A note on simplicial cliques

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Abstract

Motivated by an application in condensed matter physics and quantum information theory, we prove that every non-null even-hole-free claw-free graph has a simplicial clique, that is, a clique K such that for every vertex $v \in K$, the set of neighbours of v outside of K is a clique. In fact, we prove the existence of a simplicial clique in a more general class of graphs defined by forbidden induced subgraphs.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. A *hole* in G is an induced cycle of length at least four. By a *path* we always mean an induced path. For $v \in V(G)$ we denote by $N(v)$ the set of neighbours of v (so $v \notin N(v)$). G is *even-hole-free* if all holes in G have odd length, and G is *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$. A non-empty set $K \subseteq V(G)$ is a *simplicial clique* if K is a clique, and for every $v \in K$ we have

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that $N(v) \setminus K$ is a clique. The unique element of a simplicial clique of size one is called a *simplicial vertex*.

This paper is motivated by a question from condensed matter physics and quantum information theory concerning so-called *spin models*, i.e. models of interacting qubits (two-level quantum systems). Each spin model is defined by a Hamiltonian operator, and to every such Hamiltonian one can associate a graph, called its *frustration graph*. In [4] a new method is given that allows us to "solve a model" (meaning in this case to find the spectrum and the eigenvectors of the Hamiltonian) whose frustration graph is even-hole-free, claw-free, and has a simplicial clique. This augments earlier results of [1] where it is shown that models whose frustration graphs are line-graphs are solvable using certain classical tools. The solution method of [4] uses only the structure of the frustration graph, and it is an extension of both [5] and [6]. The authors of [4] raised a question:

1.1. Question: *Does every non-null even-hole-free claw-free graph have a simplicial clique?*

In other words, does their new solvability result hold for all models whose frustration graphs are even-hole-free and claw-free? In this note we answer their question affirmatively (the "dome of an edge" is defined before the statement of (2.2)); in fact we prove a stronger result:

1.2. *Let G be a non-null even-hole-free claw-free graph.*

1. *If G is chordal, then G has a simplicial vertex.*
2. *For every hole H of G there is an edge ab of H such that the dome of ab is a simplicial clique.*

In particular, G has a simplicial clique.

We have an even stronger result, describing explicitly the structure of all such graphs, but the proof is harder, and we do not present it here. The main result of this paper is 2.2 which is a strengthening of 1.2, and we will explain it in Section 2.

We remark that the answer to 1.1 becomes negative if we omit either the assumption that the graph is even-hole-free or that the graph is claw-free. The complement of a cycle of odd length at least seven is a claw-free graph with no simplicial clique. Moreover, the square of a cycle of length at least nine is an example of a C_4-free claw-free graph with no simplicial clique. (The *square* of a graph G is the graph obtained from G by making every vertex adjacent to all its second neighbours.) And here is an example of an even-hole-free graph rather than just C_4-free. Let k be an odd positive integer. The following is a construction of an even-hole-free graph G_k with $2k$ vertices and with no simplicial clique. Let the vertex set of G_k be the union of k disjoint pairs of adjacent vertices $\{a_i, b_i\}$ where $i \in \{1, \ldots, k\}$. For $i = \{1, \ldots, k-1\}$ add the edges $a_i a_{i+1}, a_i b_{i+1}, b_i a_{i+1}$; add also the edges $a_k a_1, a_k b_1, b_k a_1$. There are no more edges in G. It is easy to check that G_k is even-hole-free and has no simplicial clique.

In [2] an algorithm is presented that finds a simplicial clique in a claw-free graph if one exists. The authors of [4] also asked if that algorithm can be simplified when the input is known to be even-hole-free. An easy corollary of our main result 1.2 is such a simpler, but slower, algorithm 4.1. In fact 4.1 works under the more general assumptions of 2.2.

2 A strengthening

The goal of this section is to present our main result 2.2.

Let G be a graph. For $X \subseteq V(G)$ we denote by $G[X]$ the graph induced by G on X. For $A \subseteq V(G)$ and $x \in V(G) \setminus A$, we say that x is *complete* to A if x is adjacent to every element of A, and that x is *anticomplete* to A if x is non-adjacent to every element of A. Two disjoint subsets $A, B \subseteq V(G)$ are *complete* to each other if every vertex of B is complete to A, and *anticomplete* to each other if every vertex of B is anticomplete to A.

Next we define a few types of graphs. A graph is called *chordal* if it has no holes. A *jewel* is a graph consisting of a hole $H = h_1\text{-}\dots\text{-}h_k\text{-}h_1$ with $k \geq 4$ and a vertex $v \notin V(H)$ such that $N(v) \cap V(H) = \{h_1, h_2, h_3, h_4\}$. A *line wheel* is a graph consisting of a hole $H = h_1\text{-}\dots\text{-}h_k\text{-}h_1$ with $k \geq 6$ and a vertex $v \notin V(H)$ such that there exists $i \in \{4, \dots, k-2\}$ with $N(v) \cap V(H) = \{h_1, h_2, h_i, h_{i+1}\}$. A *short prism* is a graph consisting of a hole $h_1\text{-}h_2\text{-}h_3\text{-}h_4\text{-}h_1$ and a path $p_1\text{-}\dots\text{-}pk$ such that $\{p_1, \dots, p_k\} \cap \{h_1, h_2, h_3, h_4\} = \emptyset$, p_1 is adjacent to h_1 and to h_2, and p_k is adjacent to h_3 and to h_4, and there are no other edges between $\{p_1, \dots, p_k\}$ and $\{h_1, h_2, h_3, h_4\}$. Finally, the *seven-antihole* is the complement of a cycle of seven vertices. These graphs are depicted in Figure 1.

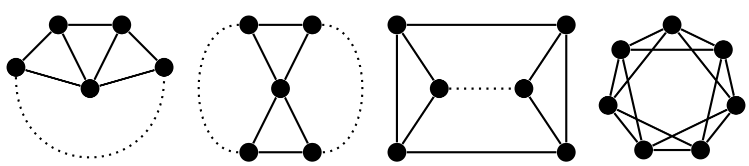

Figure 1: A jewel, a line wheel, a short prism and a seven-antihole (here dotted lines represent paths).

In what follows, whenever graph containment is mentioned, we will mean containment as an induced subgraph. We say that a graph G is *clean* if G is claw-free and contains no jewel, line wheel, short prism or seven-antihole. Note that clean graphs may contain even holes.

First we show:

2.1. *If G is claw-free and even-hole-free, then G is clean.*

Proof. Since G is even-hole free, and in particular C_4-free, G does not contain short prisms or seven-antiholes. Since a jewel contains a hole H of length k, and a hole h_1-v-h_4-h_5-$\dots$-h_k-h_1 of length $k-1$, G does not contain a jewel. Finally, at least one of the holes H, h_2-$\dots$-h_i-v-h_2, and h_{i+1}-$\dots$-h_1-v-h_{i+1} is even, and so G does not contain a line wheel. This proves 2.1. $\square$

We need one more definition. Let ab be an edge of a graph G. Let $X(ab) = \{a,b\} \cup (N(a) \cap N(b))$. The *dome* of ab is the set of all vertices $y \in X(ab)$ such that $N(y) \setminus X(ab)$ is a clique. We call the set $X(ab) \setminus Y(ab)$ the *dome* of ab. We can now state our main result.

2.2. *Let G be a non-null clean graph.*

1. *If G is chordal then G has a simplicial vertex.*
2. *For every hole H of G there is an edge ab of H such that the dome of ab is a simplicial clique.*

In particular, G has a simplicial clique.

In view of 2.1 we immediately deduce 1.2.

3 The proof of the main theorem

In this section we prove 2.2. We start with a lemma.

3.1. *Let G be a clean graph, let H be a hole in G, and let $v \in V(G) \setminus V(H)$. Then one of the following holds:*

1. *v is anticomplete to $V(H)$.*
2. *$|V(H)| = 5$ and v is complete to $V(H)$.*
3. *v has exactly two neighbours in H and they are consecutive.*
4. *v has exactly three neighbours in H and they form a path of H.*

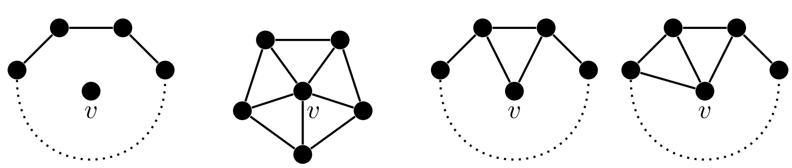

Figure 2: Outcomes of 3.1 (here dotted lines represent paths).

Proof. The outcomes of 3.1 are depicted in Figure 2. Write $H = h_1$-$\dots h_k$-h_1. We may assume that v has a neighbour in $V(H)$, for otherwise 3.1.1 holds. If v is complete to $V(H)$, then, since G is claw-free and since $G[V(H) \cup \{v\}]$ is not a jewel, it follows that $k = 5$ and so 3.1.2 holds. Thus we may assume that v has a non-neighbour in $V(H)$, say v is adjacent to h_1 and not to h_k. Since G is claw-free, v is adjacent to h_2, and $|N(v) \cap V(H)| \le 4$. We may assume that v has a neighbour in $V(H) \setminus \{h_1, h_2, h_3\}$, for otherwise 3.1.3 or 3.1.4 holds. Since G is claw-free, $N(v) \cap (V(H) \setminus \{h_1, h_2, h_3\})$ is a clique, and therefore $|N(v) \cap (V(H) \setminus \{h_1, h_2, h_3\})| \leq 2$. Let $i \in \{4, \ldots, k-1\}$ be minimum such that v is adjacent to h_i. Since $\{v, h_{i-1}, h_i, h_{i+1}\}$ is not a claw, it follows that either

- $i = 4$ and v is adjacent to h_3, or
- v is adjacent to h_{i+1} and v has no other neighbours in $V(H) \setminus \{h_1, h_2, h_i, h_{i+1}\}$.

In the former case $G[V(H) \cup \{v\}]$ is a jewel. Thus we may assume that the latter case holds. Since $|N(v) \cap V(H)| \leq 4$, it follows that $N(v) \cap V(H) = \{h_1, h_2, h_i, h_{i+1}\}$. But now $G[V(H) \cup \{v\}]$ is a line wheel, a contradiction. This proves 3.1. $\square$

Now we turn to the proof of 2.2.

Proof. If G has no hole, then G is chordal, and therefore has a simplicial vertex [3], and 2.2 holds. Thus we may assume that G has a hole. For an integer $k \geq 4$ a subset $W \subseteq V(G)$ is *k-structured* if (here the addition is mod k):

- W is the disjoint union of k non-empty cliques $K_1, \ldots, K_k$,
- for every $i \in \{1, \ldots, k\}$ every $v \in K_i$ has a neighbour in K_{i-1} and a neighbour in K_{i+1}, and
- if $i, j \in \{1, \ldots, k\}$ and $i \neq j \pm 1$ then K_i is anticomplete to K_j.

We call the partition $(K_1, \ldots, K_k)$ a *k-structure of W*.

Let H be a hole of G. Then H has length $k \geq 4$, and $V(H)$ is a k-structured set. If possible, choose H with $k \geq 5$. Let $W \subseteq V(G)$ be a k-structured set with k-structure $(K_1, \ldots, K_k)$, where each K_i contains exactly one vertex of H, and such that W is inclusion-wise maximal with this property. In what follows addition and subtraction of indices of the k-structure is mod k.

(1) *Let $i \in \{1, \ldots, k\}$. If $a, b \in K_i$ and $N(b) \cap K_{i+1} \not\subseteq N(a) \cap K_{i+1}$, then $N(b) \cap K_{i-1} \subseteq N(a) \cap K_{i-1}$.*

We may assume $i = 1$. If for each $j \in \{2, k\}$ there exists $a_j \in (N(b) \setminus N(a)) \cap K_j$, then $\{b, a_k, a, a_2\}$ is a claw in G, a contradiction. This proves (1).

(2) *Let $i \in \{1, \ldots, k\}$. For every $a, b \in K_i$ either $N(a) \cap K_{i+1} \subseteq N(b) \cap K_{i+1}$, or $N(b) \cap K_{i+1} \subseteq N(a) \cap K_{i+1}$.*

We may assume $i = 1$. Suppose there exist $a' \in (N(a) \setminus N(b)) \cap K_2$ and $b' \in (N(b) \setminus N(a)) \cap K_2$. Since K_1 and K_2 are cliques, a is adjacent to b, and a' is adjacent to b'. Now a-a'-b'-b-a is a hole of length four. Let $C = N(a) \cap K_k$ and $C' = N(a') \cap K_3$. By (1) b is complete to C, and b' is complete to C'. Switching the roles of a and b, we deduce that b is anticomplete to $K_k \setminus C$, and, similarly, b' is anticomplete to $K_3 \setminus C'$. Since the graph $G[\bigcup_{j=3}^{k} K_j]$ is connected, there is a path $P = p_1\text{-}\dots\text{-}p_t$ in $G[\bigcup_{j=3}^{k} K_j]$ with $p_1 \in C$ and $p_t \in C'$; we may assume P is chosen with t minimum. Then $p_2, \dots, p_{t-1} \notin C \cup C'$, and therefore $V(P) \setminus \{p_1, p_t\}$ is anticomplete to $\{a, b, a', b'\}$. But now $G[a, a', b, b', p_1, \dots, p_t]$ is a short prism in G, a contradiction. This proves (2).

(3) *For every $i \in \{1, \dots, k\}$, K_i is complete to at least one of K_{i-1}, K_{i+1}.*

We may assume $i = 1$. Suppose there exist $u \in K_k$, $v, w \in K_1$ and $z \in K_2$ (where possibly $v = w$) such that u is not adjacent to w, and v is not adjacent to z.

First we claim that v, w can be chosen in such a way that uv and wz are edges. Suppose not; then we may assume that v is non-adjacent to both u and z. Since $(K_1, \dots, K_k)$ is a k-structure, there exist $n_u, n_z \in K_1$ such that u is adjacent to n_u, and z is adjacent to n_z (possibly $n_z = w$). Since v is anticomplete to $\{u, z\}$, it follows from (1) that n_u is non-adjacent to z, and n_z is non-adjacent to u. But now we can choose $v = n_u$ and $w = n_z$, and the claim holds.

In view of the claim in the previous paragraph we assume that uv and wz are adjacent (and in particular $v \neq w$). Let $u' \in K_k$ be a neighbour of w; by (2) u' is adjacent to v. Now the set $T = \bigcup_{i=3}^{k-1} K_i$ is non-empty and connected, and both u' and z have neighbours in T. Consequently, there is a path R from u' to z with $V(R) \setminus \{u', z\} \subseteq T$. Let $x \in V(R)$ be the neighbour of u'. Then $x \in K_{k-1}$, and by (1) x is adjacent to u. It follows from the definition of a k-structure that $V(R) \setminus \{z, x, u'\}$ is anticomplete to $\{u, u', v, w\}$. But now the hole $z\text{-}w\text{-}v\text{-}u\text{-}x\text{-}R\text{-}z$ together with the vertex u' forms a jewel in G, a contradiction. This proves (3).

(4) *Let $j \in \{1, \dots, k\}$. For every $i \in \{2, \dots, k-2\}$, $a_j \in K_j$ and $a_{i+j} \in K_{j+i}$, there is a path P from a_j to a_{j+i} with $V(P) \subseteq \bigcup_{t=j}^{j+i} K_t$ and using exactly one vertex from each of $K_j, \dots, K_{j+i}$.*

We may assume $j = 1$. The proof is by induction on i. Suppose first that $i = 2$. Since by (3) K_2 is complete to at least one of K_1, K_3, it follows that a_1 and a_3 have a common neighbour $a_2 \in K_2$. Now $a_1\text{-}a_2\text{-}a_3$ is the required path.

Now assume that $i > 2$, let $a_{j+i-1} \in K_{j+i-1}$ be a neighbour of a_{j+i}. By the inductive hypothesis there is a path P from a_1 to a_{j+i-1} with $V(P) \subseteq \bigcup_{r=j}^{i+j-1} K_r$ using exactly one vertex from each of $K_j, \dots, K_{i+j-1}$. Now $a_j\text{-}P\text{-}a_{j+i-1}\text{-}a_{j+i}$ is the required path. This proves (4).

(5) *Let $v \in V(G) \setminus W$. For $i \in \{1, \dots, k\}$ let $N_i = K_i \cap N(v)$ and $M_i = K_i \setminus N_i$. The following hold for every $i \in \{1, \dots, k\}$:*

1. *N_i is anticomplete to at least one of M_{i-1}, M_{i+1}.*
2. *If $k > 4$, then M_i is anticomplete to at least one of N_{i-1}, N_{i+1}.*

We may assume $i = 1$. By (3) we may assume that K_1 is complete to K_2.

We first prove the first statement. We may assume that there exists $m_2 \in M_2$, for otherwise the claim holds (N_1 is anticomplete to M_2 because $M_2 = \emptyset$). Now if $n_1 \in N_1$ has a neighbour $m_k \in M_k$, then $\{n_1, v, m_2, m_k\}$ is a claw, a contradiction. This proves that N_1 is anticomplete to M_2, and (5).1 follows.

Next we prove the second statement. We may assume that there exist $m_1 \in M_1$ and $n_2 \in N_2$ such that m_1 is adjacent to n_2. Let $n_k \in N_k$, then $n_k \in N(m_1)$. By (4) there exists a path P from n_2 to n_k with $V(P) \subseteq \bigcup_{i=2}^{k} K_i$ with $|V(P) \cap K_i| = 1$ for every $i \in \{2, \ldots, k\}$. But now we get a contradiction applying 3.1 to the hole m_1-n_2-P-n_k-m_1 and the vertex v. This proves (5).2 and completes the proof of (5).

(6) *Let $v \in V(G) \setminus W$ and for $i \in \{1, \ldots, k\}$, let $N_i = N(v) \cap K_i$ and $M_i = K_i \setminus N_i$. Either*

- *N_i is non-empty for at most two consecutive values of i (mod k) or*
- *$k = 5$, v is complete to W, and K_i is complete to K_{i+1} for every $i \in \{1, \ldots, 4\}$.*

First we claim that we can choose j, l with $N_j \neq \emptyset$ and $N_l \neq \emptyset$, and such that $j = l \pm 2$. If $N_i \neq \emptyset$ for every $i \in \{1, \ldots, k\}$, then the claim holds. If $N_i = \emptyset$ for every $i \in \{1, \ldots, k\}$, then the claim holds. Thus we may assume that some N_is are empty, and some are not. By shifting the indices, we may assume $N_1 \neq \emptyset$ and $N_k = \emptyset$. We may assume that $N_t \neq \emptyset$ for some $t \in \{3, \ldots, k-1\}$ for otherwise (6) holds with $i = 1$. Let $n_1 \in N_1$ and $n_t \in N_t$. By (4) there exists a path P from n_1 to n_t such that $V(P) \subseteq \bigcup_{j=1}^{t} K_j$ and P uses exactly one vertex from each of $K_1, K_2, \ldots, K_t$. Also by (4) there exists a path Q from n_t to n_1 such that $V(Q) \subseteq K_1 \cup \bigcup_{j=t}^{k} K_j$ and Q uses exactly one vertex from each of $K_t, K_{t+1} \ldots, K_k, K_1$. Now $F = n_1$-P-n_t-Q-n_1 is a hole, and $n_1, n_t \in V(F)$. Since $t \geq 3$ and $N_k = \emptyset$, applying 3.1 to F and v we deduce that the fourth outcome of 3.1 holds and $t = 3$. Now we can set $j = 1$ and $l = t$. This proves the claim.

By the claim of the previous paragraph (shifting the indices so that $j = k$ and $l = 2$) we may assume that N_k and N_2 are both non-empty. For $i \in \{2, k\}$ let $a_i \in N_i$. By (3) a_k and a_2 have a common neighbour $a_1 \in K_1$.

Suppose $\bigcup_{i=3}^{k-1} N_i = \emptyset$. Since W is maximal, $(K_1 \cup \{v\}, K_2, \ldots, K_k)$ is not a k-structure for $W \cup \{v\}$, and therefore there exists $a_1' \in M_1$. By symmetry we may assume that K_1 is complete to K_2. Let $a_3' \in K_3$ be a neighbour of a_2; then $a_3' \in M_3$, contrary to (5).1. This proves that $\bigcup_{i=3}^{k-1} N_i \neq \emptyset$.

Suppose $k = 4$. Then there is symmetry between K_1 and K_3, and we deduce that $N_i \neq \emptyset$ for every $i \in \{1, \ldots, 4\}$. By (3) we may assume that K_1 is complete to K_2, and K_3 to K_4. Now 3.1 implies that there is no hole n_1-n_2-n_3-n_4-n_1 with $n_i \in N_i$ for every $i \in \{1, \ldots, 4\}$, and consequently either N_1 is anticomplete to N_4, or N_3 is anticomplete to N_2. By symmetry we may assume that N_1 is anticomplete to N_4. Let $n_1 \in N_1$ and $n_4 \in N_4$, and let $m_1 \in K_1$ be adjacent to n_4 and $m_4 \in K_4$ be adjacent to n_1. Then $m_1 \in M_1$ and $m_4 \in M_4$. By (2) applied to m_1 and n_1, we deduce that m_1 is adjacent to m_4. If there exists $m_2 \in M_2$, then $\{n_1, v, m_2, m_4\}$ is a claw, a contradiction. This proves (using symmetry) that $M_2 \cup M_3 = \emptyset$. Let $n_2 \in K_2$, and let $n_3 \in K_3$ be adjacent to n_2. Then $n_2 \in N_2$ and $n_3 \in N_3$. But now $G[v, m_4, n_2, n_4, n_1, n_3, m_1]$ is a seven-antihole, a contradiction. This proves that $k \geq 5$.

By (5).2 it follows that $a_1 \in N_1$. We claim that $k = 5$ and v is complete to W. Suppose v has a non-neighbour in W. Since $\{v, a_k, a_2, x\}$ is not a claw for any $x \in \bigcup_{i=4}^{k-2} K_i$, it follows that $\bigcup_{i=4}^{k-2} N_i = \emptyset$.

We may assume that there is $a_3 \in N_3$. Since $\{v, a_2, a_k, a_3\}$ is not a claw, it follows that a_2 is adjacent to a_3. By (4) there is a path P from a_3 to a_k with $V(P) \subseteq \bigcup_{j=3}^{k} K_j$ and using exactly one vertex from each of $K_3, \ldots, K_k$. Now $F = a_k$-a_1-a_2-a_3-P-a_k is a hole, and 3.1 implies that $k = 5$ and v is complete to $V(F)$. We have proved that $N_i \neq \emptyset$ for every $i \in \{1, \ldots, 5\}$ thus restoring the symmetry of the 5-structure. Since for $n_1 \in N_1$, $n_2 \in N_2$ and $n_4 \in N_4$, $\{v, n_1, n_2, n_4\}$ is not a claw, we deduce (using symmetry) that N_i is complete to N_{i+1} for every $i \in \{1, \ldots, 5\}$.

Next suppose that both M_5 and M_2 are non-empty. By (3) we may assume that K_1 is complete to K_2. By (5).1, N_1 is anticomplete to M_5. Since every vertex of M_5 has a neighbour in K_1, it follows that $M_1 \neq \emptyset$. By (5).2 M_1 is anticomplete to N_5, but now $m_1 \in M_1$ and $n_1 \in N_1$ contradict (2). By symmetry we may assume M_i is non-empty for at most two consecutive values of i, and that $M_1 \cup M_2 \cup M_3 = \emptyset$. By (5).2 N_4 is anticomplete to M_5, and similarly M_4 is anticomplete to N_5. By symmetry we may assume that $M_4 \neq \emptyset$. But now $m_4 \in M_4$ and $n_4 \in N_4$ contradicts (2). This proves that $k = 5$ and v is complete to W. To complete the proof of (6) assume for a contradiction that there exist $i \in \{1, \ldots, 4\}$, $k_i \in K_i$ and $k_{i+1} \in K_{i+1}$ such that k_i is non-adjacent to k_{i+1}. We may assume $i = 1$. Then $\{v, k_1, k_2, k_4\}$ is a claw in G, a contradiction. This proves (6).

For $i \in \{1, \ldots, k\}$ let $K_{i,i+1}$ be the set of all vertices of $V(G) \setminus W$ that have a neighbour in K_i, a neighbour in K_{i+1}, and no neighbour in $W \setminus (K_i \cup K_{i+1})$. The outcomes of (6) are summarized in Figure 3.

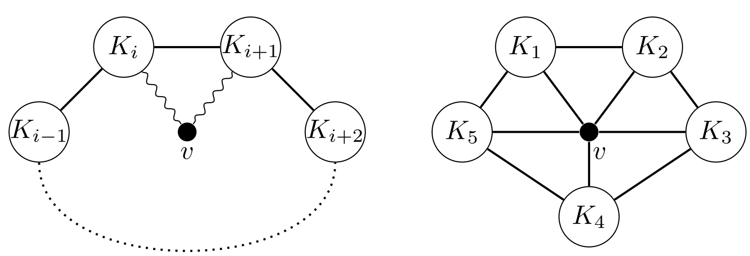

Figure 3: Outcomes of (6) (here wiggly lines represent possible adjacency, and the dotted arc represents the remainder of the k-structure).

Assume that $K_{i,i+} \neq \emptyset$. The following statements hold

(7)

1. *$K_i \cup K_{i+1} \cup K_{i,i+1}$ is a clique.*
2. *If $u \in V(G) \setminus W$ is complete to W, then u is anticomplete to $K_{i,i+1}$.*
3. *$K_{i,i+1}$ is anticomplete to $K_{i-1,i}$.*

Let $v \in K_{1,2}$. For $i \in \{1,2\}$ let $N_i = K_i \cap N(v)$, and let $M_i = K_i \setminus N_i$. By (5).1 N_1 is anticomplete to M_2, and N_2 is anticomplete to M_1. If there exists $m_1 \in M_1$, then n_1, m_1 contradict (2) for every $n_1 \in N_1$. Thus $M_1 = \emptyset$, and by symmetry $M_2 = \emptyset$. This proves that $K_{1,2}$ is complete to $K_1 \cup K_2$.

Suppose $k_1 \in K_1$ is non-adjacent to $k_2 \in K_2$. Let P be a path from k_2 to a vertex $k_k \in K_k$ as in (4), such that $|V(P) \cap K_i| = 1$ for every $i \in \{2, 3, \dots, k\}$. By (3) K_1 is complete to K_k, and so $F = k_k\text{-}k_1\text{-}v\text{-}k_2\text{-}P\text{-}k_k$ is a hole. Let $k_1' \in K_1 \cap N(k_2)$, then $G[V(F) \cup \{k_1'\}]$ is a jewel in G, a contradiction. This proves that K_1 is complete to K_2.

Since $\{k_1, k_k, a, b\}$ is not a claw for any $k_1 \in K_1$, $k_k \in K_k \cap N(k_1)$ and $a, b \in K_{1,2}$, it follows that $K_{1,2}$ is a clique, and therefore $K_1 \cup K_2 \cup K_{1,2}$ is a clique. By symmetry, $K_1 \cup K_2 \cup K_{1,2}$ is a clique for every i. This proves (7).1.

Next suppose that u is complete to W and u is adjacent to $w \in K_{1,2}$. By (6) $k = 5$. Let $k_3 \in K_3$ and $k_5 \in K_5$. Then $\{u, w, k_3, k_5\}$ is a claw, a contradiction. This proves (7).2.

Finally, suppose that $w_k \in K_{k,1}$ is adjacent to $w_2 \in K_{1,2}$. Let $k_1 \in K_1$, $k_2 \in K_2$ and $k_k \in K_k$, and let P be a path from k_2 to k_k as in (4) such that $|V(P) \cap K_i| = 1$ for every $i \in \{2, 3, \dots, k\}$. Then $F = k_2\text{-}P\text{-}k_k\text{-}w_k\text{-}w_2\text{-}k_2$ is a hole and $G[V(F) \cup \{k_1\}]$ is a jewel, a contradiction. This proves (7).3 and completes the proof of (7).

For $i \in \{1, \dots, k\}$, let $V(H) \cap K_i = \{h_i\}$. Choose $i \in \{1, \dots, k\}$ such that K_i is complete to K_{i+1} and, if possible, such that $K_{i,i+1} \neq \emptyset$; we may assume $i = 1$. Set $a = h_1$ and $b = h_2$, and let $K = K_1 \cup K_2 \cup K_{1,2}$.

By (6), every vertex of $X(ab) = \{a, b\} \cup (N(a) \cap N(b))$ either belongs to K or is complete to W (and $k = 5$). Since if $y \in X(ab)$ is complete to W, then y has two non-adjacent neighbours in $V(G) \setminus X(ab)$, it follows that K contains the dome of ab. By (7).1 K is a clique.

We prove that K is a simplicial clique, and therefore K equals the dome of ab. Suppose K is not a simplicial clique, and let $v \in K$ and $u, w \in V(G) \setminus K$ be such that u and w are adjacent to v, and u is non-adjacent to w. Suppose first that u is complete to W. By (6) $k = 5$ and for every i K_i is complete to K_{i+1}. By (7).2 $v \notin K_{1,2}$. For $i \in \{1, \dots, 5\}$ let $k_i \in K_i$. We may assume that $v = k_1$. Since u is non-adjacent to w, it follows that $w \notin W$. Since $G[k_1, u, k_3, w, k_2]$ is not a jewel in G, it follows that w is not complete to W. By (5).1 (since k_1 is complete to $K_2 \cup K_5$) w has a neighbour in at least one of K_2, K_5. By

(6) $w \in K_{1,2} \cup K_{5,1}$. Since $w \notin K$, it follows that $w \in K_{5,1}$. By (7).1 K_5 is complete to K_1. Since $K_{1,5} \neq \emptyset$, and since K_1, K_2 where chosen with $K_{1,2} \neq \emptyset$ if possible, it follows that there exists $k \in K_{1,2}$. By (7).2 u is non-adjacent to k, and by (7).3 w is non-adjacent to k. But now $\{k_1, u, k, w\}$ is a claw in G, a contradiction. This proves that u is not complete to W. By symmetry, w is not complete to W.

Now suppose $v \in K_1$. Since $u, w \notin K$, it follows from (6) that $u, w \in K_k \cup K_{k,1}$. But then u is adjacent to w by (7).1, a contradiction. This proves that $v \notin K_1$, and, by symmetry, $v \notin K_2$.

It follows that $v \in K_{1,2}$. Moreover, applying the previous argument to an arbitrary vertex of $K_1 \cup K_2$ in the role on v, we deduce that no vertex of $K_1 \cup K_2$ is complete to $\{u, w\}$. Since $\{v, u, w, p\}$ is not a claw for $p \in K_1 \cup K_2$ it follows that every vertex of $K_1 \cup K_2$ has a neighbour in $\{u, w\}$. Since $u, w \notin K$, (7).1 implies that each of u, w is anticomplete to at least one of K_1, K_2. By switching u and w if necessary, we may assume that u has a neighbour in K_1 and is anticomplete to K_2. By (5).1 u has a neighbour in K_k, but now $u \in K_{k,1}$ is adjacent to $v \in K_{1,2}$, contrary to (7).3. Thus we have found an edge of H whose dome is a simplicial clique. This proves 2.2. □

4 The Algorithm

In this section we use 2.2 to design a simple algorithm that finds a simplicial clique in a clean graph.

4.1. *There is an algorithm with the following specifications.*
Input: *A non-null clean graph G.*
Output: *A simplicial clique in G.*
Running time: $O(|V(G)|^5)$.

Proof. Let $|V(G)| = n$. First, for every vertex $v \in V(G)$ check if $N(v)$ is a clique. If the answer is yes for some v, then output a simplicial clique $\{v\}$. This step can be done in time $O(n^3)$.

Now for every edge ab compute $X(ab), Y(ab)$ and the dome of ab, and check if the dome of ab is a simplicial clique. This step can be done in time $O(n^5)$.

We now use 2.2 to prove that the algorithm will return a simplicial clique of G. If G is a chordal graph, then by the first statement of 2.2 the first step of the algorithm will return a simplicial clique; otherwise there is a hole in G, and so by the second statement of 2.2, the second step of the algorithm will return a simplicial clique. This proves 4.1. □

We remark that the algorithm of 4.1 can be used to produce, in time $O(|V(G)|^2)$, a list of at most $|V(G)|^2$ sets one of which is guaranteed to be a simplicial clique. The rest of the running time is spent checking whether each of the sets is a simplicial clique.

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