The percolation density $\theta(p)$ is analytic

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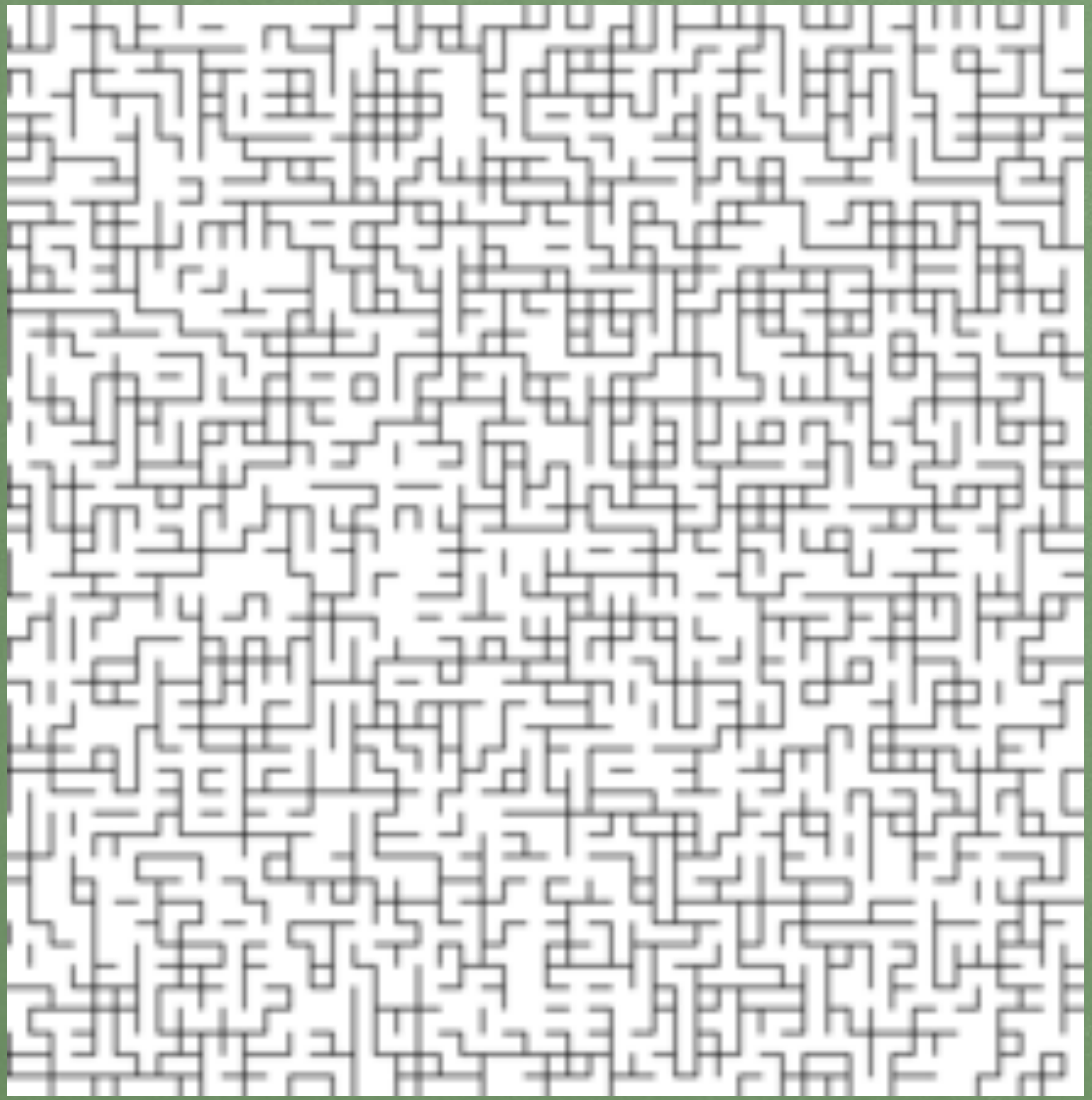
Joint work with Christoforos Panagiotis

Bernoulli bond Percolation



Each edge
— present with probability p
and

— absent with probability 1-p
independently

$p_c := \sup\{p \mid \mathbb{P}_p(\text{ cluster } C_o \text{ of } o \text{ is infinite }) = 0\}$

Motivation

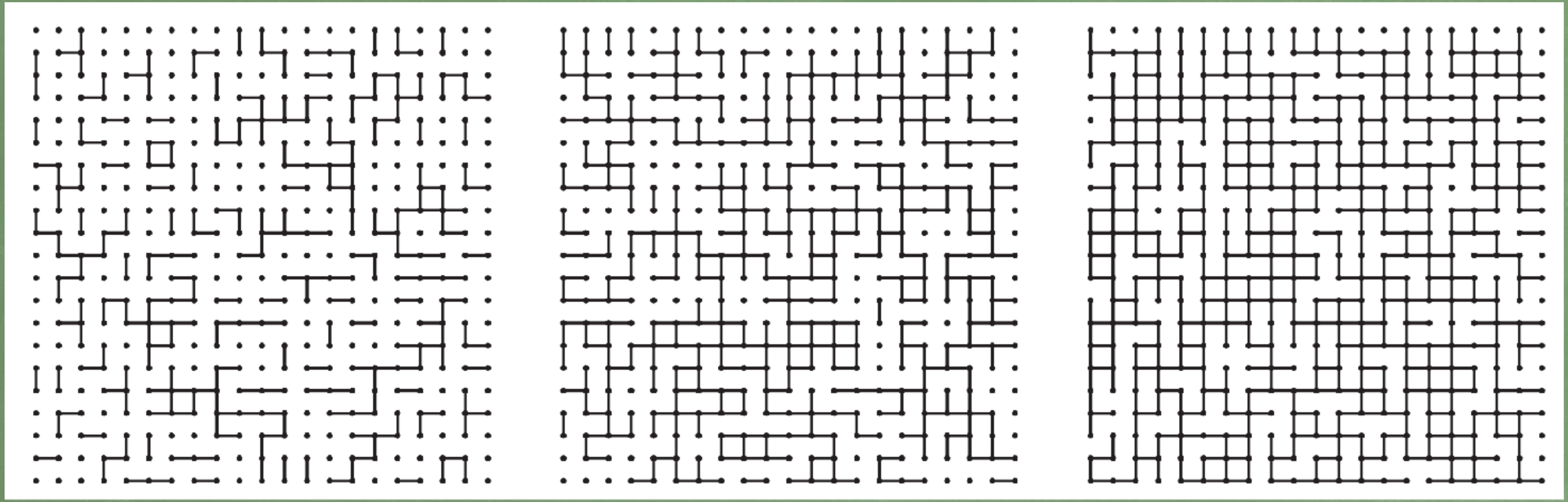
Introduced by physicists Broadbent & Hammersley '57 as a toy model of statistical mechanics

Many deep rigorous results by mathematicians

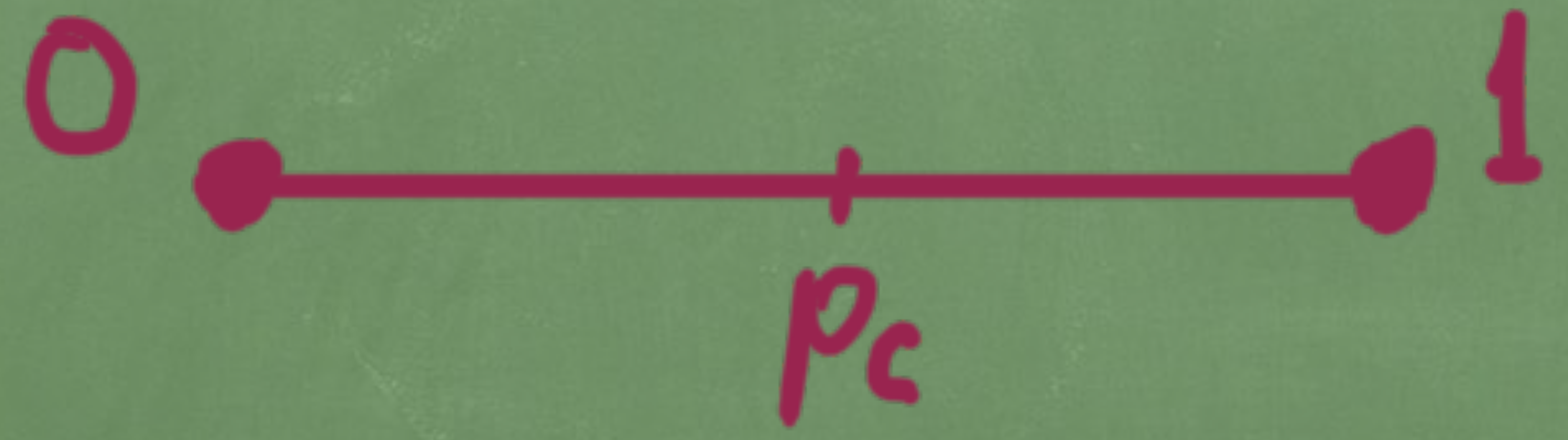
Varying the underlying graph unleashes an interesting interplay between geometry & probability

Rich connections to other models (Ising, GFF, loop O(n))

The 3 regimes



subcritical: $p<p_c$ critical: $p=p_c$ supercritical: $p>p_c$



But is p_c the only phase transition?

Exponential decay

Theorem (Aizenman & Barsky '87/ Menshikov '86)

For every $p < p_c$ *there is* $c_p > 1$ *such that*

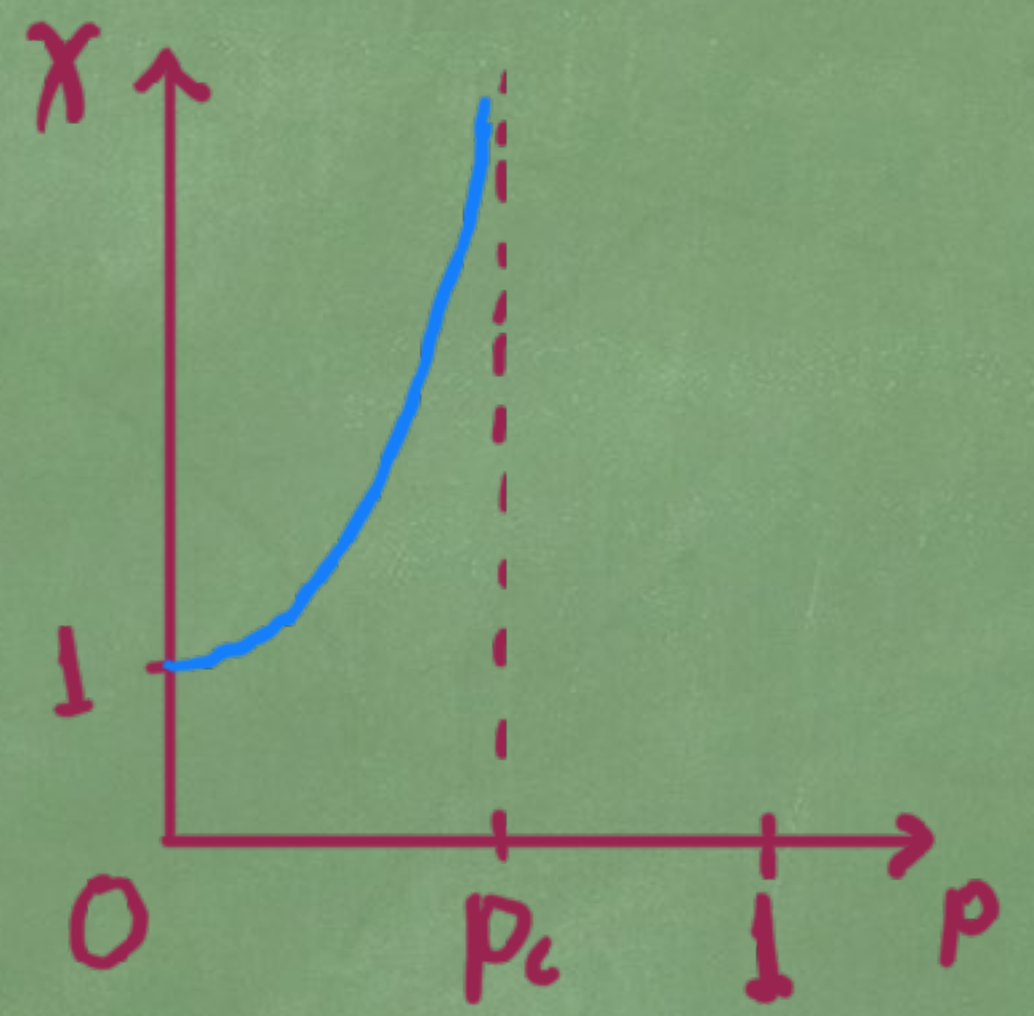
$$\mathbb{P}_p(|C_o| \geq n) \leq c_p^{-n}.$$

Analyticity of X(p)

$$\chi(p) := \mathbb{E}_p(|C_o|)$$

Theorem (Kesten '82)

$\chi(p)$ is an analytic function of p for $p \in [0, p_c)$ when G is a lattice in $\mathbb{R}^d$.



Proof:

$$\chi(p) = \sum_{\substack{o \in S \subset G \\ \text{finite, connected}}} |S| \cdot \mathbb{P}_p(C_o = S) = \sum_{n \in \mathbb{N}} n \cdot \sum_{|S|=n} \mathbb{P}_p(C_o = S)$$

$$= \sum_{n \in \mathbb{N}} n \cdot \mathbb{P}_p(|C_o| = n)$$

polynomial: $\sum p^{n_i}(1-p)^{m_i}$

Complex analysis basics

Theorem (Weierstrass): Let $f = \sum f_n$ be a series of analytic functions which converges uniformly on each compact subset of a domain $\Omega \subset \mathbb{C}$. Then f is analytic on Ω.

Weierstrass M-test: Let (f_n) be a sequence of functions such that there is a sequence of 'upper bounds' M_n satisfying

$$|f_n(z)| \leq M_n, \forall x \in \Omega \quad \text{and} \quad \sum M_n < \infty.$$

Then the series $\sum f_n(x)$ converges uniformly on Ω.

Analyticity of X(p)

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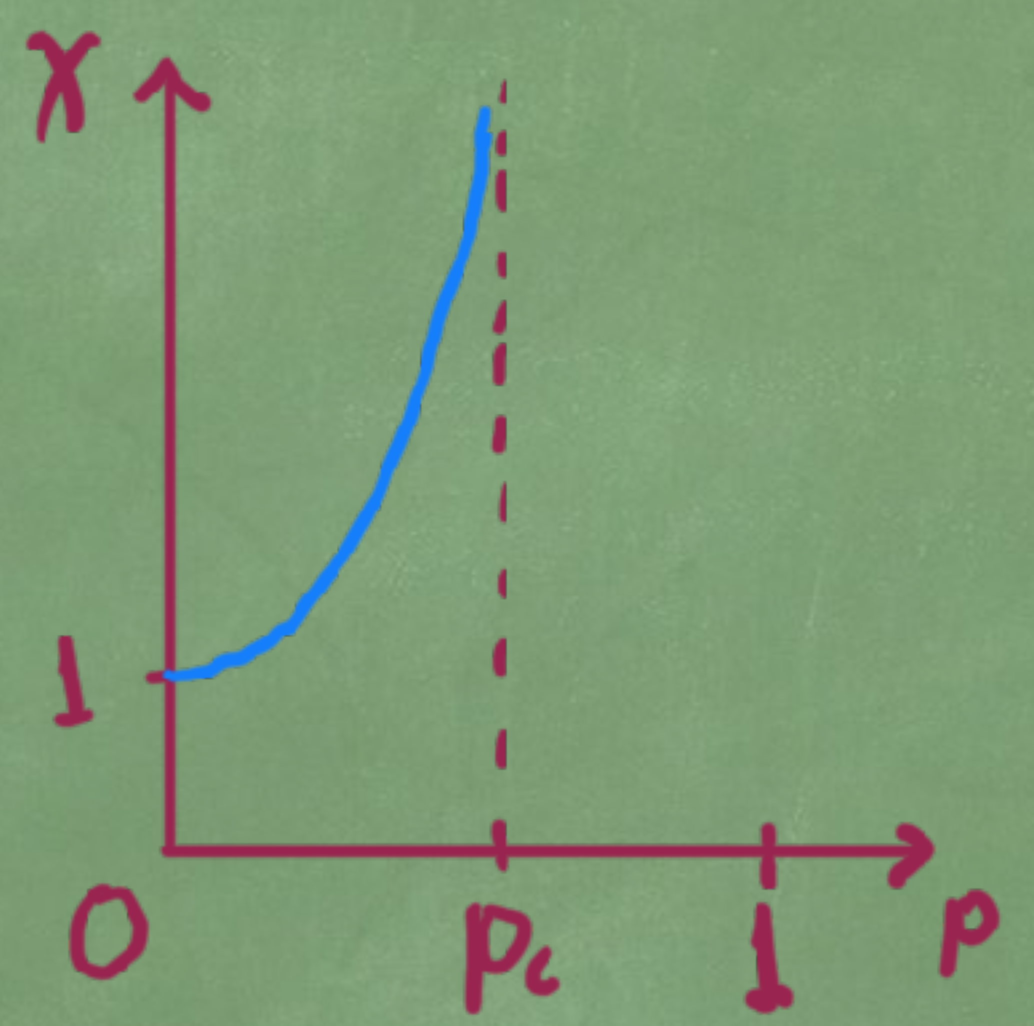

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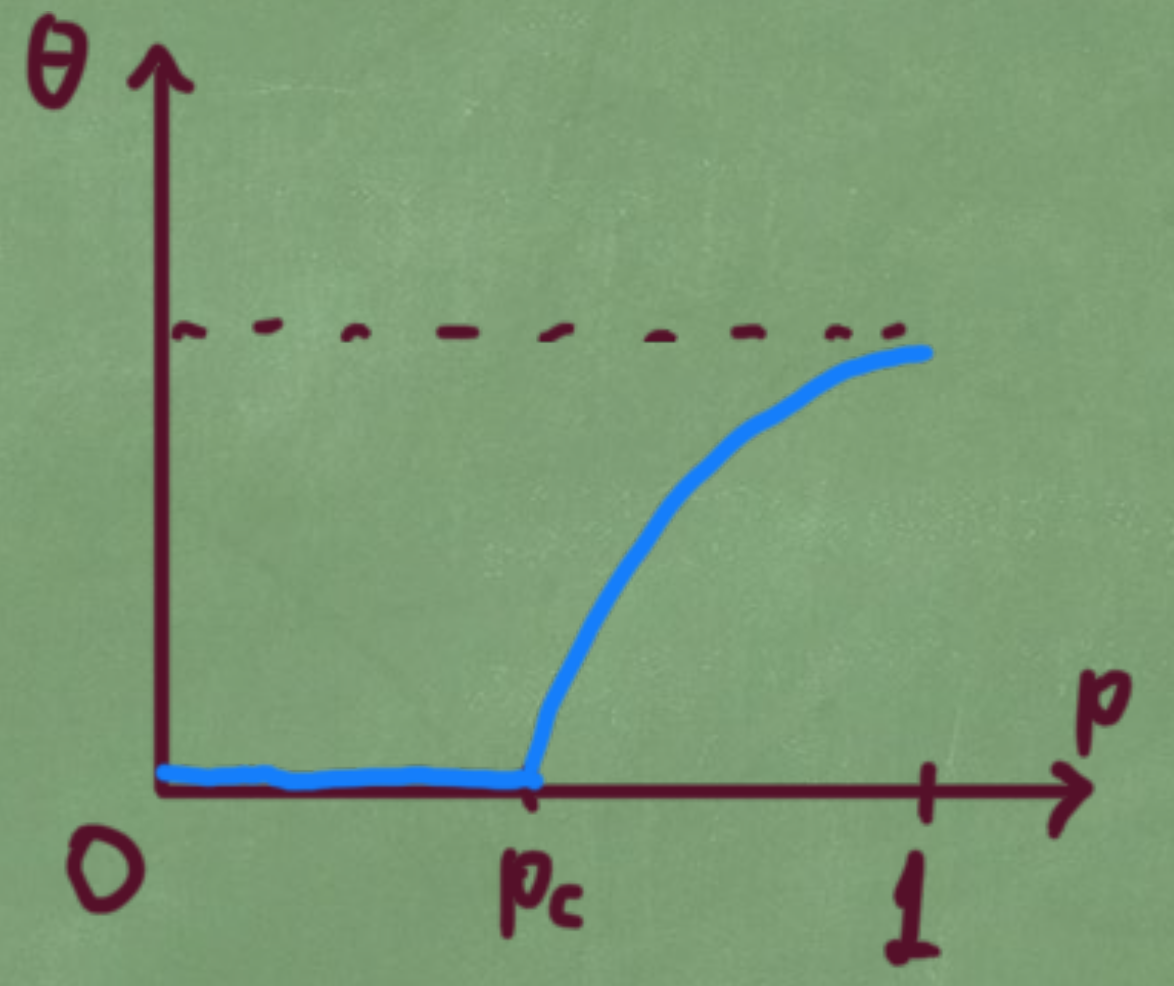
Analyticity of θ(p)



$$\theta(p) := \mathbb{P}_p(|C_o| = \infty)$$

Question (Kesten ’81): Is $\theta(p)$ analytic for $p > p_c$?

Appearing in the textbooks
Kesten ’82, Grimmett ’96, Grimmett ’99.

— θ(p) is infinitely differentiable [Chayes, Chayes & Newman ’87]
— θ(p) is analytic near p=1 [Braga, Proccaci & Sanchis ’02]

Analytic vs. C_∞ functions

Bob:

— What was the difference between C_∞ and analytic again?

Analise:

— The latter has a convergent Taylor series.

Bob:

— Isn't almost every C_∞ function analytic?

Analise:

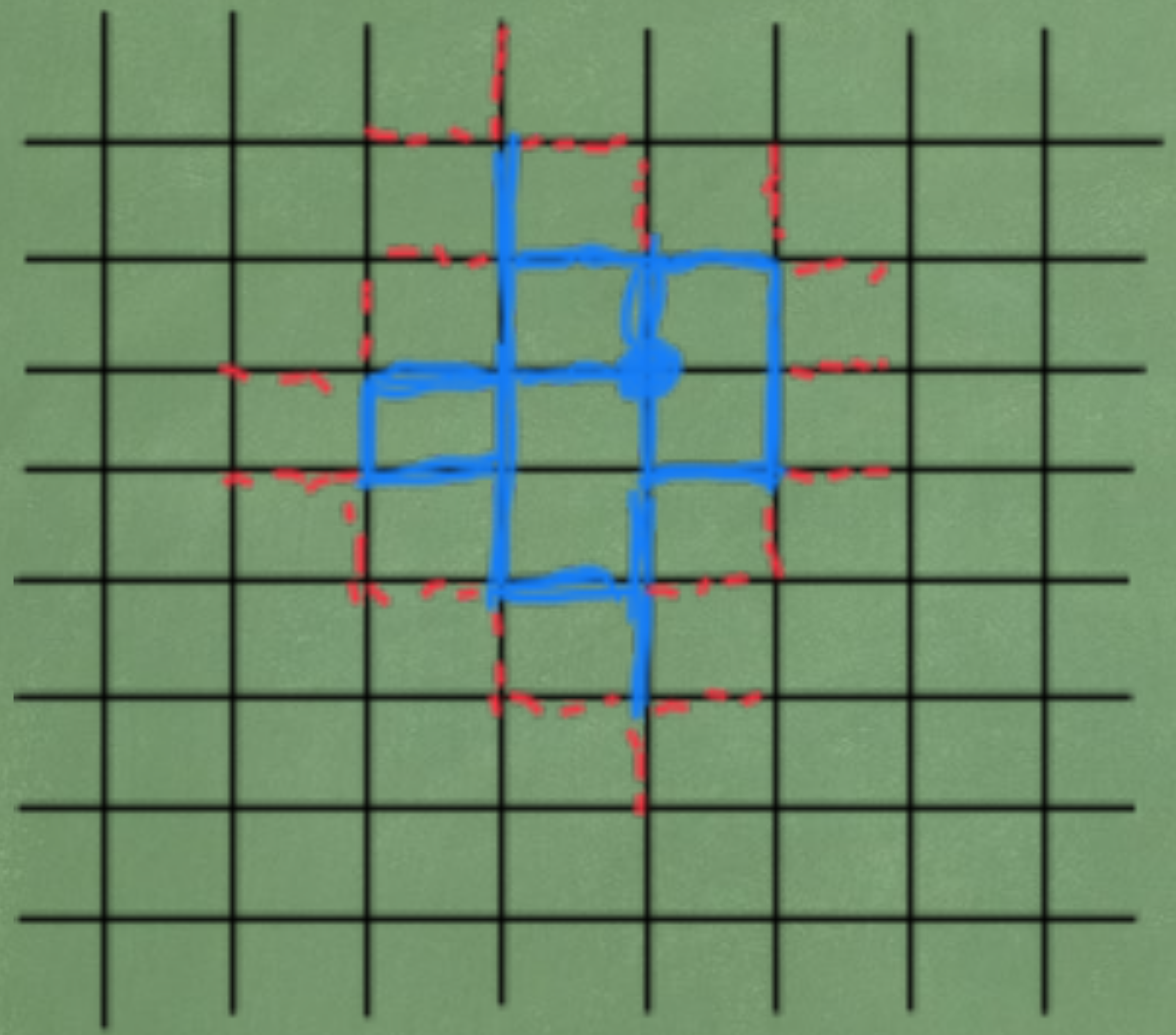
— Quite the contrary: the nowhere analytic functions are a dense G_δ subset of the C_∞ functions! [Cater '84]

Griffiths singularities

[Griffiths '69] introduced models, constructed by applying the Ising model on 2-dimensional percolation clusters, in which the free energy is **infinitely differentiable** but **not analytic**.

This phenomenon is now called a **Griffiths singularity**

Interlude: Peierls's argument



$$1-\theta(p) \le \sum_{n\in\mathbb{N}} \sum_{\substack{\text{cycles } c \\ \text{around } 0 \\ |c|=n}} (1-p)^n$$

$$= \sum_n c_n (1-p)^n$$

$$\Rightarrow \dots \quad p_c(\mathbb{Z}^2) \le 2/3$$

Analyti

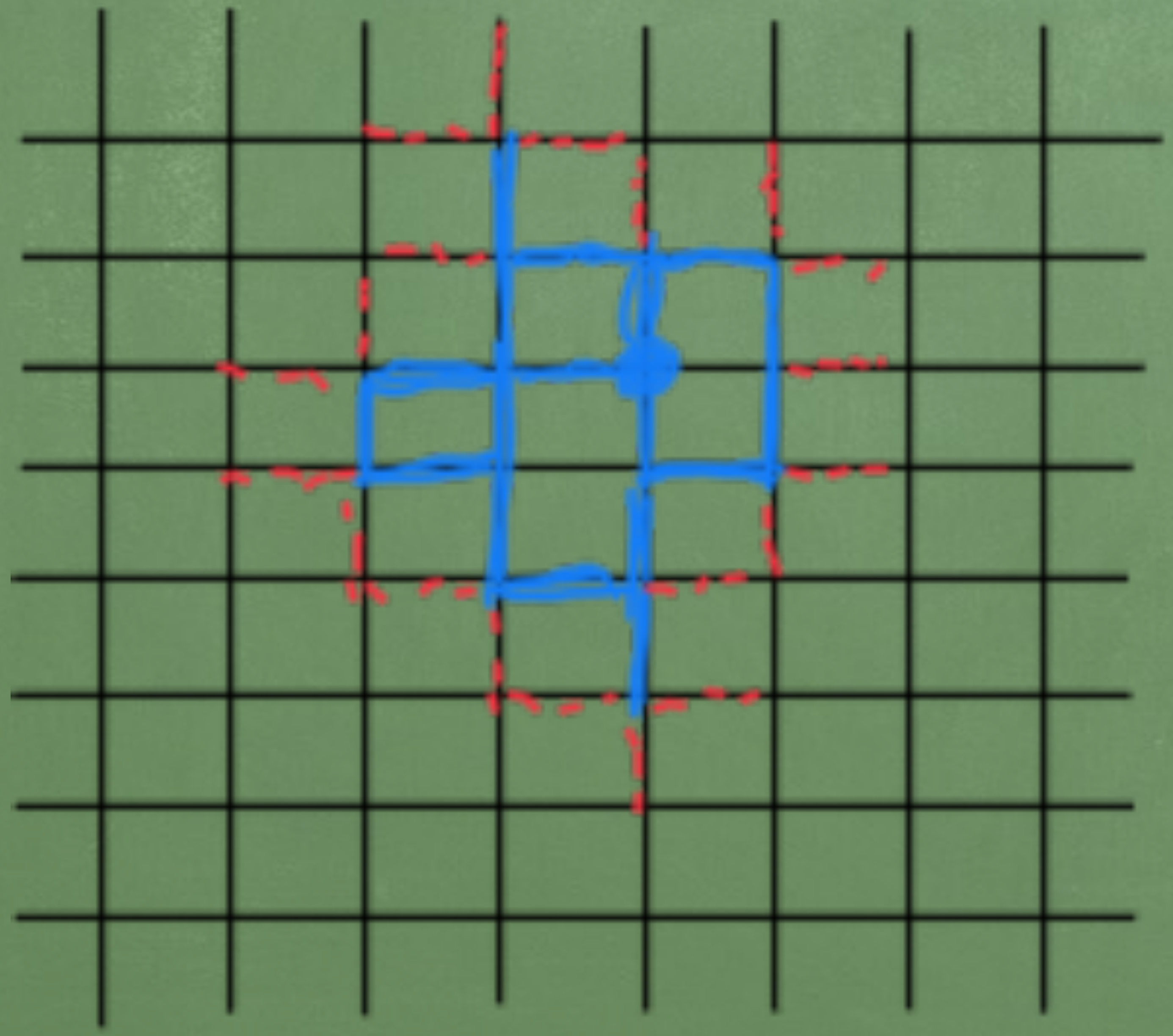
Theorem (Hardy & Ramanujan 1918)

The number of partitions of the integer n is of order $\exp(\sqrt{n})$.

Theorem (G & Panagiotis ’18+)

$\theta(p)$ *is analytic for* $p > p_c$ *on any planar lattice.*

$$1-\theta(p) = \sum_{S} \mathbb{P}_p(C_0 = S) = \sum_{n \in \mathbb{N}} \mathbb{P}(|C_0| = n) \quad e^{-\alpha\sqrt{n}} \text{ [ADS '80]}$$



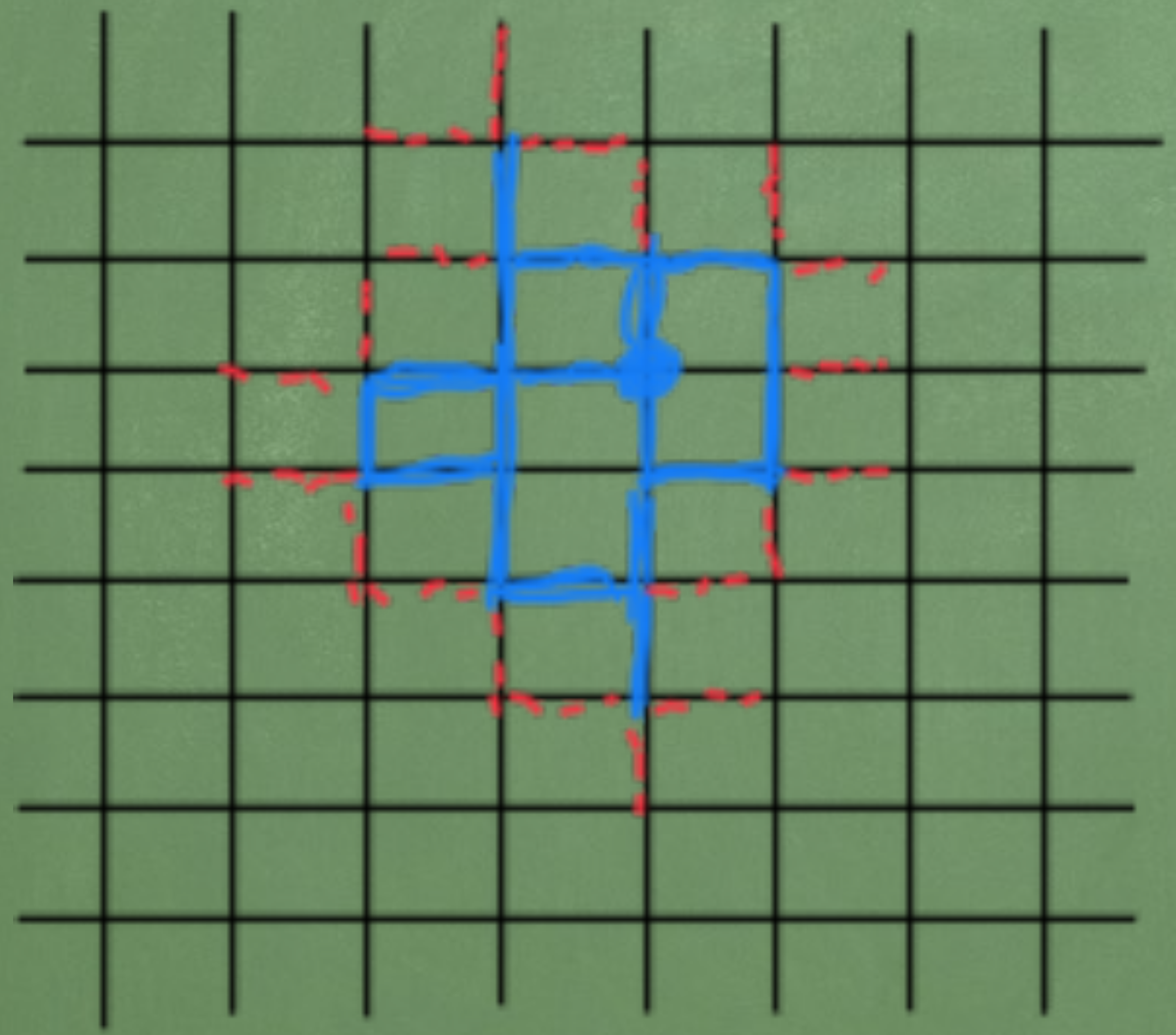
IEP

$$\sum_{n \in \mathbb{N}} \sum_{\substack{\text{multi-interfaces } I \\ |I| = n}} (-1)^{c(I)} \mathbb{P}_p(I \text{ occurs})$$

Analyticity of θ(p)

Theorem (G & Panagiotis '18+)

$\theta(p)$ is analytic for $p > p_c$ on any planar lattice.



Ingredients:

— elementary complex analysis

— better interfaces

— Inclusion-Exclusion Principle

— Weak Hardy-Ramanujan

— BK inequality

— Exponential decay (in dual)

— More combinatorics

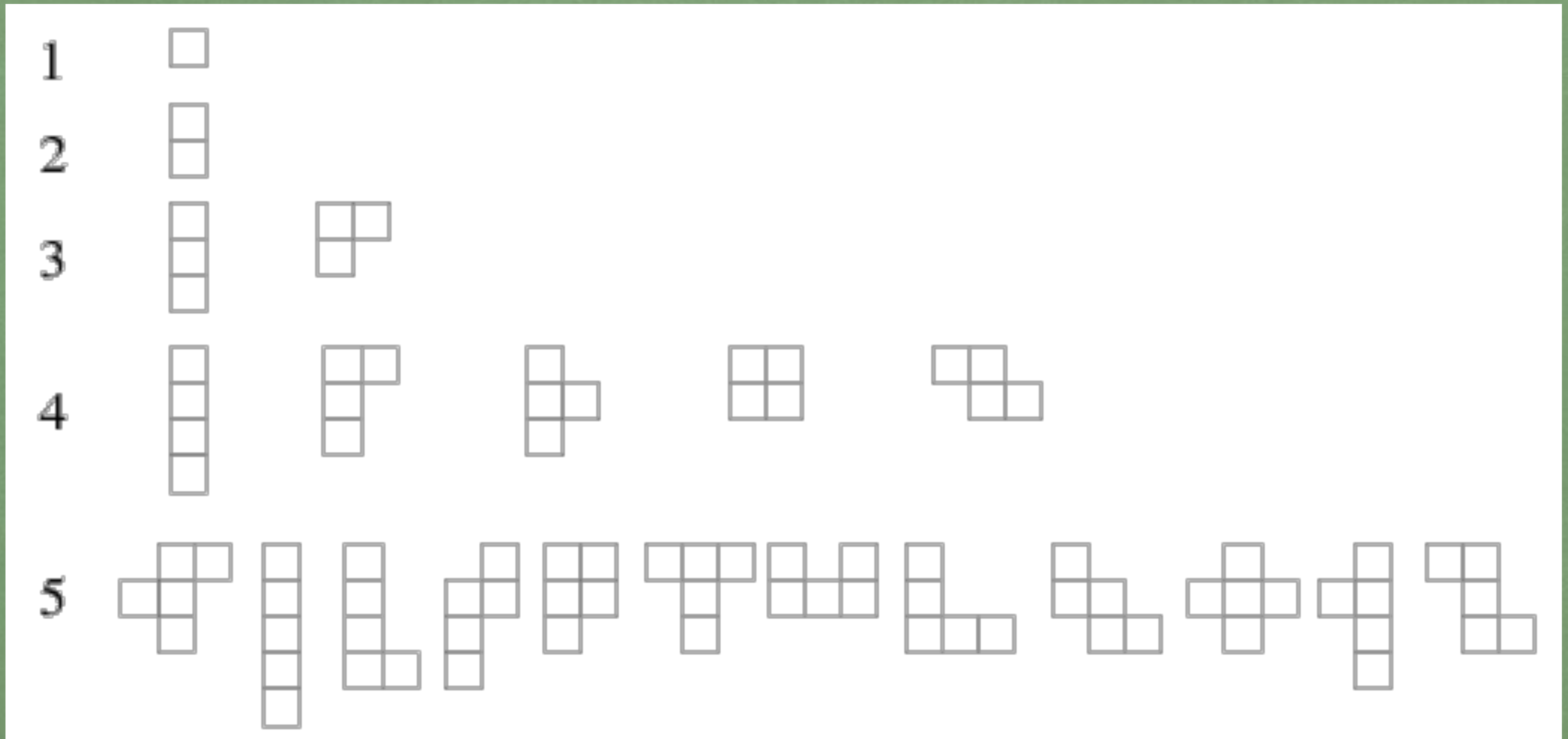
Further results

- θ analytic for $p > p_c$ for continuum percolation – *asked by [Last et al. '17]*
- θ analytic for $p > p_c$ on regular trees, and on almost every Galton-Watson tree.
  – *asked by [Michelen, Pemantle & Rosenberg]*
- θ analytic for p near 1 on all finitely presented Cayley graphs.
- θ analytic for p near 1 on all non-amenable graphs.
  –*Extended to* $p \in (p_c, 1]$ *by [Hermon & Hutchcroft '19+]*
- For certain families of planar triangulations for which [Benjamini et al. '96, '15, '18] conjectured that $p_c^{site} \leq 1/2$, we prove $p_c^{bond} \leq 1/2$ (and analyticity of θ).

Chapter II: Polyominoes and growth rates of interfaces

Polyominoes

A **polyomino**, aka. **lattice animal**, is a connected, induced, subgraph of $\mathbb{Z}^2$.



Their exponential growth rate

$a(\mathbb{Z}^2) := \lim_{n\to\infty}(\sharp\{\text{ polyominoes of size } n\})^{1/n}$

is not known.

Kesten's argument

$$1-\theta(p) = \sum_{0 \in S \subset G} \mathbb{P}(C_0 = S)$$

$$= \sum_{n \in \mathbb{N}} \sum_{|S|=n} p^n (1-p)^{|\partial S|} = \sum_{n \in \mathbb{N}} a_n p^n (1-p)^{r(n)}$$

$$\geq \sum_n \underset{\substack{\rotatebox{90}{$\leq$} \\ \alpha(G)^n}}{a_n} p^n (1-p)^{\Delta n}$$

$$\Rightarrow \alpha(G) \leq \frac{1}{p(1-p)^{\Delta}} \leq \dots \leq \Delta e$$

Can we do better?

The growth rates b_r

$c_{n,r,\varepsilon} := \#\left\{ \begin{array}{l} \text{interfaces of size } n \\ \text{and boundary size 'roughly' } rn \end{array} \right\}$

i.e. " " in $((r-\varepsilon)n, (r+\varepsilon)n)$

$b_{r,\varepsilon} :=$ their growth rate $= \lim_{\varepsilon \to 0} \limsup_{n\to\infty} c_{n,r,\varepsilon}^{1/n}$

$b(G) := \max_r b_r =$ growth rate of all interfaces

The growth rates b_r

... refining Kesten's argument, we obtain:

$$b_{r(p)}(G) \leq f(r(p))$$

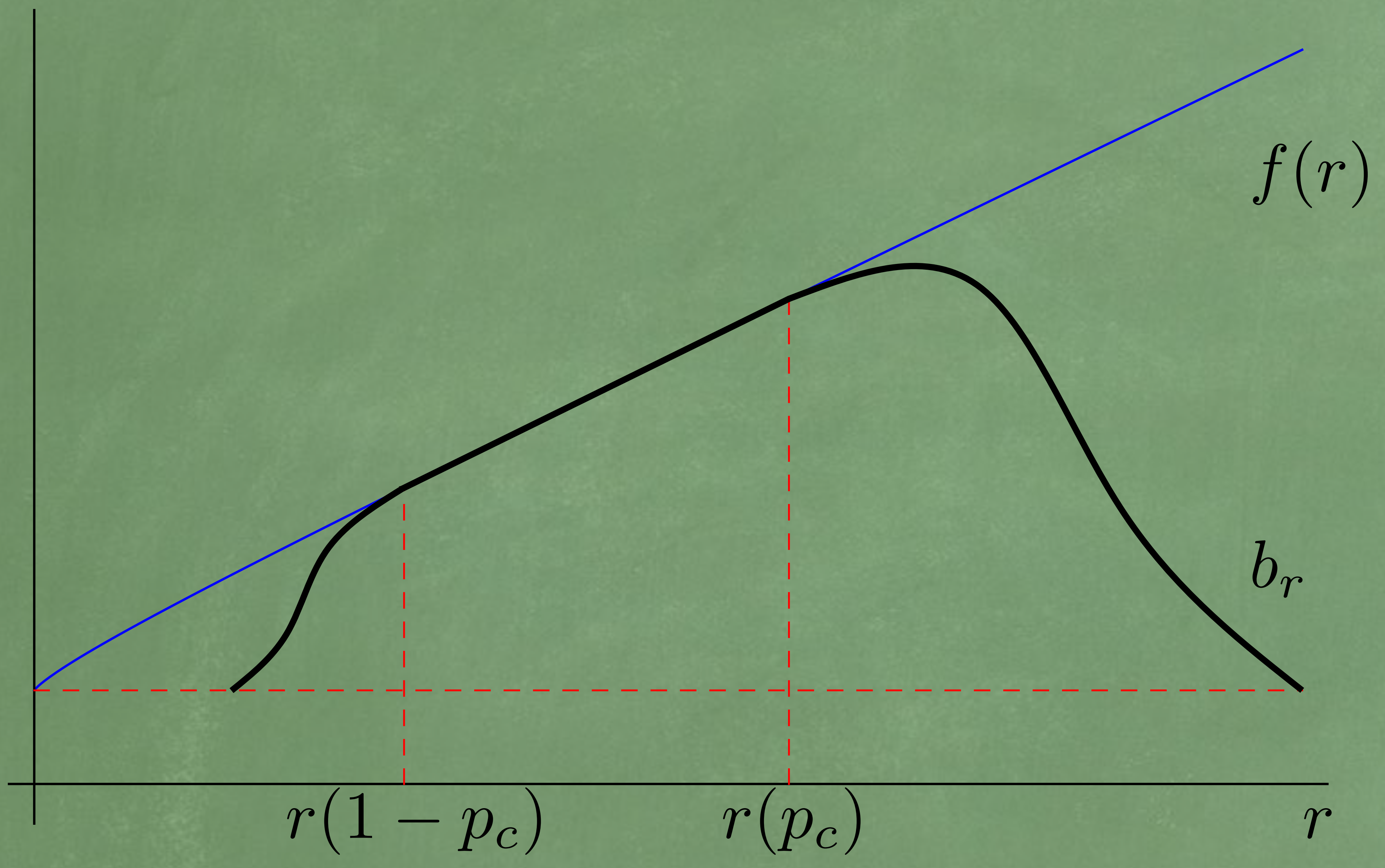
where $f(r) := \frac{(1+r)^{1+r}}{r^r}$ and $r(p) := \frac{1-p}{p}$

are universal.

Equality holds iff exponential decay fails!

For lattice animals obtained by [Delyon '80] and [Hammond '05]

More on b_r



$$b_{r(p)}(G) \leq f(r(p))$$

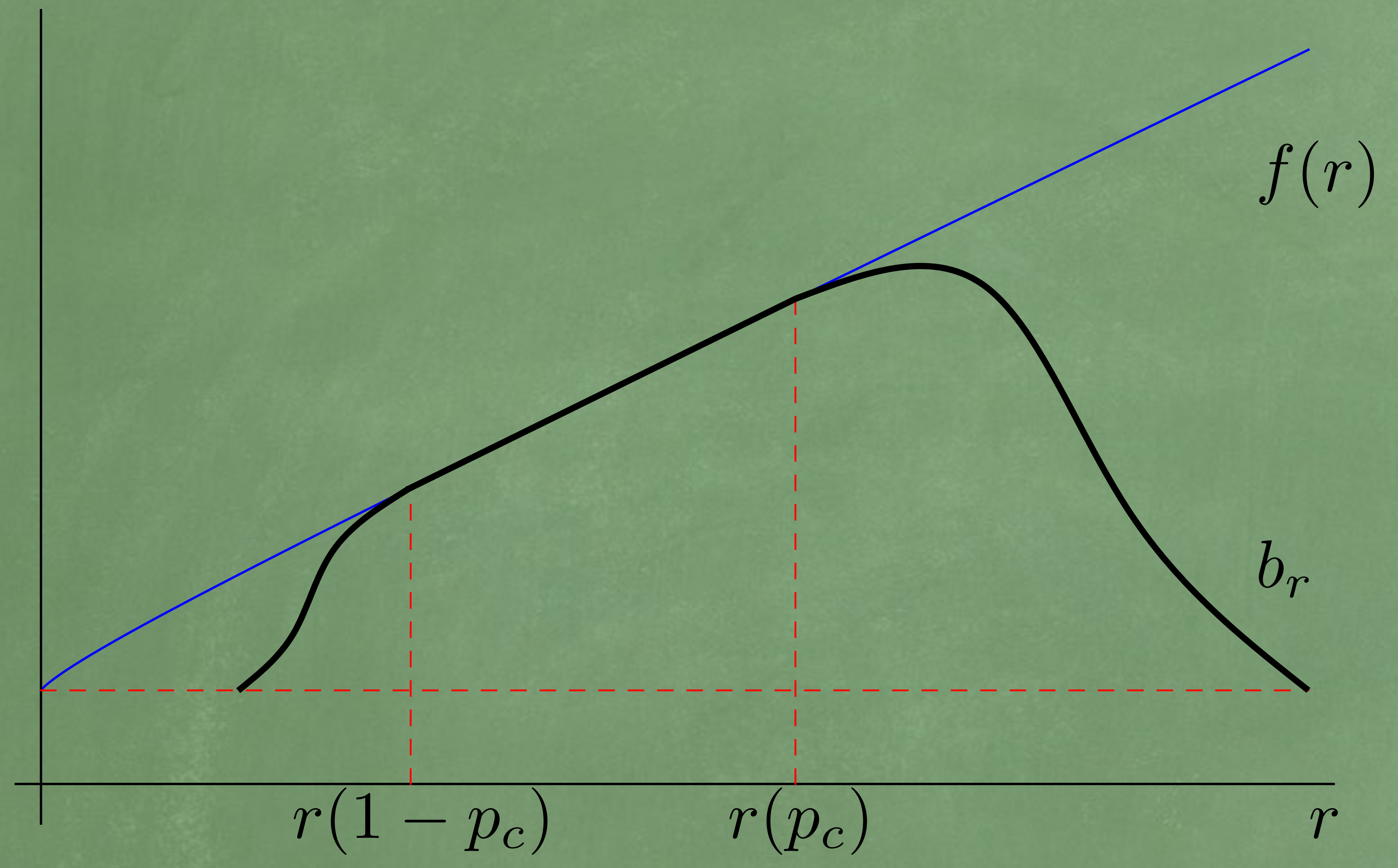
$$b_r = (b_{1/r})^r$$

$$a(G) \geq b(G) \geq f(r(p_c(G)))$$

Further results

- $p_c(\mathbb{Z}^3) > 0.2522$
  –using bounds of [Barequet & Shalah '19+]
- $a(\mathbb{Z}^d) \leq 2de - 5e/2 + O(1/\log(d))$
  –improves on bounds of [Barequet & Shalah '19+]
- as a result, we obtain
  $p_c(\mathbb{Z}^d) \geq \frac{1}{2d} + \frac{2}{(2d)^2} - O(1/d^2 \log(d))$
- Using upper bounds on $p_c(\mathbb{Z}^d)$ from [Heydenreich & Matzke '19+], we obtain $a(\mathbb{Z}^d) \geq 2de - 3e$
  –asked by [Barequet, Barequet & Rote '10], nonrigorously obtained by [Peard & Gaunt '95]
- $p_c < 1/2$ for plane graphs of minimum degree ≥ 7 [Haslegrave & Panagiotis '19+]
  –answers a question of [Benjamini & Schramm '96]

Analyticity of θ(p)



$$b_r = (b_{1/r})^r$$

Kesten's question

Question (Kesten '81): Is $\theta(p)$ analytic for $p > p_c$?

Theorem (Panagiotis & G '20+)

Yes.

—C. Panagiotis and A. Georgakopoulos. Analyticity of the percolation density in all dimensions. arXiv:2001.09178
—A. Georgakopoulos and C. Panagiotis. On the exponential growth rates of lattice animals and interfaces, and new bounds on p_c. arXiv:1908.03426
—A. Georgakopoulos and C. Panagiotis. Analyticity results in Bernoulli percolation. arXiv:1811.07404

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