Thresholds

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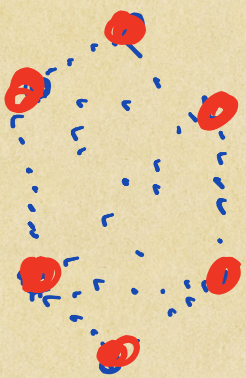
Random Discrete Structures: An Example

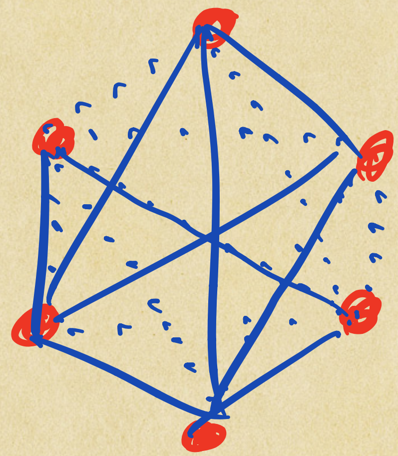
$G(n,p)$: the Erdős-Rényi random graph

n-vertices, each edge w/ prob. p

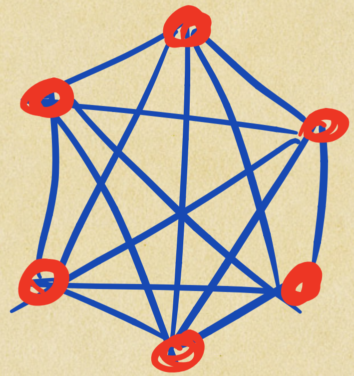
$p=0$



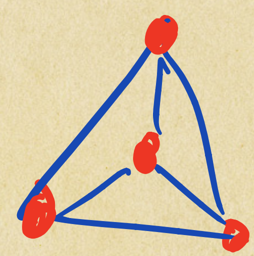
$p=\frac{1}{2}$

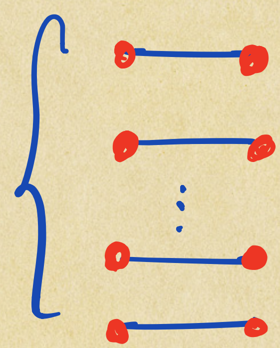


$p=1$



Two Questions about $G(n,p)$

Q1: When does $G(n,p)$ contain a $K_4 =$  ?

Q2: When does $G(n,p)$ contain a perfect matching $\frac{n}{2}$  ?

An easy calculation for Q1

X = # copies of K_4 in $G(n,p)$

Compute $\mathbb{E}[X]$ i.e. the expectation.

If $\mathbb{E}[X] \ll 1$, $\mathbb{P}[X>0] \ll 1$.

$$\mathbb{E}[X] = \binom{n}{4} p^6 \simeq n^4 p^6$$

So $\mathbb{E}[X] \ll 1$ when $p \ll n^{-2/3}$

$\Rightarrow G(n,p)$ is unlikely to have K_4's when $p \ll n^{-2/3}$

In fact, computing $\mathbb{E}[X^2]$ shows:

$G(n,p)$ is likely to contain a K_4 when

$$p \gg n^{-2/3}$$

$$\boxed{\text{The threshold for } G(n,p) \text{ to contain } K_4 \text{ is } n^{-2/3}}$$

Thresholds

X - finite set

$\mathcal{F}$ - increasing family on X $(\neq \emptyset, 2^X)$

μ_p - p-biased measure $\left(\mu_p(S) = p^{|S|}(1-p)^{|X \setminus S|}\right)$

$$\boxed{p_c(\mathcal{F}) = p \text{ s.t. } \mu_p(\mathcal{F}) = 1/2}$$

__Theorem__ (Bollobás-Thomason)

$$\mu_p(\mathcal{F}) \to \begin{cases} 1 & p/p_c \to \infty \\ 0 & p/p_c \to 0 \end{cases}$$

__Q1__: $X = \{\text{edges of } K_n\}$, $\mathcal{F} = \{G : G \supseteq K_4\}$

$$\underline{\underline{p_c(\mathcal{F}) \asymp n^{-2/3}}}$$

Another easy calculation for Q2

$X = \#$ perfect matchings in $G(n,p)$

What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \frac{n!}{\left(\frac{n}{2}\right)!\, 2^{n/2}} \cdot p^{n/2} \approx \left(\frac{np}{e}\right)^{n/2}$$

So $\mathbb{E}[X] \ll 1$ when $p \ll 1/n$

$$\Rightarrow \quad P_C(\text{"perfect matching"}) \gtrsim 1/n$$

<u>Truth</u> : p_c ("perfect matching") $\simeq \frac{\log n}{n}$.

(Due to Erdős-Rényi, from 60s)

Why is $p_c \gtrsim \frac{\log n}{n}$?

Given $\mathcal{F}$, say $\exists$ a "witness" $\mathcal{G} \subseteq 2^X$ s.t.

① $\mathcal{F} \subseteq \langle \mathcal{G} \rangle$ increasing family above $\mathcal{G}$

② $\sum_{S \in \mathcal{G}} p^{|S|} \leq 1/2$

① & ② $\Rightarrow$ $p_c(\mathcal{F}) \geq p$

Expectation-Thresholds

$$\boxed{q(\mathcal{F}) = \max\{p : \exists\, \mathcal{G} \text{ w/ } 1 \ \& \ 2\}}$$

<u>Trivial</u>: $p_c(\mathcal{F}) \geq q(\mathcal{F})$

Conj (Kahn-Kalai): $p_c(\mathcal{F}) \lesssim q(\mathcal{F}) \log |X|$

- Implies many hard results
  - → "Shamir's" problem
  - → Bounded-degree trees.
- Obstacle: How do we find the right witness $\mathcal{G}$?

Fractional Expectation-Thresholds

Replace $\mathcal{G} : 2^X \to \{0,1\}$ by a

$$g : 2^X \to [0,1]$$

$$\boxed{q^*(\mathcal{F}) = \max\{p : \exists \text{ such a } g\}}$$

As before: $p_c \geq q^* \geq q$

Conj (Talagrand)

$$p_c(\mathcal{F}) \lesssim q^*(\mathcal{F}) \log \underline{\underline{\ell(\mathcal{F})}}$$

largest size of a minimal element

~~Conj (Talagrand)~~ ; now Thm (FKNS)

- Still gives all applications of KK
- In all examples: $q \approx q^*$

Conj (Talagrand): $q(\mathcal{F}) \simeq q^*(\mathcal{F})$

Advantage of q^{*} over q: Duality!

$$q^{*}(\mathcal{F}) \approx \min \left\{ p : \mathcal{F} \supseteq \langle \mathcal{H} \rangle \text{ \& } \mathcal{H} \text{ is } p\text{-spread} \right\}$$

Spread

$\mathcal{H} \subseteq 2^X$, multiset/hypergraph, p-spread if

$$\boxed{|\mathcal{H} \cap \langle S \rangle| \leq p^{|S|} |\mathcal{H}| \quad \forall S \subseteq X}$$

What we prove:

If $\mathcal{H}$ is p-spread, then a random set of density $10\, p \log |X|$ will be in $\langle \mathcal{H} \rangle$ w.h.p.

Builds On:

Alweiss-Lovett-Wu-Zhang
(on the sunflower conjecture)

Application to Q2

Want $p_c(\text{"perfect matching"}) \lesssim \frac{\log n}{n}$

$\hookrightarrow$ Enough to show $q^*(\text{"p.m."}) \lesssim 1/n$

$\hookrightarrow$ Enough to find a $1/n$-spread $\mathcal{H}$ s.t. "p.m." $\supseteq \langle \mathcal{H} \rangle$. $\|$ Easy!

$\Rightarrow p_c(\text{"p.m."}) \asymp \frac{\log n}{n}$ (classical)

BUT: Trivially generalises from random graphs to hypergraphs

$p_c(\text{"r-graph p.m."}) \asymp \frac{\log n}{n^{r-1}}$.

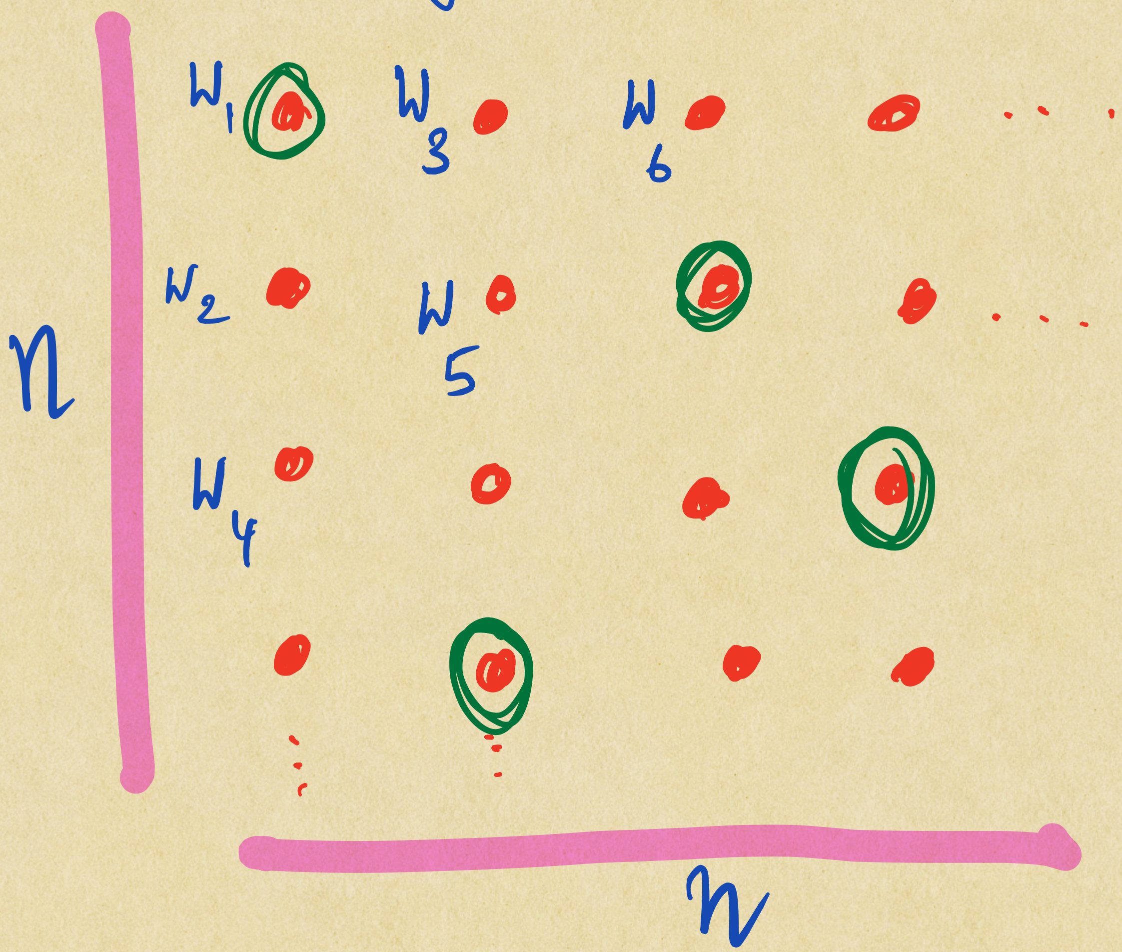
$\hookrightarrow$ HARD (Johansson-Kahn-Vu)

Other Applications: Random (hyper)-graphs

- Bounded-degree Trees.
- Hamilton Cycles.
- Bounded-degree Graphs.

One final application: Statistical Physics.

2-d assignment problem



Want n pts

1 per row & col

s.t. $Z_2 = \sum' \text{wts}$

is minimum.

Easy : $\mathbb{E}[Z_2] \gtrsim 1$

Hard : $\mathbb{E}[Z_2] \sim \pi^2/6$ (Aldous)

In 3-d: Each pt of $[n]^3$ given an Exp(1) weight.

Want: n pts meeting each plane exactly once s.t. $Z_3 = \sum$ wts is min.

$$\mathbb{E}[Z_3] \gtrsim ?$$

Conjecture (Frieze - Sorkin)

$$\mathbb{E}[Z_3] \asymp 1/n.$$

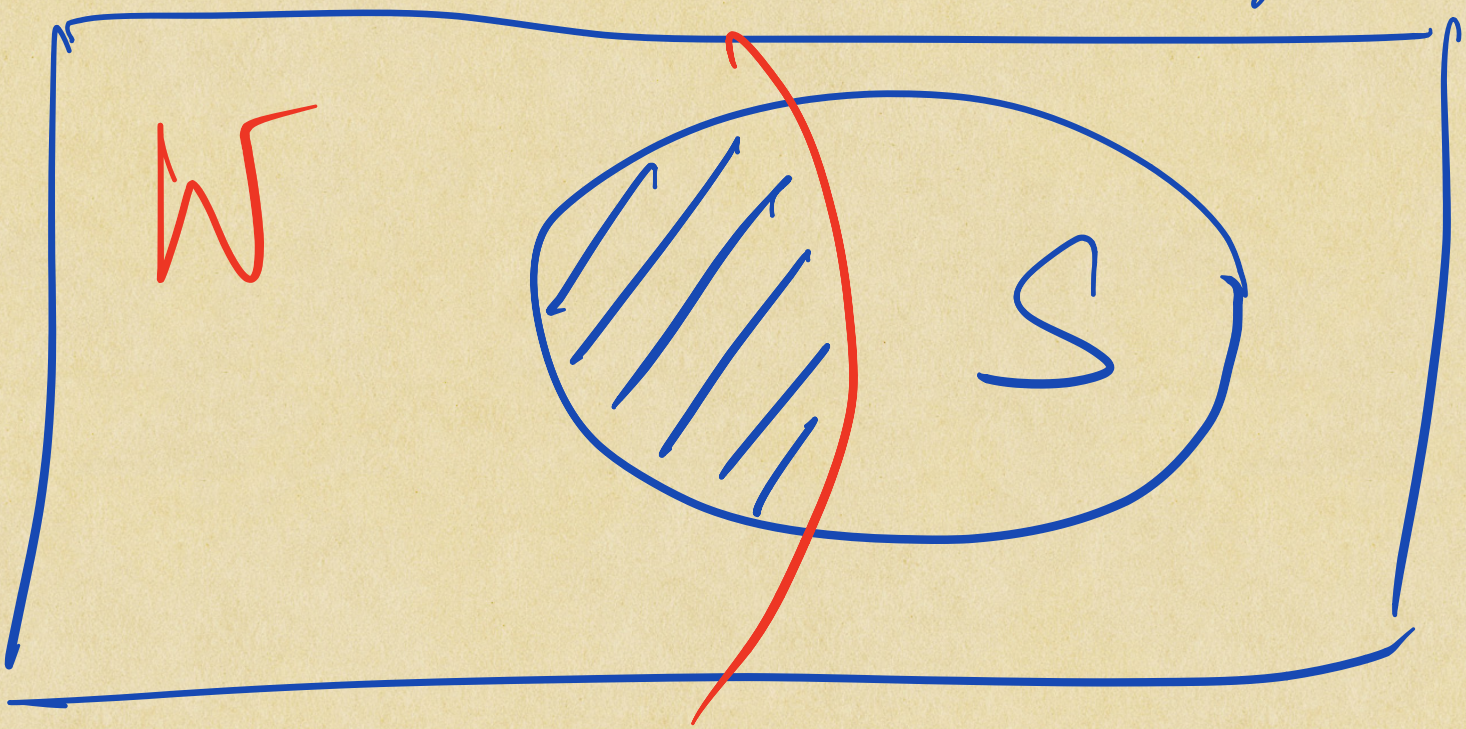
Thm (FKNS): $\mathbb{E}[Z_r] \asymp 1/n^{r-1}$

A Sketch (of a Sketch)

Goal: Random set lands in $\langle \mathcal{H} \rangle$

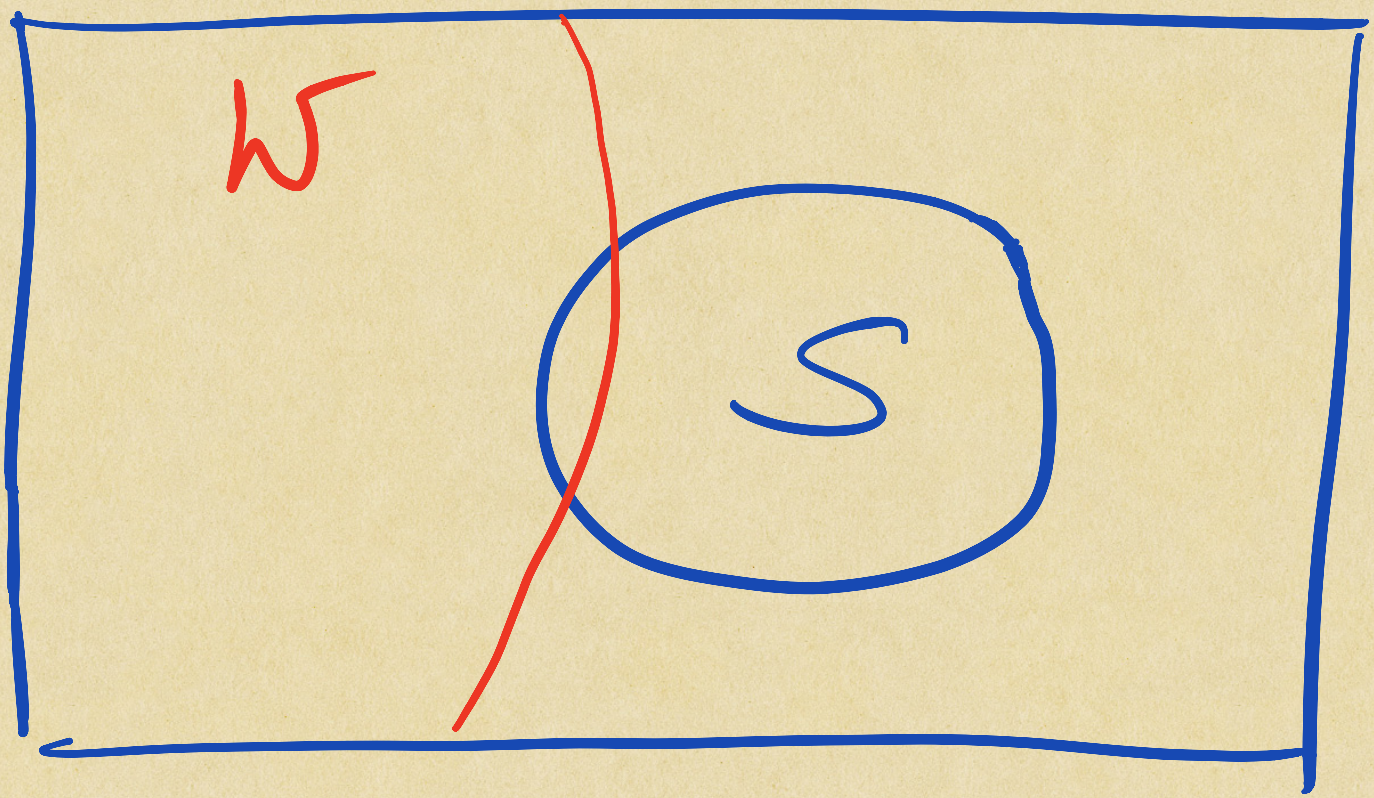
Dream: W – random "nibble"

S – typical member of $\mathcal{H}$.



<u>Unfortunately</u> :

Typically



But:

Also typically

Great: $(S \setminus W) \to (S' \setminus W)$

Open Problems

- $q \simeq q^{\star} \equiv$ Kahn-Kalai
- Universality phenomena.