(k, \epsilon)$ then there is a triangle-free graph $G$ of order $n$ such that no degree in $G$ is repeated more than $k$ times and*

$$\mathrm{ind}(G) \leq (1 + \epsilon)|G|/k.$$

Erdős, Faudree, Reid, Schelp and Staton [3] investigated $K_r$-free graphs with few repeated degrees. They showed that, for $r \geq 5$ and $k \geq 2$, there exist $K_r$-free graphs of order $n$ with independence number $o(n)$ and no degree repeated more than $k$ times, but that no such graphs exist for $r = 4$ and $k = 3$. In [1] it is proved that there exist $K_4$-free graphs of order $n$ with independence number $o(n)$ and no degree repeated more than 5 times. This leaves open only the case $k = 4$.

## §2. Proof of Theorem 1

We will make use of the following immediate consequence of the Max-Flow Min-Cut Theorem.

**Lemma 2.** *Suppose $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and $e_1 \geq e_2 \geq \cdots \geq e_n > 0$ are two sequences such that*

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} e_i. \tag{1}$$

*Then there is a bipartite graph with bipartition $(V_1, V_2)$ and degrees $d_1, \ldots, d_n$ in $V_1$ and $e_1, \ldots, e_n$ in $V_2$ iff, for all $1 \leq i, j \leq n$, we have*

$$\sum_{h=1}^{i} d_h - \sum_{h=n-j+1}^{n} e_h \leq ij. \tag{2}$$

□

The other result that we shall need below is the following lemma about triangle-free graphs (we note that we could prove a much stronger result, but this is all we shall need).

**Lemma 3.** *Let $k \geq 2$ be an integer and let $\epsilon > 0$. If $n$ is even and sufficiently large then there exists a $k$-partite triangle-free graph with $n$ vertices in each vertex class such that every vertex has degree $\lceil \log n \rceil$ and the largest independent set has size at most $(1+\epsilon)n$.*

**Proof.** Let $n = 2n_0$, and let $p$ be the maximal integer such that

$$\binom{k}{2} p + 1 < \lceil \log n \rceil.$$

Fix $0 < c < 1/2$ such that $2ck < \epsilon$. Let $G$ be the random $k$-partite graph with vertex classes $V_1, \ldots, V_k$, each of size $n_0$, obtained by taking the union of $p$ independent random matchings between each pair of vertex classes. Clearly

$$\Delta(G) \leq \binom{k}{2} p < \lceil \log n \rceil.$$

If $\mathrm{ind}(G) > (1+ck)n_0$ then there exists $i \neq j$ and sets $W_i \subset V_i$ and $W_j \subset V_j$ such that $|W_i| \geq cn_0$, $|W_j| \geq cn_0$ and $e(W_i, W_j) = 0$. Let $X(G)$ be the number of such pairs with $|W_i| = |W_j| = \lceil cn_0 \rceil$. Then

$$\begin{aligned}
\mathbb{E}(X(G)) &= \binom{k}{2}\binom{n_0}{cn_0}^2 \left(\left(1-\frac{1}{n_0}\right)^p\right)^{\lceil cn_0 \rceil^2} \\
&< k^2 \frac{1}{2\pi n_0 c(1-c)c^{2cn_0}(1-c)^{2(1-c)n_0}} e^{-(1/n_0)pc^2 n_0^2} \\
&< \left(\frac{(1+o(1))e^{-pc^2}}{c^{2c}(1-c)^{2(1-c)}}\right)^{n_0} \\
&< 2^{-n_0} \qquad (3)
\end{aligned}$$

for $n$ sufficiently large, since $p \to \infty$ as $n_0 \to \infty$. Thus, with probability $1 - o(1)$, $G$ contains no independent set of size $(1+ck)n_0$.

Now let $Y(G)$ denote the number of triangles in $G$. Then

$$\begin{aligned}
\mathbb{E}(Y(G)) &= \binom{k}{3} n_0^3 \left(1 - \left(1 - \frac{1}{n_0}\right)^p\right)^3 \\
&< \binom{k}{3} n_0^3 \left(\frac{p}{n_0}\right)^3 \\
&= \binom{k}{3} p^3 \\
&< \lceil \log n \rceil^3. \qquad (4)
\end{aligned}$$

Thus, with probability greater than $1/2$, $G$ contains at most $\lceil \log n \rceil^3$ triangles. We deduce from (3) and (4) that we can find some $G_0$ with

$$\mathrm{ind}(G_0) \leq (1+ck)n_0$$

and

$$Y(G_0) < \lceil \log n \rceil^3.$$

Now pick one edge from each triangle in $G$ and delete it. Note that this increases $\mathrm{ind}(G)$ by less than $\lceil \log n \rceil^2 < ckn_0$, provided $n_0$ is sufficiently large. We get a triangle-free $k$-partite graph $G_1$ with vertex classes $V_1, \ldots, V_k$ such that

$$|V_1| = \cdots = |V_k| = n_0$$

and

$$\Delta(G) \le \binom{k}{2} p < \lceil \log n \rceil - 1$$

and

$$\mathrm{ind}(G) < (1 + 2ck) n_0.$$

Let $G_2$ be another copy of $G_1$, with vertex classes $W_1, \ldots, W_k$. We will add edges between $V_i$ and $W_i$, for each $i$, so as to get a $\lceil \log n \rceil$-regular graph. Suppose $V_i = \{v_1, \ldots, v_{n_0}\}$ and $W_i = \{w_1, \ldots, w_{n_0}\}$, and let $d_j = \lceil \log n \rceil - d(v_j)$, for $j = 1, \ldots, n$. Now $1 \le d_j \le \lceil \log n \rceil$ for each $j$, so it follows easily from Lemma 3 that there is a bipartite graph $B_i$ with degrees $d_1, \ldots, d_{n_0}$ in each vertex class. We add this graph $B_i$ between $V_i$ and $W_i$, for each $i$, and call the resulting graph $H$. We now have a $\lceil \log n \rceil$-regular graph. For $i = 1, \ldots, k$, let

$$S_i = V_i \cup W_{i+1},$$

where we define $W_{k+1} \equiv W_1$. Then each $S_i$ is an independent set of size $n$, $H$ is triangle-free and

$$\mathrm{ind}(H) \le \mathrm{ind}(G_1) + \mathrm{ind}(G_2) < (1 + 2ck)n < (1 + \epsilon)n.$$

$\square$

Armed with these lemmas, we are now ready to prove Theorem 1.

**Proof of Theorem 1.** We prove the theorem for numbers $n$ of a specific form; the graphs we obtain can easily be modified for other values of $n$.

(i) For $k = 2$ the theorem is easily seen to be true. We define the bipartite graph $B_n$ with vertex classes $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ by

$$x_i y_j \in B_n \text{ iff } i \le j.$$

Then no degree is repeated more than twice, and it is easily seen that $\mathrm{ind}(B_n) = |B_n|/2$.

(ii) For $k \geq 3$ we need a more complicated construction than for the bipartite case. Pick integers $m$ and $l$ such that

$$mn^2 = (k-2)l^2,$$

which means

$$l \sim n\sqrt{m/(k-2)}.$$

We can do this for arbitrarily large values of $l$, $m$ and $n$. Let $G_0$ be a $\lceil \log n \rceil$-regular triangle free $k$-partite graph with vertex classes $V_1, \ldots, V_k$ such that $|V_1| = \cdots = |V_k| = n$, and $\mathrm{ind}(G) < (1+\epsilon)n$. We know from Lemma 3 that such a graph exists. Now, for $i = 1, \ldots, 2m$ let $G_i$ be a copy of $G_0$ with vertex classes $V_1^i, \ldots, V_k^i$. For $i = 1, \ldots, k$, let $R_0^i$ be an independent set of size $kl$, and let $R_1^i, \ldots, R_k^i$ be independent sets of size $l$. We will write superscripts mod $k$, so for instance $V_j^{k+1} \equiv V_j^1$. Note that the largest independent set in this graph has size at most $2m(1+\epsilon)n + 2k^2 l < 2mn(1+3k^2\epsilon)$, provided $m$ is sufficiently large ($m > k^3$ will do). Thus the independence condition is fulfilled. We will add further edges to get a triangle-free graph in which no degree is repeated more than $k$ times.

For $i = 1, \ldots, k$ we add edges

$$\bigcup \{K(V_a^i, V_b^{i+1}) : m+1 \leq a < b \leq 2m\} \cup K(\bigcup_{j=m+1}^{2m} V_j^i, \bigcup_{j=1}^{k} R_j^{i+1}),$$

and

$$\bigcup \{K(R_a^i, R_b^{i+1}) : 1 \leq a < b \leq k\} \cup \bigcup_{h \neq i} K(R_h^i, R_h^{i+1}).$$

(The condition $h \neq i$ ensures that we do not get triangles when $k = 3$.)

For each $i$, let

$$A_i = R_0^i \cup \bigcup_{j=1}^{m} V_j^i$$

and

$$B_i = \bigcup_{j=1}^{k} R_j^i \cup \bigcup_{j=m+1}^{2m} V_j^i.$$

Then, for each $i$, $A_i \cup B_i$ is an independent set. There are $kl$ vertices in $A_i$ with degree 0 and $mn$ vertices with degree

$$\lceil \log n \rceil. \tag{5}$$

There are $mn$ vertices in $B_i$ with degree

$$(m-1)n + \lceil \log n \rceil + kl, \tag{6}$$

$2l$ vertices with degree

$$mn + kl \tag{7}$$

and $(k-2)l$ vertices with degree

$$mn + (k+1)l. \tag{8}$$

Now the sums of the degrees in $A_i$ is

$$mn\lceil \log n \rceil,$$

while the sum of the degrees in $B_i$ is

$$\begin{aligned} mn((m-1)n + \lceil \log n \rceil + kl) + 2l(mn+kl) + (k-2)l(mn+(k+1)l) \\ = mn\lceil \log n \rceil + (mn+kl)^2 - mn^2 + kl^2 - 2l^2 \\ = mn\lceil \log n \rceil + (mn+kl)^2, \end{aligned} \tag{9}$$

provided $mn^2 = (k-2)l^2$, which is true by our choice of $m$ and $l$. For each $i$, We will add a bipartite graph between $A_i$ and $B_i$ so that the vertices in $A_i$ have degrees

$$1, \ldots, mn + kl$$

and the vertices in $B_i$ have degrees

$$mn + kl + 1, \ldots, 2mn + 2kl.$$

If we can do this, then no degree in the resulting graph is repeated more than $k$ times. It follows from (5)-(8) that it is enough to find a bipartite graph with degrees

$$1, \ldots, kl, kl - \lceil \log n \rceil + 1, \ldots, kl + mn\lceil \log n \rceil \tag{10}$$

in one class, and

$$\begin{aligned}
&n-\lceil\log n\rceil+1,\ldots,n-\lceil\log n\rceil+mn,\\
&mn+1,\ldots,mn+2l,\\
&mn+l,\ldots,mn+(k-1)l
\end{aligned}\tag{11}$$

in the other class.

We claim that this is possible. We will prove this by a straightforward, although rather tedious, application of Lemma 2. Note first that it follows from (9) that the sequences (10) and (11) have the same total, so (1) is satisfied. We now check condition (2).

Rearranging (10), we get

$$\begin{aligned}
&1,\ldots,kl-\lceil\log n\rceil,\\
&kl-\lceil\log n\rceil+1,kl-\lceil\log n\rceil+1,\ldots,kl,kl\\
&kl+1,kl+2,\ldots,kl+mn-\lceil\log n\rceil,
\end{aligned}\tag{12}$$

and rearranging (11) we get

$$\begin{aligned}
&n-\lceil\log n\rceil+1,\ldots,mn,\\
&mn+1,mn+1,\ldots,mn+n-\lceil\log n\rceil,mn+n-\lceil\log n\rceil,\\
&mn+n-\lceil\log n\rceil+1,\ldots,mn+l+1,\\
&mn+l,mn+l,\ldots,mn+2l,mn+2l,\\
&mn+2l+1,\ldots,mn+(k-1)l.
\end{aligned}\tag{13}$$

Let us relabel (12) and (13) as $e_N,\ldots,e_1$ and $d_N,\ldots,d_1$ respectively, where $N=mn+kl$. Note that in (2), for a given value of $i$, the only value of $j$ we need check is $j(i)=\min\{h:e_h\geq i\}$. For $i=1,\ldots,kl-\lceil\log n\rceil$, we have

$$j(i)=N-i;$$

for $i=kl-\lceil\log n\rceil+1,\ldots,kl$ we have

$$j(i)=N-2i+kl-\lceil\log n\rceil;$$

and for $i = kl + 1, \dots, mn + kl$, we have

$$j(i) = \max\{N - i - \lceil \log n \rceil, 1\}.$$

It is now an easy but tedious calculation to check that (12) and (13) satisfy (2).

□

We remark that it is straightforward to give bounds for $n_0(k, \epsilon)$ in Theorem 1 (and so bound the $o(1)$ term in $(1 + o(1))n/k$). We do not, however, know the best possible bound. In particular, it would be of interest to determine whether for every $k$ there is some $c(k)$ such that for every $n$ there is a triangle-free graph of order $n$ with no degree repeated more than $k$ times and independence number at most $n/k + c(k)$.

The proof we have given for Theorem 1 is essentially probabilistic, so it does not give a construction. It would be interesting to find a constructive proof of the theorem, although this might not be particularly easy.

There are many further questions of the same type: given a graph $H$ and integer $k$, what can we say about the independence number of graphs that have no degree occurring more than $k$ times and contain no induced copy of $H$?

## References


[1] B. Bollobás, Degree multiplicities and independent sets in $K_4$-free graphs, *to appear.*

[2] P. Erdős, S. Fajtlowicz and W. Staton, Degree sequences in triangle-free graphs, *Discrete Math.* **92** (1991), 85-88.

[3] P. Erdős, R. Faudree, T.J. Reid, R. Schelp and W. Staton, Degree sequence and independence in $K(4)$-free graphs, *Discrete Math.* **141** (1995), 285-290.