Infinite-Bin Model and the Longest Increasing Path in an Erdős-Rényi random graph

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Université Sorbonne Paris Nord

Oxford Discrete Mathematics and Probability Seminar

Cooking recipe

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Problem

Compute the time necessary for the apple pie to be made depending on the number of cooks.

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Compute the time necessary for the apple pie to be made depending on the number of cooks.

A cooking recipe

with a single cook

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Step	Alice
1	1
2	2
3	3
4	4
5	5
6	6
7	7

The recipe takes an amount of time equal to its number of steps.

A cooking recipe

with two cooks

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Step	Alice	Bob
1	1	2
2	4	3
3	5	
4	6	
5	7	

Some task can be parallelized, allowing for a reduction of the number of steps needed to realize the recipe.

A cooking recipe

with three cooks

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Step	Alice	Bob	Craig
1	1	2	4
2		3	5
3	6		
4	7		

Increasing the number of cooks allows to decrease the number of step neeeded.

A cooking recipe

with three cooks or many more...

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
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Step	Alice	Bob	Craig
1	1	2	4
2		3	5
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Increasing the number of cooks allows to decrease the number of step neeeded... up to a point.

Formalizing the problem

The dependecies of the tasks of the recipe are represented as an oriented graph (without cycles).

- The vertices of the graph represent the different tasks.
- Edges denote dependencies.

Lemma

The minimal number of steps needed to realize the project is equal to the length of the longest path in the oriented graph.

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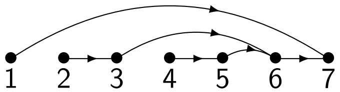

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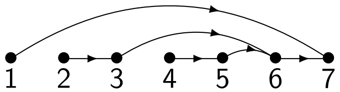

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The minimal number of steps needed to realize the project is equal to the length of the longest path in the oriented graph.

Outline

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The Barak-Erdős graph

Definition

The Barak-Erdős graph is a directed version of the Erdős-Rényi graph in which every edge $\{i, j\}$ is directed from i to j if $i < j$.
In other words, given $p \in [0, 1]$ and $n \in \mathbb{N}$, for any $1 \leq i < j \leq n$, put a edge from i to j with probability p, independently from any other edge.

Figure: A Barak-Erdős graph

We take interest in the length $L_n(p)$ of the longest increasing path in this graph.

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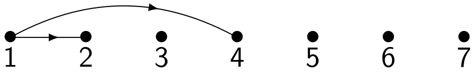

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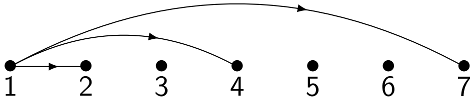

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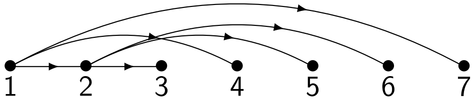

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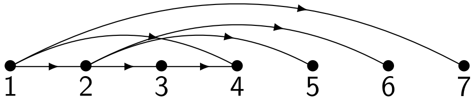

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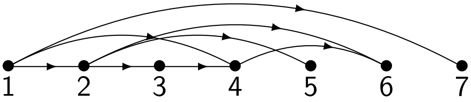

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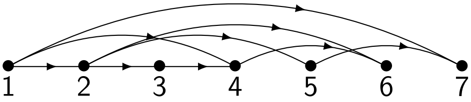

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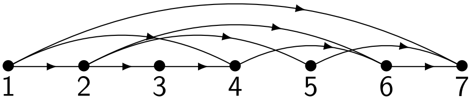

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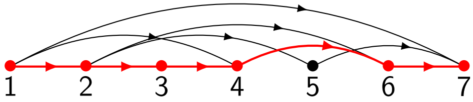

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Some references

- Model introduced by Barak and Erdős in 1984.
- The length of the longest increasing path is one of the most studied features of this model.
- Applications span over a wide array of fields:
  - Performance evaluation of computer systems (Gelenbe-Nelson-Philips-Tantawi '86, Isopi-Newman '94);
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Existing results

Existence of a limiting function

Theorem (Newman '92)

There exists a function C such that for any $p \in [0,1]$,

$$\lim_{n \to +\infty} \frac{L_n(p)}{n} = C(p) \quad \textit{in probability.}$$

Moreover, C is continuous, increasing and $C'(0) = e$.

Existing results

Existence of a limiting function

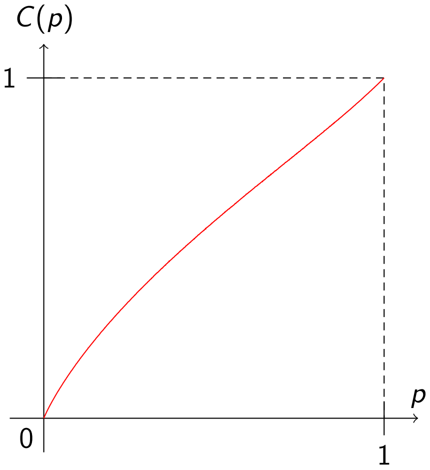

Figure: Graph of the function C

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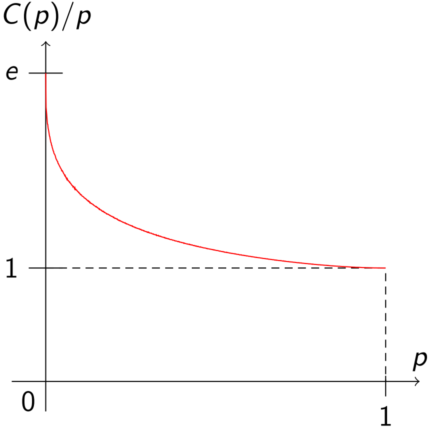


Figure: Graph of the function $p \mapsto C(p)/p$

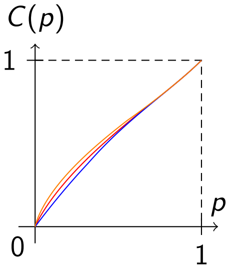
Existing results

Bounds on the function C

Theorem (Foss-Konstantopoulos '03)

There exist two explicit functions L and U such that $L(p) < C(p) < U(p)$ for any $p \in (0,1)$. This in particular yields, as $p \to 1$,

$$C(1-p) = 1 - (1-p) + (1-p)^2 - 3(1-p)^3 + 7(1-p)^4 + O(1-p)^5.$$



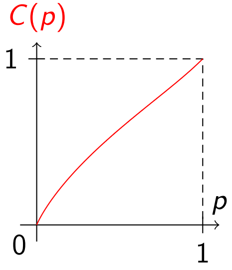
Contribution from infinite-bin models theory I

Improved bounds in a neighbourhood of 1

Theorem (M.-Ramassamy)

There exist sequences of upper bounds (U_k) and lower bounds (L_k) that converge to C exponentially fast for any $p > 0$.

In particular, the Taylor expansion of C can be computed explicitely to any order around $p = 1$.



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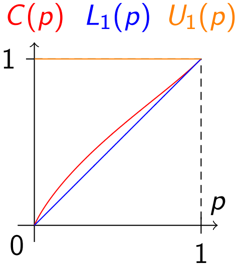

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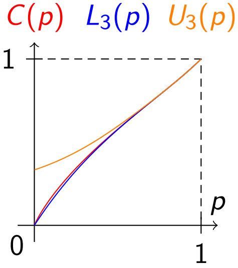

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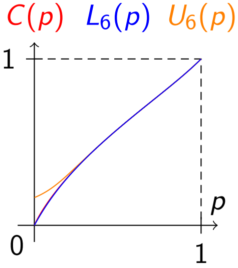

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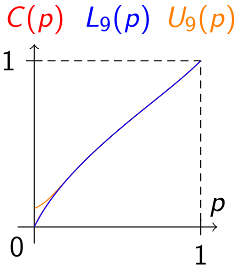

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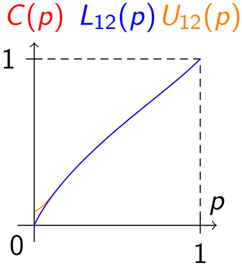

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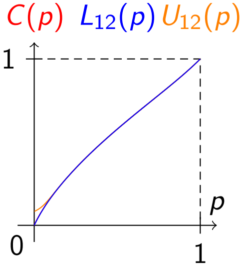

Contribution from infinite-bin models theory II

Analyticity of C in a neighbourhood of 1

Theorem (M.-Ramassamy)

The function C is analytic on $(0,1]$, and there exists an explicit sequence of integers (a_k) such that

$$C(p) = \sum_{k=0}^{+\infty} a_k (1-p)^k \text{ for all } p \geq 3/4.$$

First coefficients

$$\begin{aligned} C(p) = {} & 1 - (1-p) + (1-p)^2 - 3(1-p)^3 + 7(1-p)^4 - 15(1-p)^5 \\ & + 29(1-p)^6 - 54(1-p)^7 + 102(1-p)^8 - 197(1-p)^9 \\ & + 375(1-p)^{10} - 687(1-p)^{11} + 1226(1-p)^{12} - 2182(1-p)^{13} \\ & + 3885(1-p)^{14} - 6828(1-p)^{15} + 11767(1-p)^{16} + \cdots \end{aligned}$$

(sequence A321309 of OEIS)

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Contribution from infinite-bin models theory III

Asymptotic behaviour of C as $p \to 0$

Theorem (M.-Ramassamy)

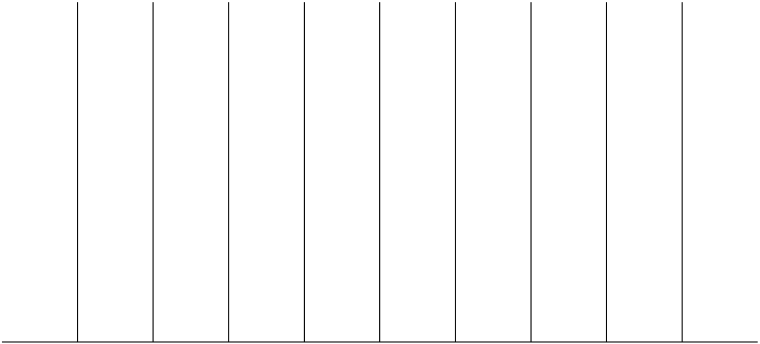
$$C(p) = ep\left(1 - \frac{\pi^2}{2}\frac{1}{(\log p)^2}\right) + o\left(\frac{p}{(\log p)^2}\right) \text{ as } p \to 0.$$

Outline

The infinite-bin model

Description

- Infinite number of bins on $\mathbb{Z}$.
- At each time n, a new ball is put to the right of the ξ_nth ball, with (ξ_j) i.i.d. sequence of random variables on $\mathbb{N}$.
- We take interest in the speed of the front.



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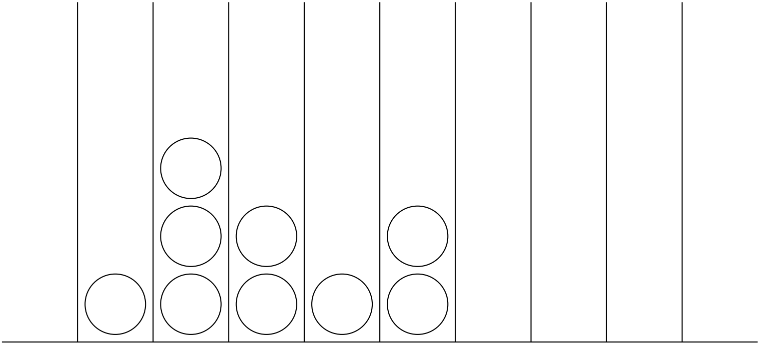

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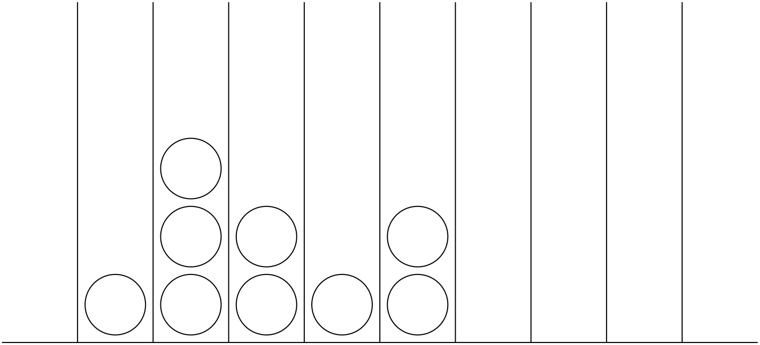

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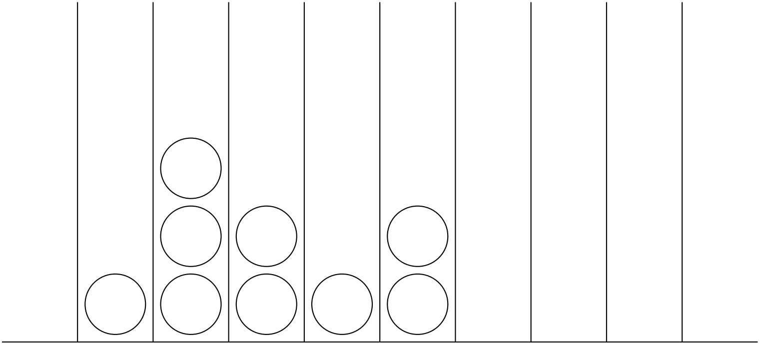

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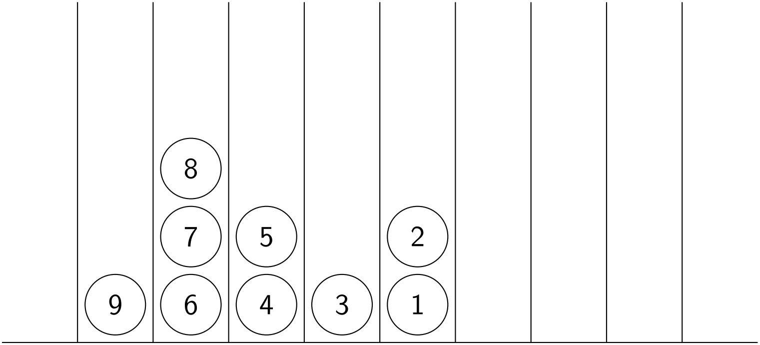

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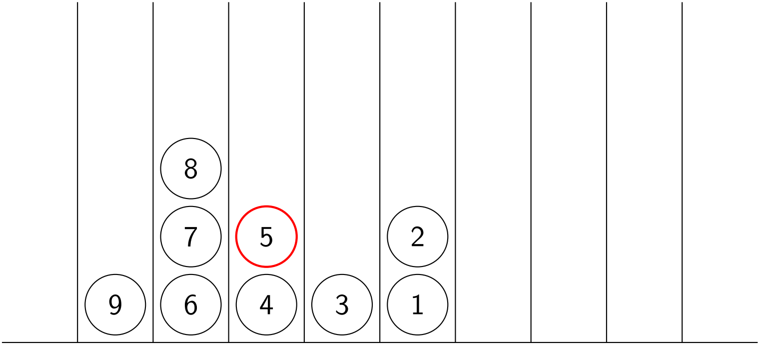

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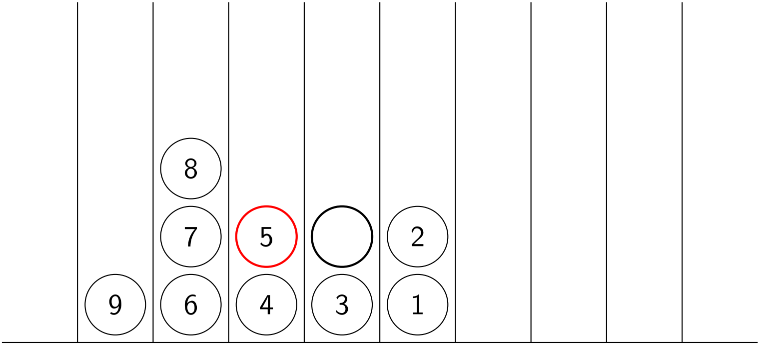

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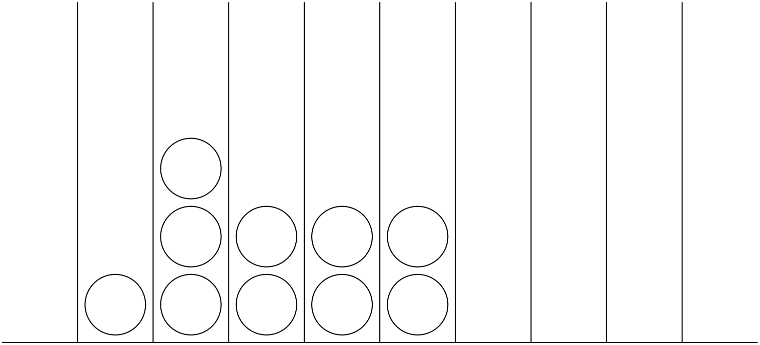

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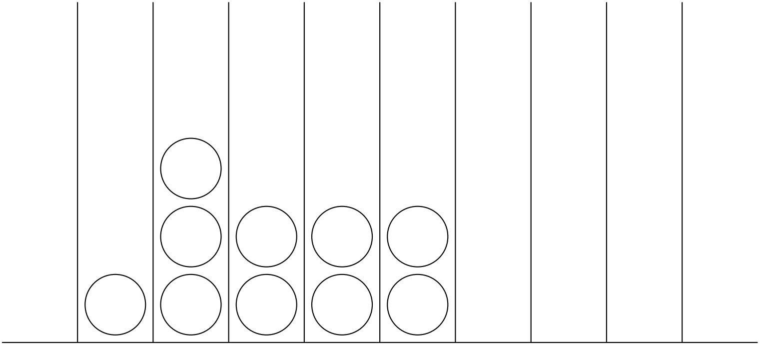

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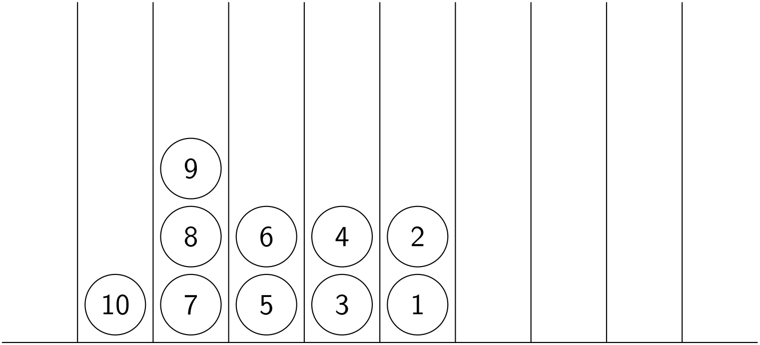

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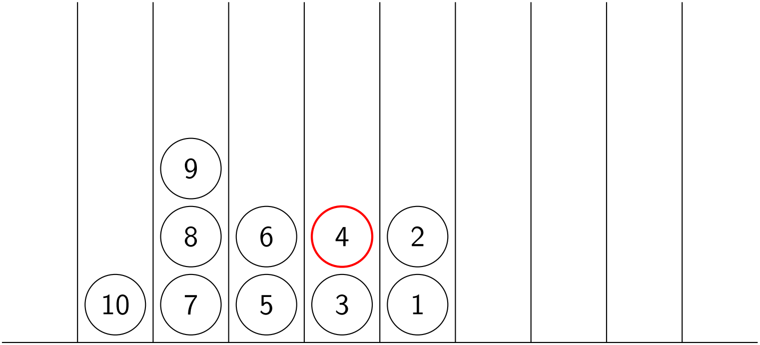

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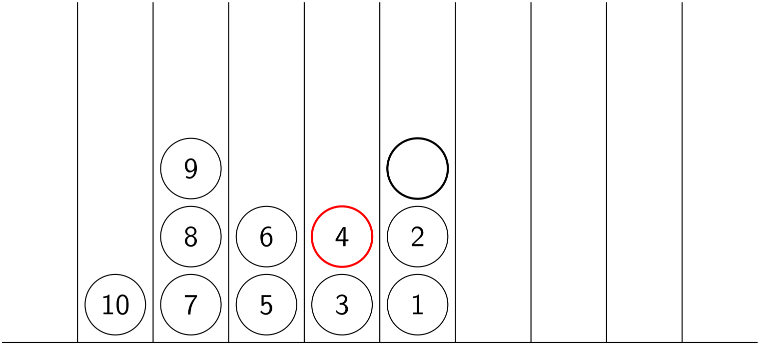

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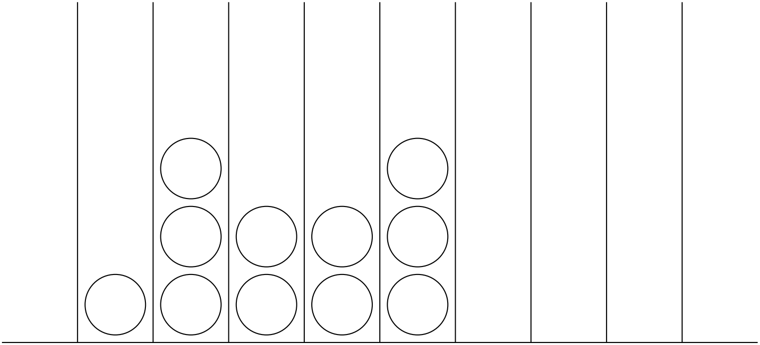

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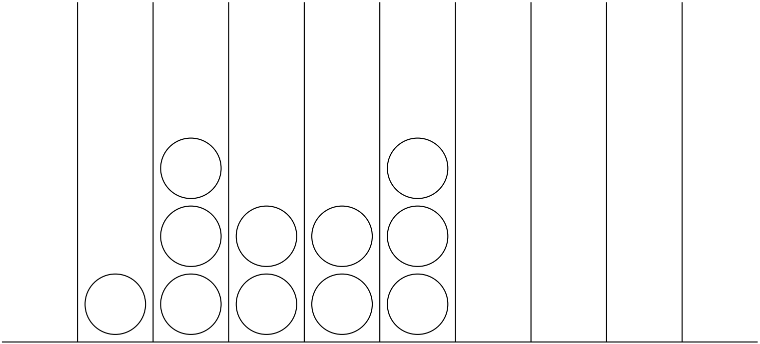

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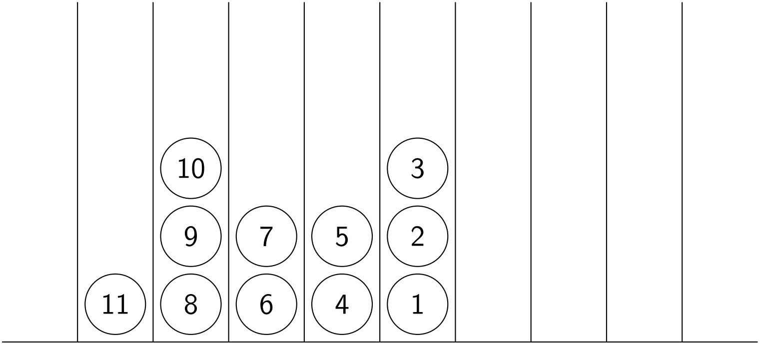

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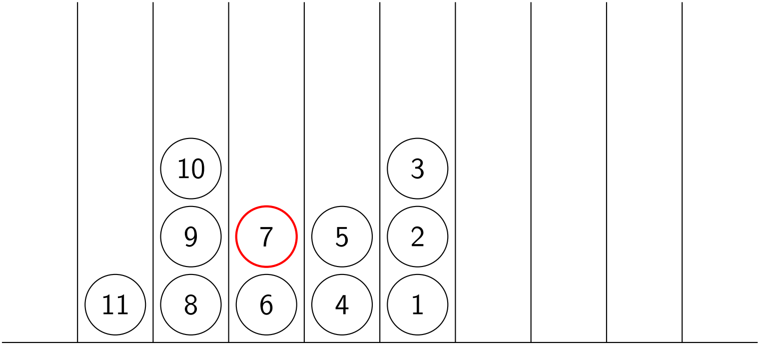

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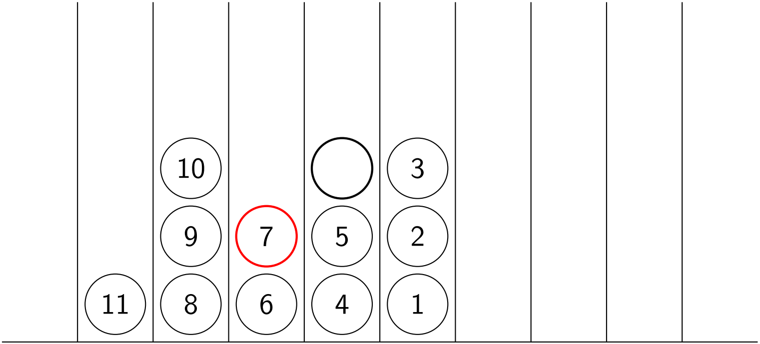

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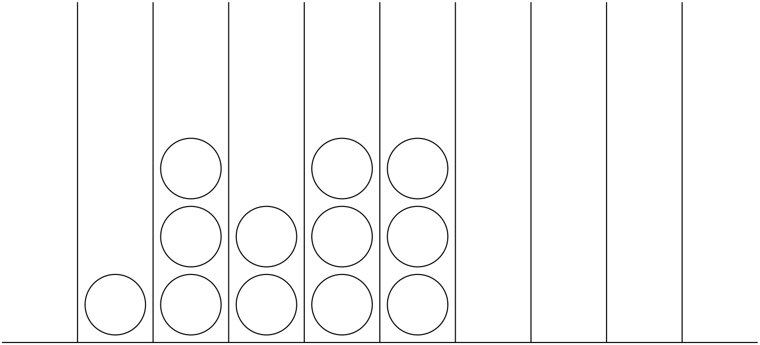

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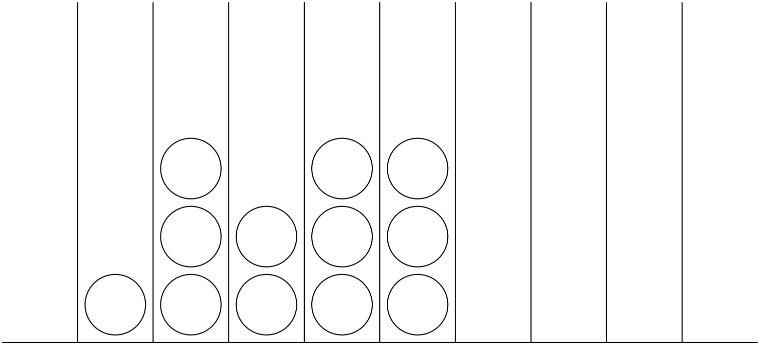

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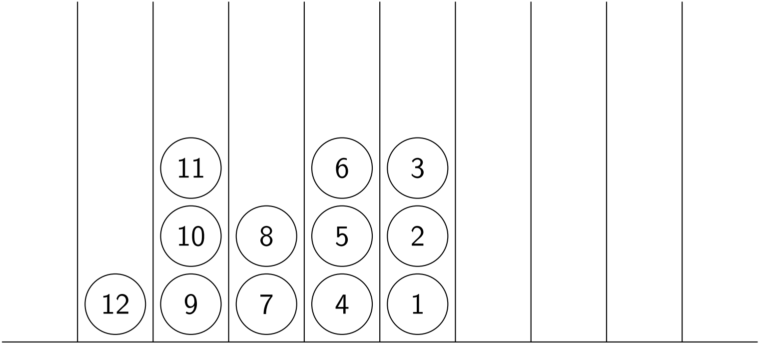

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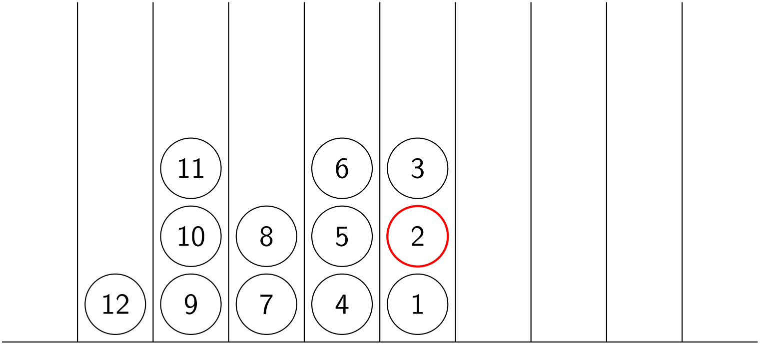

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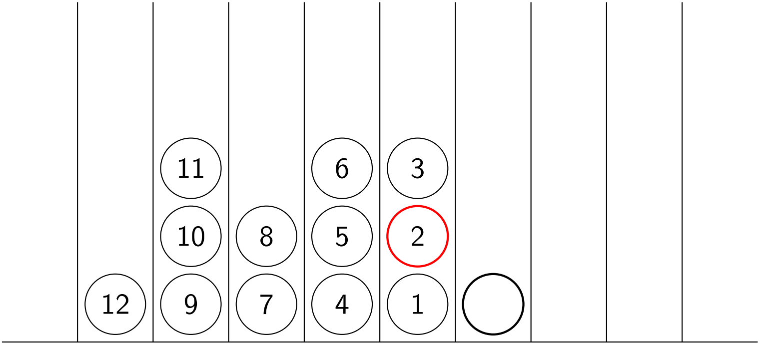

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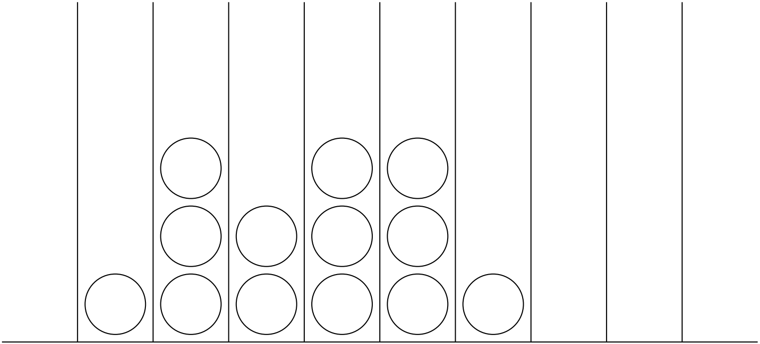

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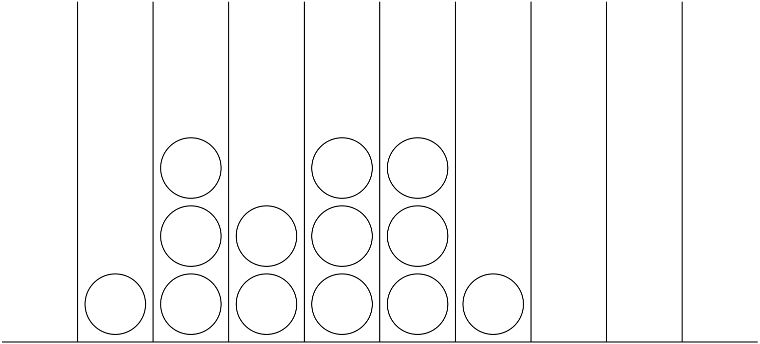

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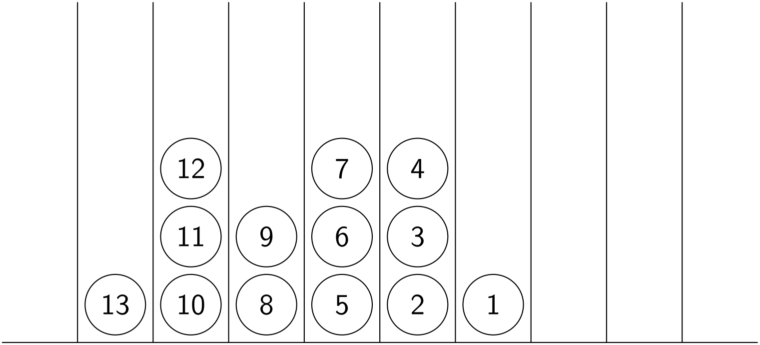

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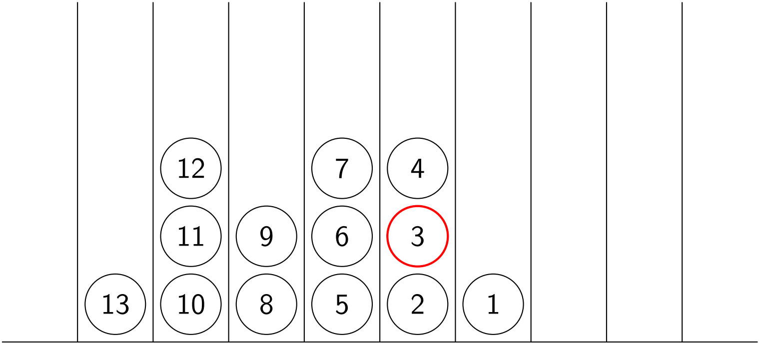

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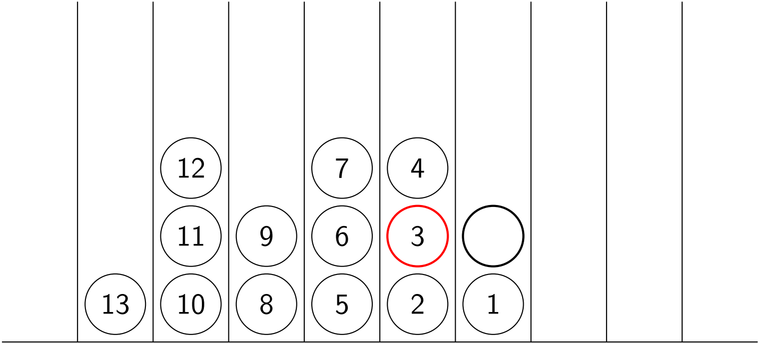

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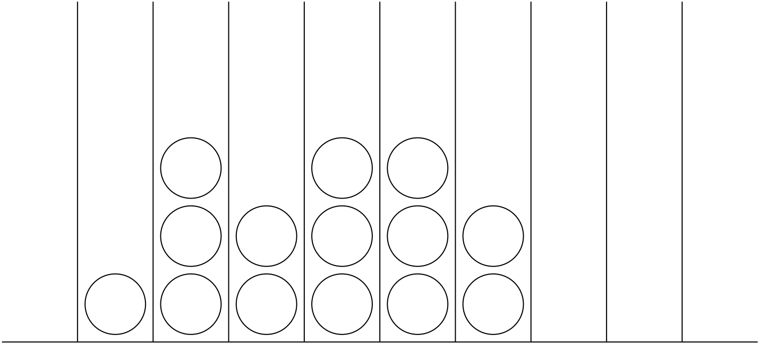

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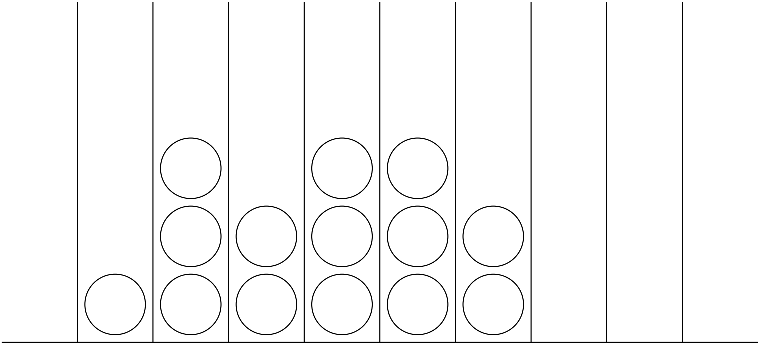

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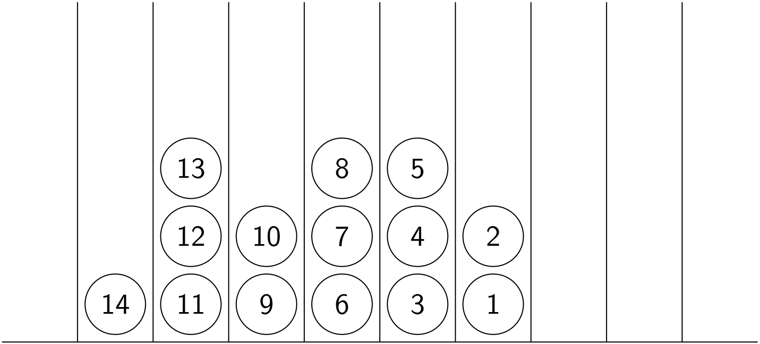

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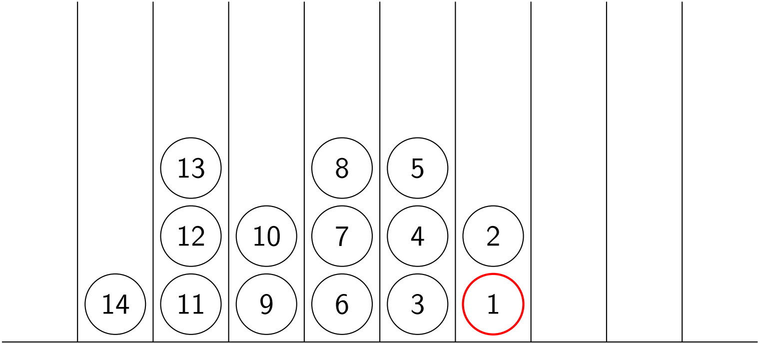

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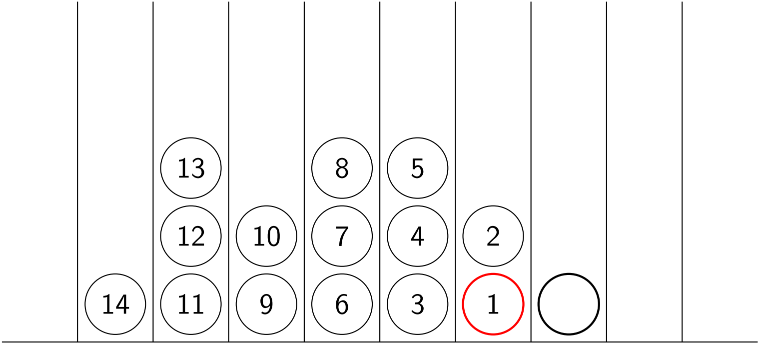

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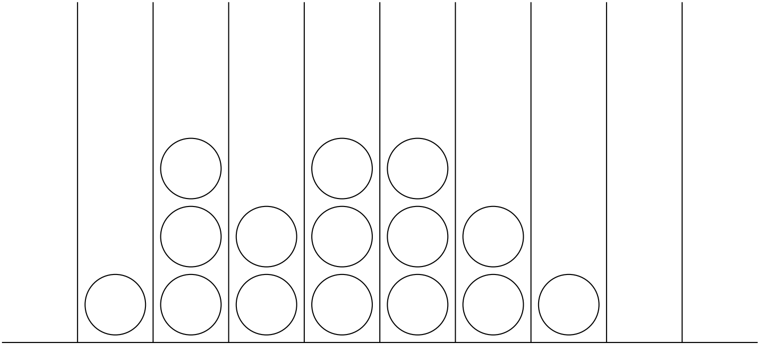

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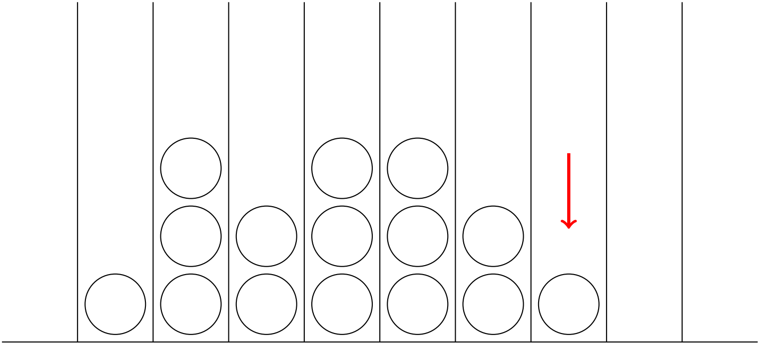

On the infinite-bin model

Existing results

- Aldous and Pitman (1993) studied a version of this model when ξ is the uniform distribution on $\{1, \ldots, N\}$.
- This general version introduced by Foss and Konstantopoulos in 2003.
- Studied using the existence of renewal event when $\mathbf{E}(\xi) < +\infty$ (Foss, Konstantopoulos, Chernysh, Ramassamy, Zachary).

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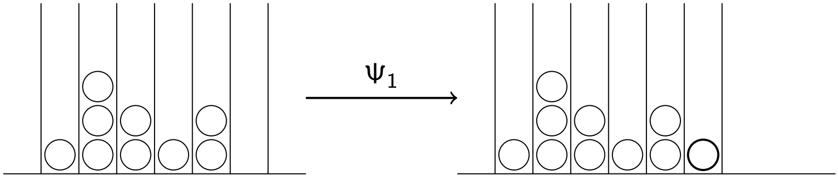
Construction of the infinite-bin model

Definition

Given X a configuration and $k \in \mathbb{N}$, we denote by $\Psi_k(X)$ the configuration with a ball added to the right of the kth rightmost ball in X.



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Given (ξ_n) i.i.d. random variables of law μ and X_0 a starting configuration, we call an IBM(μ) the process

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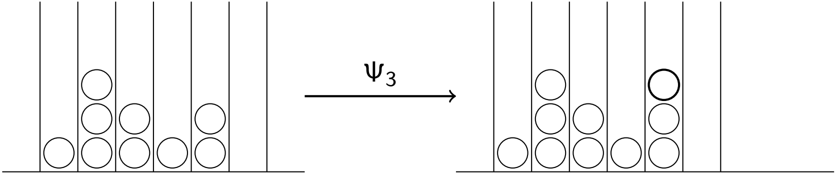

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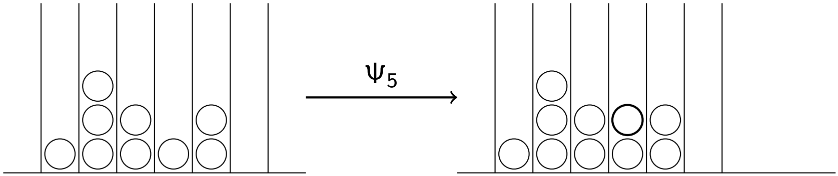

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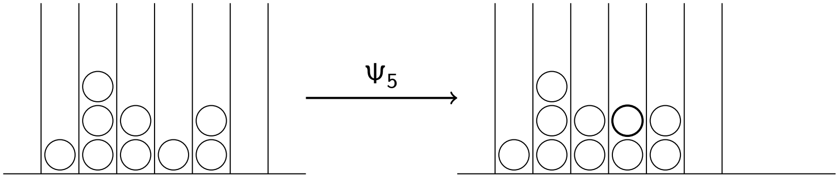

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A coupling for infinite-bin models

Partial order

Given X and Y two configurations, we say that $X \preccurlyeq Y$ if for every k, there are more balls to the right of kth urn in Y than in X.

Lemma

The function $(X, k) \mapsto \Psi_k(X)$ is decreasing with k and increasing with X.

Proposition

If (X_n), (Y_n) are two infinite-bin models defined with (ξ_n), (ζ_n), such that $X_0 \preccurlyeq Y_0$ and $\xi_k \geq \zeta_k$ for all $k \in \mathbb{N}$, then

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Speed of the infinite-bin model

Theorem (Foss-Konstantopoulos, M.-Ramassamy)

For any probability measure μ on $\mathbb{N}$, there exists $v_\mu \in [0, 1]$ such that writing F_n for the front at time n of an IBM(μ), we have

$$\lim_{n \to +\infty} \frac{F_n}{n} = v_\mu \quad a.s.$$

Proof of the existence of the speed

Proof.

- If the measure has finite support K, then the relative positions of the rightmost K balls form a Markov process.
- Hence the speed exists by ergodicity.
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Outline

Coupling the IBM and the Barak-Erdős graph

Coupling

One can couple a Barak-Erdős graph with parameter p with an IBM with geometric distribution $\mu_p(k) = p(1-p)^{k-1}$.

- Start with the empty graph, and the configuration with an infinite number of balls in bin -1.
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For any $p \in [0,1]$, we have $C(p) = v_{\mu_p}$.

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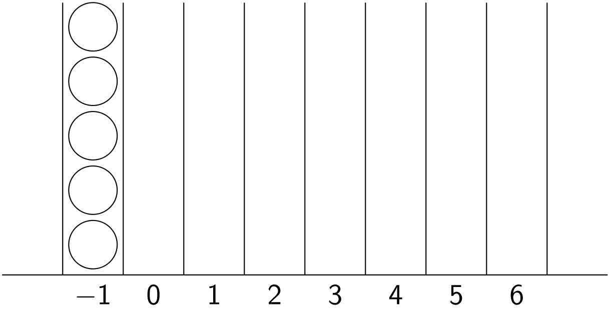

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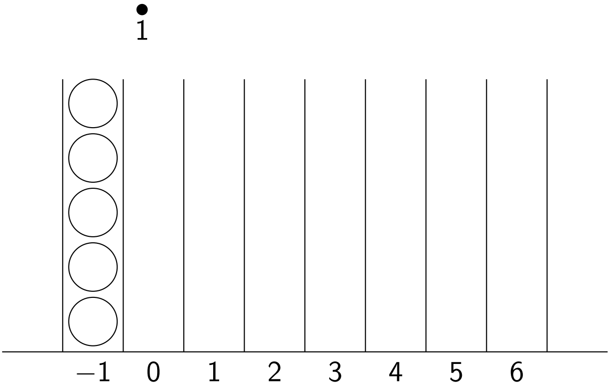

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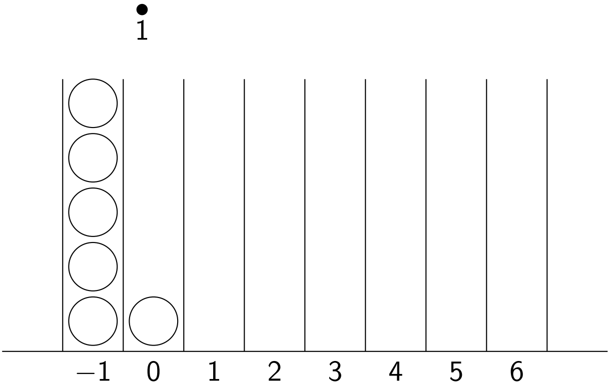

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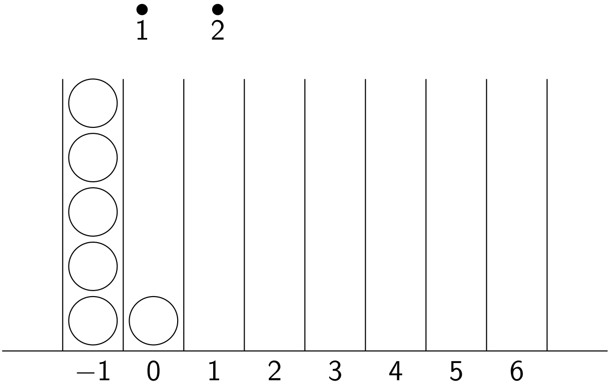

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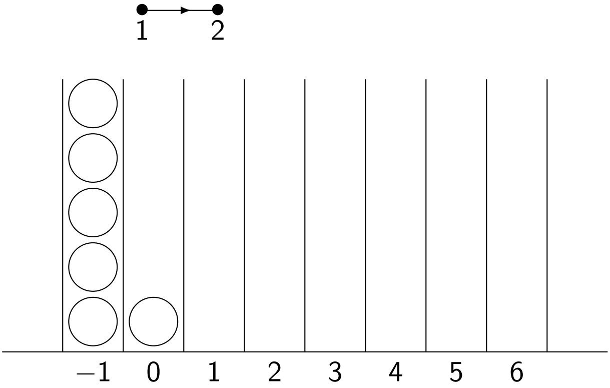

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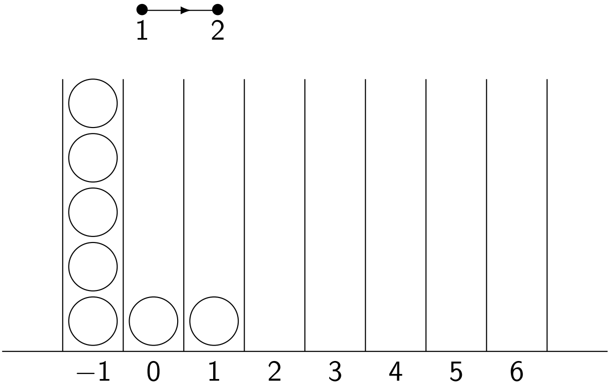

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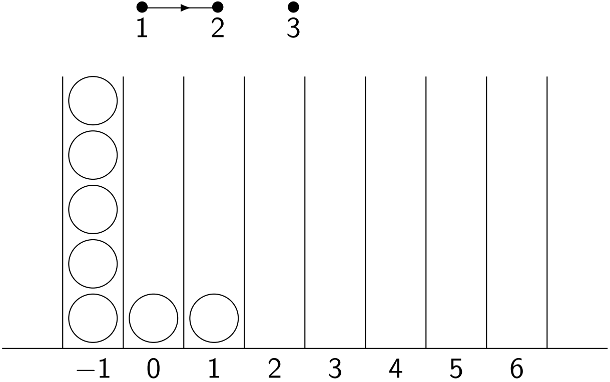

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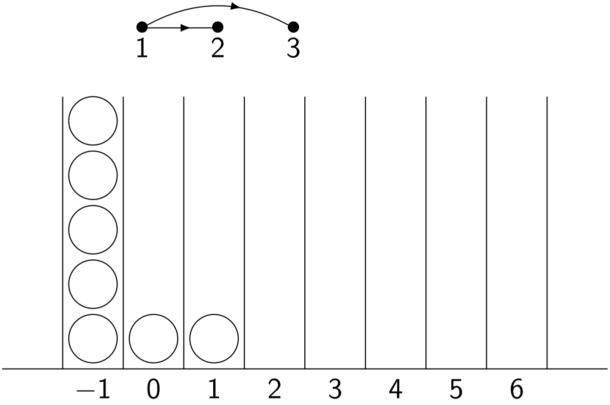

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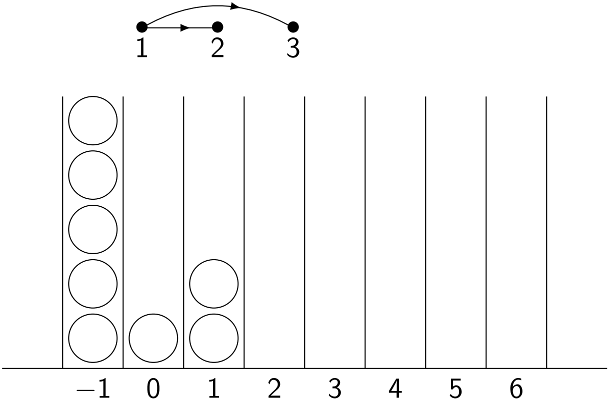

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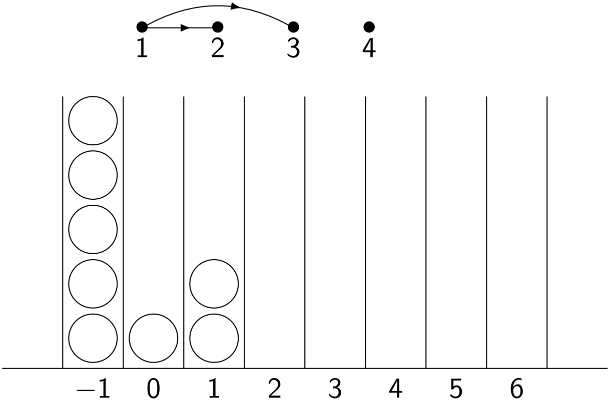

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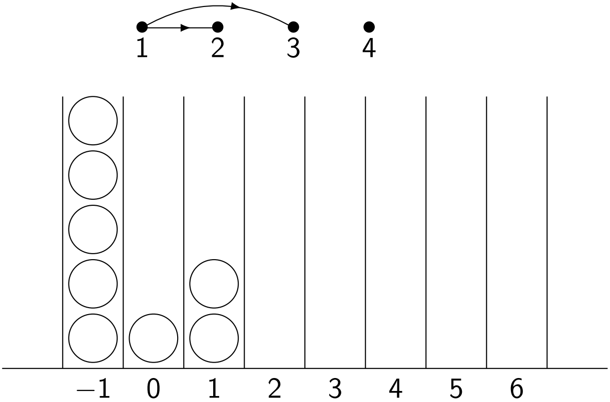

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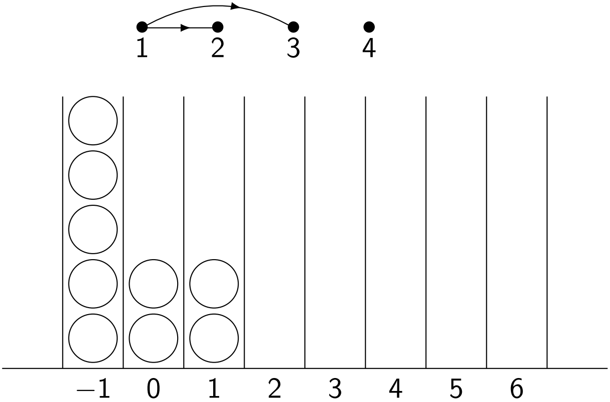

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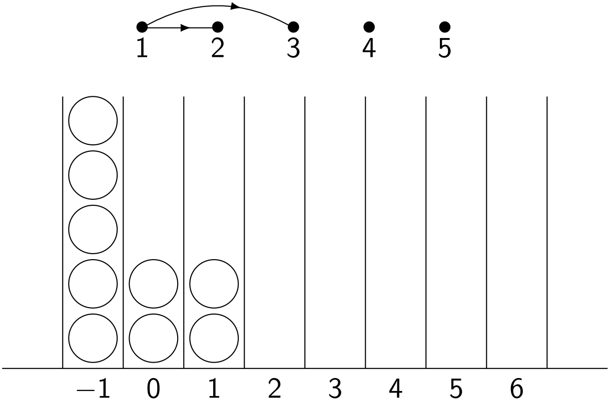

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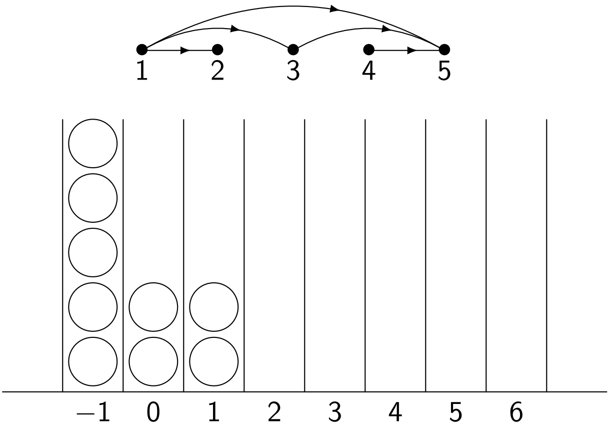

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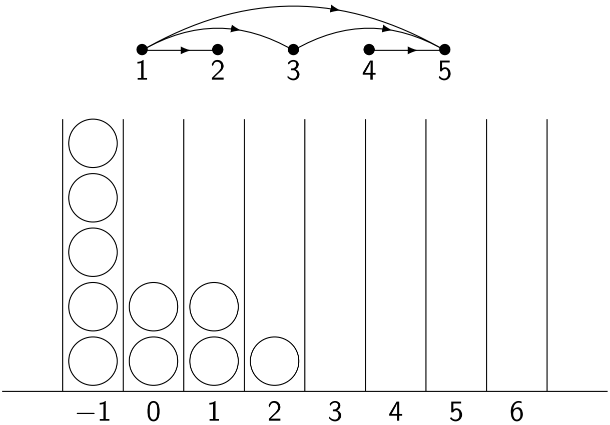

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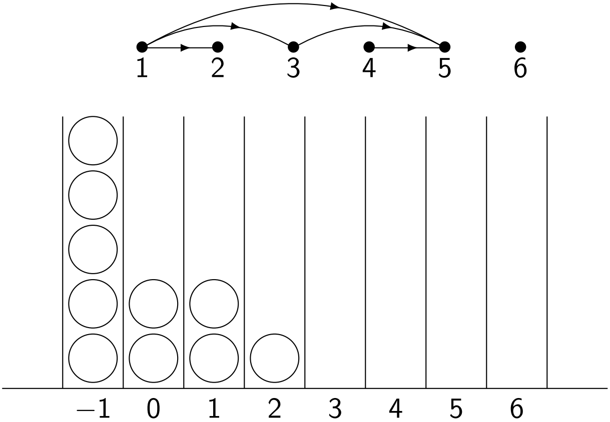

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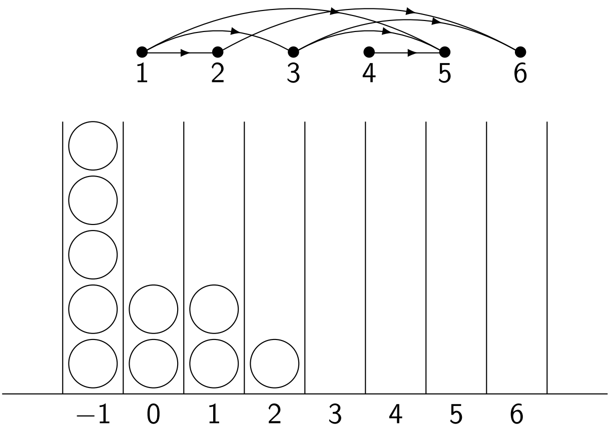

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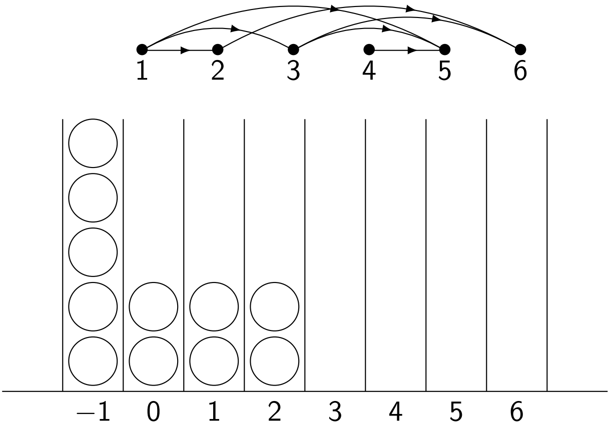

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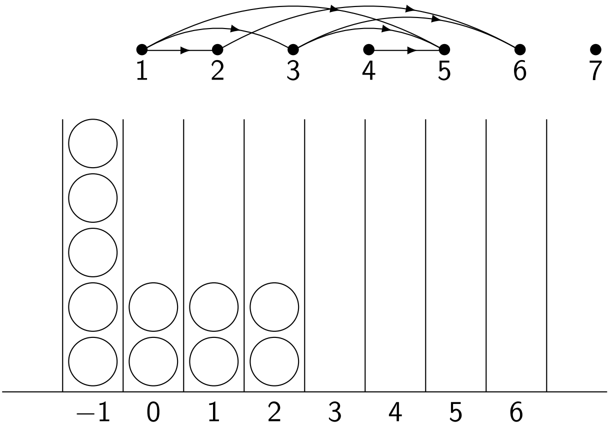

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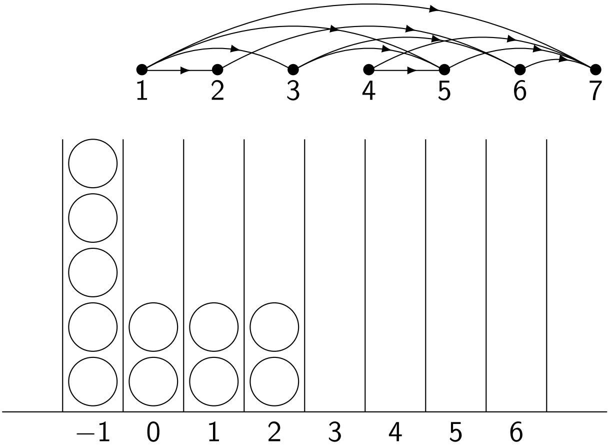

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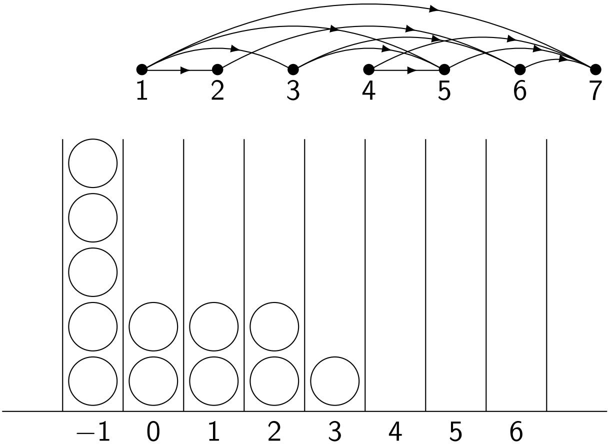

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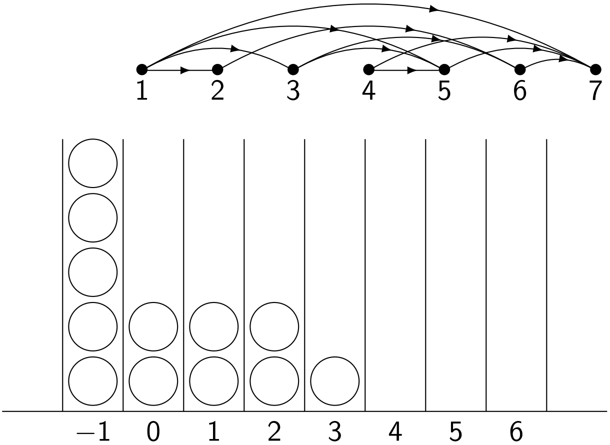

Asymptotic behaviour of C as $p \to 1$

Strategy of proof

- We use the $\mathbb{L}^1$ convergence of the position of the front at time n F_n:

$$\lim_{n \to +\infty} \frac{1}{n} \mathbf{E}(F_n) = C(p).$$

- We observe that $\mathbf{E}(F_n)$ can be computed for large p as the sum of the contributions of small complex patterns arising in the middle of long sequences of 1.
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We assume that p is close to 1. Recall that $\mu_p(k) = p(1-p)^{k-1}$.

Approximate behaviour

Up to $o(1)$ corrections, (ξ_n) looks like

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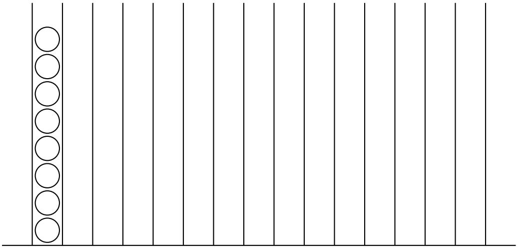

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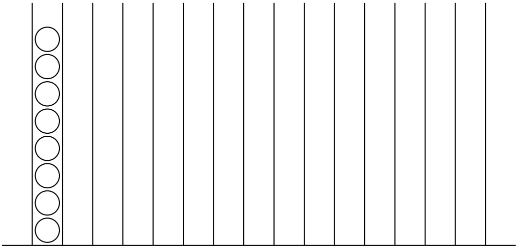

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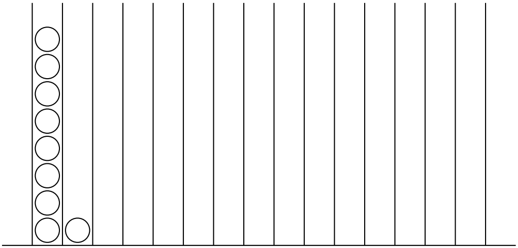

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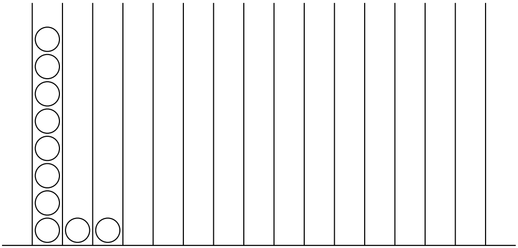

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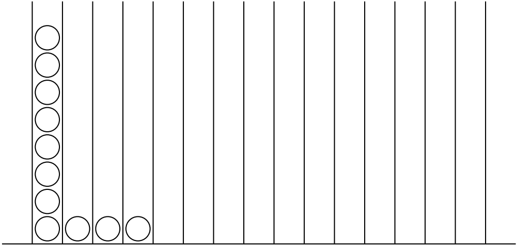

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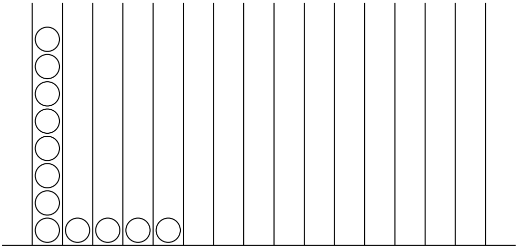

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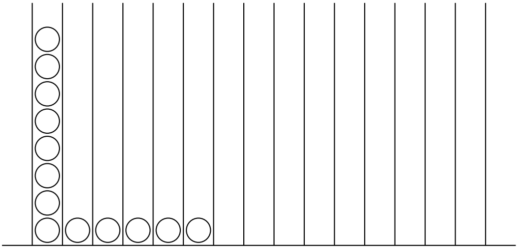

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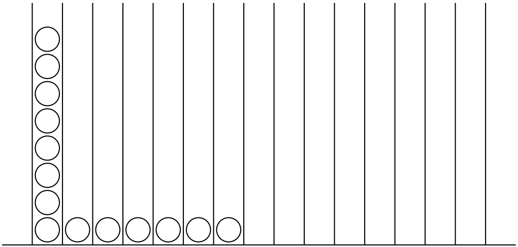

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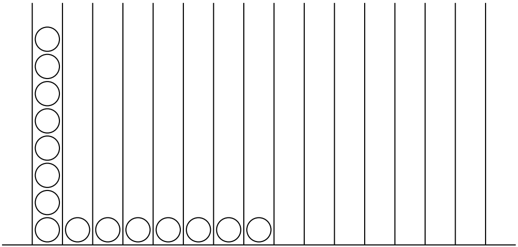

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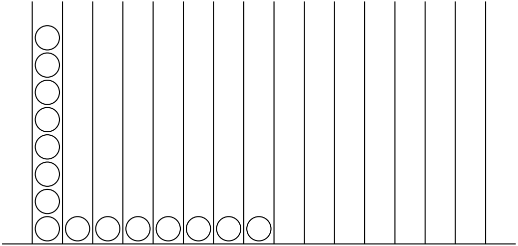

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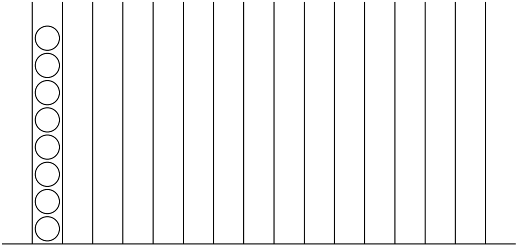

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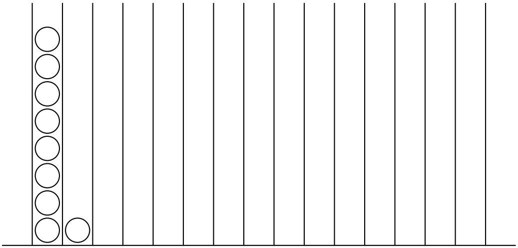

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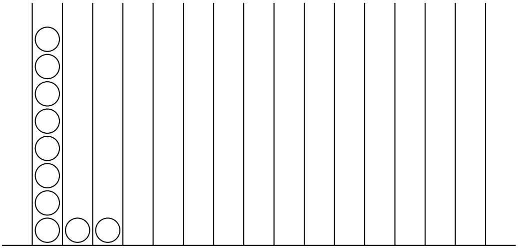

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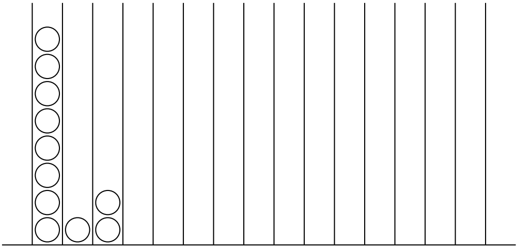

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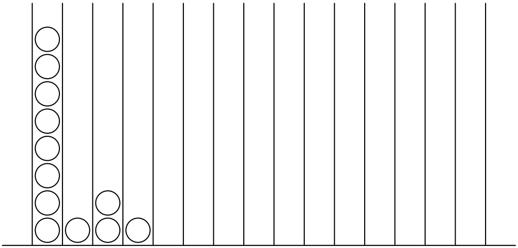

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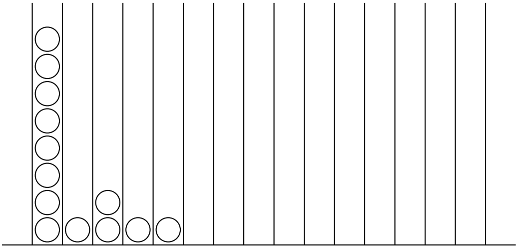
Therefore $C(p) = 1 - p(1-p) + o((1-p))$.



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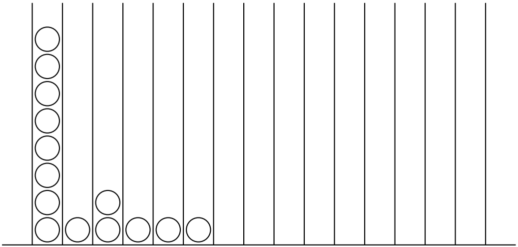

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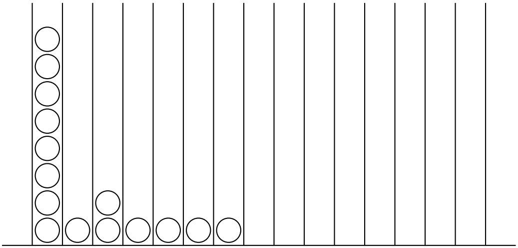

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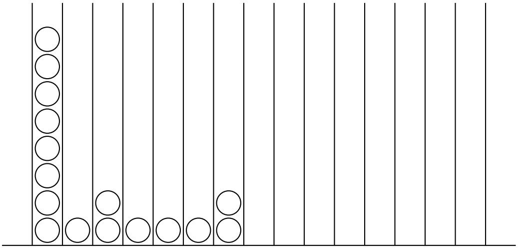

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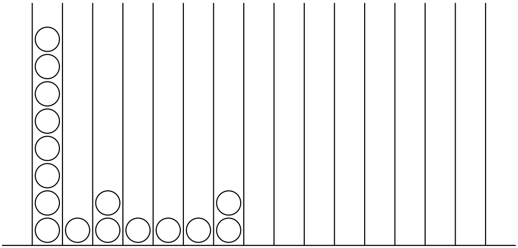

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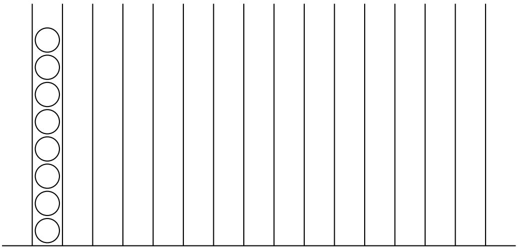

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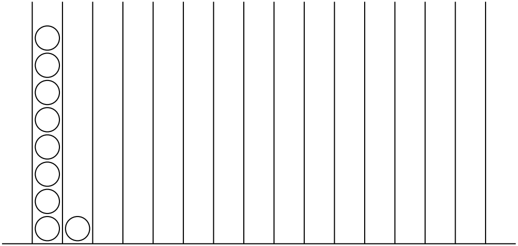

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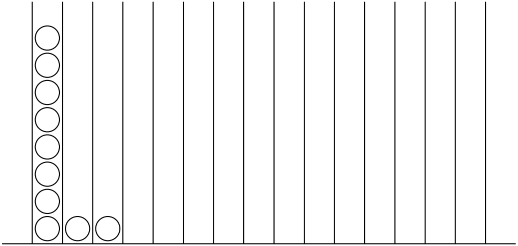

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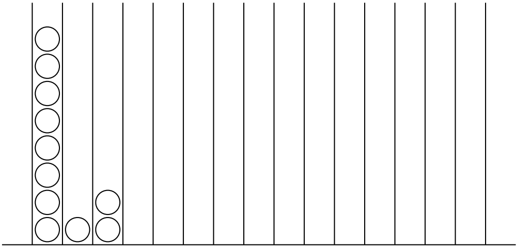

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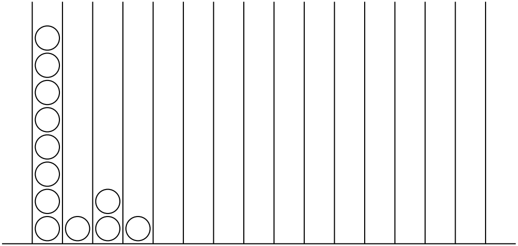

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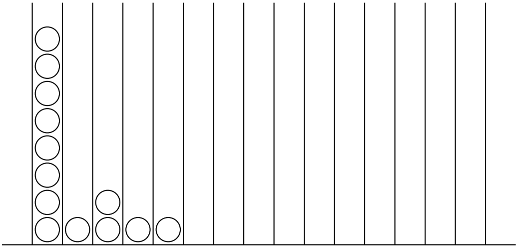

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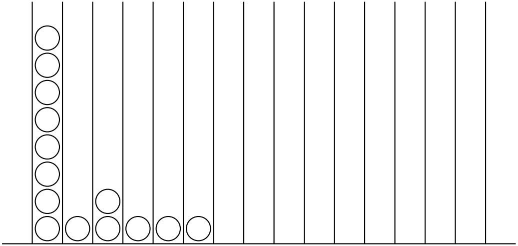

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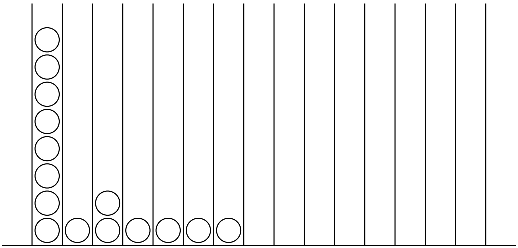

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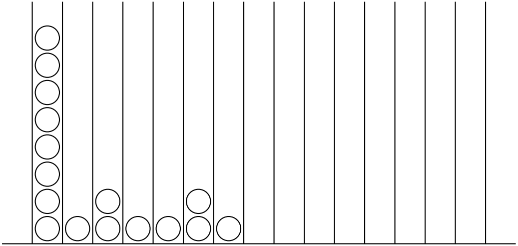

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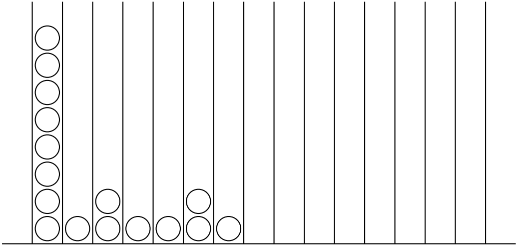

Finding the asymptotic expansion

Aim

To each finite pattern $u = (u(1), \ldots u(n)) \in \cup\mathbb{N}^n$, we would like to associate a term $\varepsilon(u) \in \{-1, 0, 1\}$ such that for any $N \in \mathbb{N}$,

$$C(p) = \sum_{n\geq 0} \sum_{u\in\cup\mathbb{N}^n} \varepsilon(u)\mathbf{P}(\xi_1 = u(1), \ldots \xi_n = u(n)) + o((1-p)^N)$$

Definition

For each finite pattern u, we denote by $d(u)$ the distance the front travels when applying successively $\Psi_{u(1)}, \ldots, \Psi_{u(n)}$.

We define ε as the solution of the following equation:

$$d(u) = \sum_{v \text{ subpattern of } u} \varepsilon(v) = \sum_{k=1}^{|u|} \sum_{j=1}^{|u|-k} \varepsilon(u(j), u(j+1), \ldots u(j+k-1)).$$

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A direct definition for ε

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For u a pattern, we write πu the pattern obtained by forgetting the last number and

$$\delta(u) = d(u) - d(\pi u) \in \{0, 1\},$$

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Analyticity of C

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For any $p \in (1/2, 1]$, *we have* $C(p) = \sum_u \varepsilon(u) p^{|u|} (1-p)^{\sum (u(j)-1)}$.

Proof.

$$\begin{aligned} C(p) &= \lim_{n\to+\infty} \frac{1}{n} \mathbf{E}(d(\xi_1, \dots \xi_n)) \\ &= \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n-k} \mathbf{E}(\varepsilon(\xi_j, \xi_{j+1}, \dots \xi_{j+k-1})) \\ &= \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (n-k) \mathbf{E}(\varepsilon(\xi_1, \xi_2, \dots \xi_k)) \\ &= \sum_{k=1}^{+\infty} \mathbf{E}(\varepsilon(\xi_1, \xi_2, \dots \xi_k)). \end{aligned}$$

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Conclusion

We were able to study the function C by coupling Barak-Erdős graphs with Infinite-Bin models. This function:

- Is analytic on $(0, 1]$;
- Behaves as $ep(1 - \pi^2/2(logp)^2)$ at $p = 0$;
- Its series expansion can be computed as a perturbation expansion.

Some open questions:

- Is $p \mapsto C(p)/p$ convex?
- Can similar computations be made with $C_k(p)$ the time taken to undertake a series of tasks with k servers.

Thank you for your attention!

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Asymptotic behaviour of C as $p \to 0$

Strategy of proof

- Using the increasing coupling, we have

$$C(p) \approx \text{speed of an IBM with uniform distribution on } \left\{1, \ldots, \left\lfloor \frac{1}{p} \right\rfloor \right\}.$$

- The speed of an IBM with uniform distribution is coupled with a branching random walk with selection.
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Bound with an infinite-bin model with uniform distribution

Notation

We write w_N the speed of an infinite-bin model with uniform distribution on $\{1, \dots, N\}$.

Upper bound

For any $p \in [\frac{1}{N+1}, \frac{1}{N}]$, we have $C(p) \leq w_N$.
Indeed, we have $\sum_{j=1}^{k} p(1-p)^{j-1} \leq (pk) \wedge 1$, thus we can couple a geometric random variable G and a uniform random variable U such that $G \geq U$ a.s.

Lower bound

For any $p \in [0, 1]$, we have $C(p) \geq Np(1-p)^N w_N$.

Conclusion

$$C(1/N) \approx w_N \quad \text{as } N \to +\infty.$$

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A N-branching random walk in continuous-time

Behaviour of the rightmost N balls

We consider the process $(X_{P_t}, t \geq 0)$, where P is an independent Poisson process of intensity N.

- At rate N an event occurs.
- With probability $1/N$, one of the N rightmost ball makes an offspring to its right.
- The leftmost ball is removed from consideration.

Alternative description

- A clock on each of the N rightmost balls will ring at rate 1 independently.
- The selected ball makes an offspring to its right.
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A N-branching random walk in continuous-time

Behaviour of the rightmost N balls

We consider the process $(X_{P_t}, t \geq 0)$, where P is an independent Poisson process of intensity N.

- At rate N an event occurs.
- With probability $1/N$, one of the N rightmost ball makes an offspring to its right.
- The leftmost ball is removed from consideration.

Alternative description

- A clock on each of the N rightmost balls will ring at rate 1 independently.
- The selected ball makes an offspring to its right.
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Brunet-Derrida behaviour of branching random walks with selection

Theorem (Bérard-Gouéré 2010)

Under some assumptions, if we denote by v_N the speed of a branching random walk with selection, there exist explicit v_∞ and $\chi > 0$ such that

$$v_N - v_\infty \sim_{N \to +\infty} -\frac{\chi}{(\log N)^2}.$$

Notation

More precisely, setting $\kappa(\theta) = \log \mathbf{E}(\sum_{|u|=1} e^{\theta V(u)})$, we have

$$v_\infty = \inf_{\theta > 0} \frac{\kappa(\theta)}{\theta} \qquad \theta_* \text{ solution of } \theta\kappa'(\theta) - \kappa(\theta) = 0$$

$$\sigma^2 = \kappa''(\theta^*) \qquad \chi = -\frac{\pi^2\sigma^2}{2}\theta^*.$$

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Conclusion

Theorem

We have $C(p) = p\left(e - \frac{\pi^2 e}{2(\log p)^2}\right)$.

Proof.

Recall that $C(1/N) \approx \frac{1}{N} v_N$.
We have $\kappa(\theta) = e^\theta$, thus:

- $v_\infty = e$;
- $\theta^* = 1$;
- $\sigma^2 = e$.

This concludes the proof. □