Scaling limits of the two- and three-dimensional uniform spanning trees

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joint with
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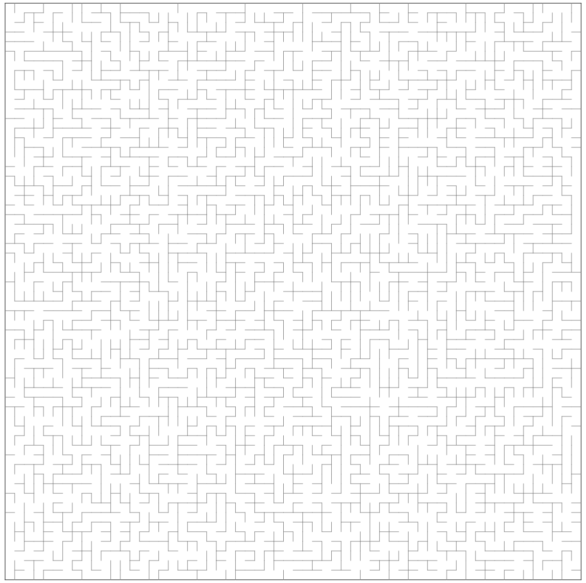
1. THE MODEL

UNIFORM SPANNING FOREST ON $\mathbb{Z}^d$



Let $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$.

A subgraph of the lattice is a **spanning tree** of Λ_n if it connects all vertices and has no cycles.

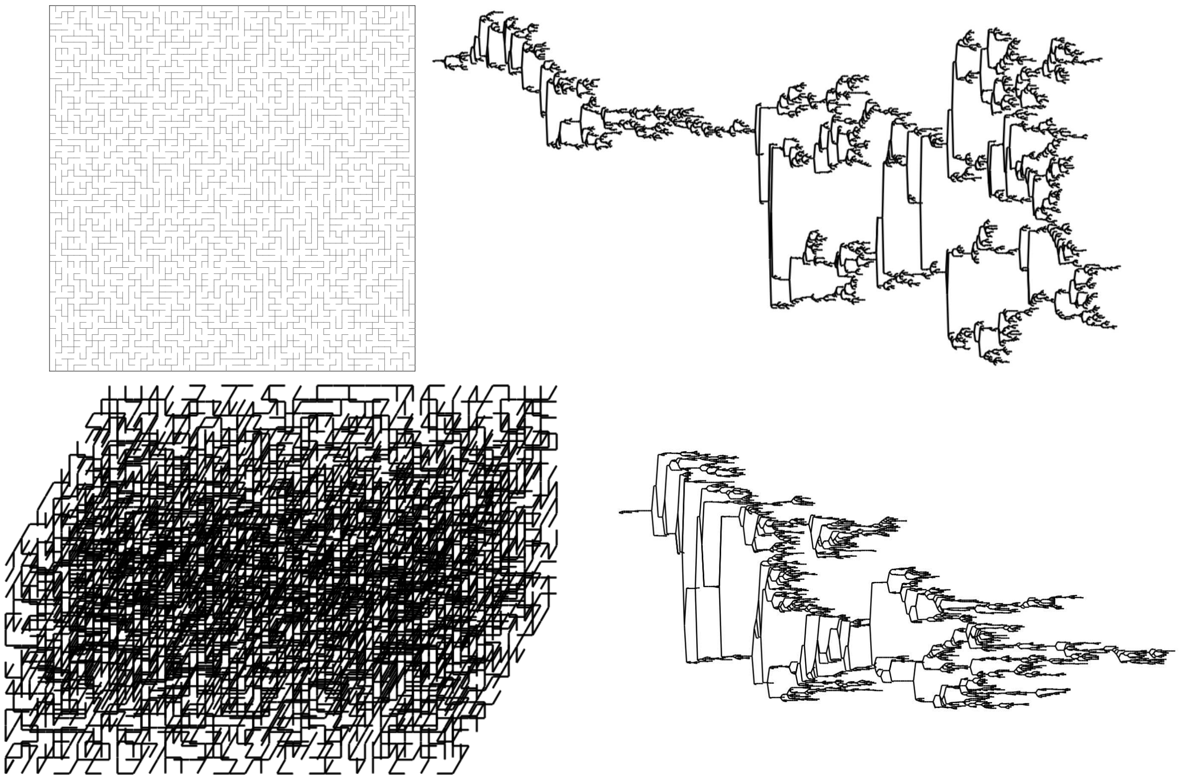
Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities.

The USF on $\mathbb{Z}^d$, $\mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.
NB. Wired/free boundary conditions unimportant.

For $d = 2, 3, 4$, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^d$, a.s. (Forest for $d > 4$.)

[Aldous, Benjamini, Broder, Häggström, Hutchcroft, Kirchoff, Lyons, Nachmias, Pemantle, Peres, Schramm, Wilson,. . .]

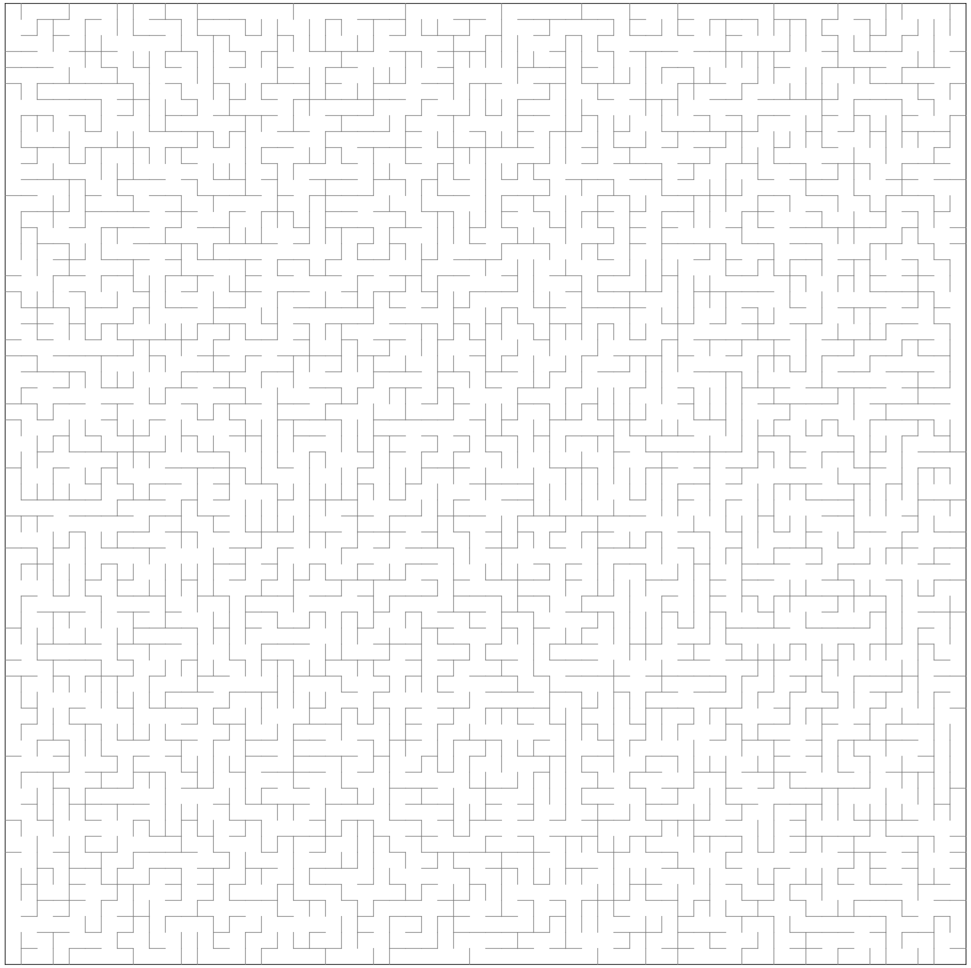
GENEALOGICAL STRUCTURE



2d animation: Bostock, adapted to 3d by C.

2. SCALING LIMITS

PATHS IN THE 2d-UST



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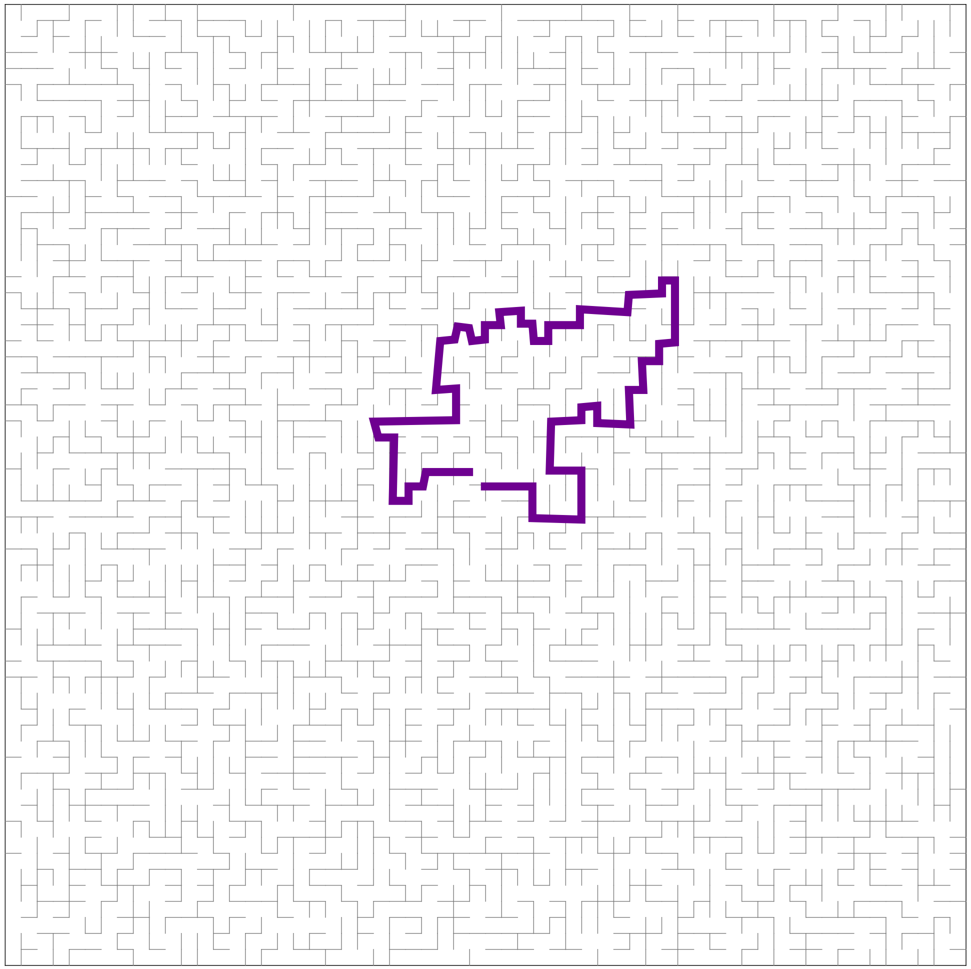

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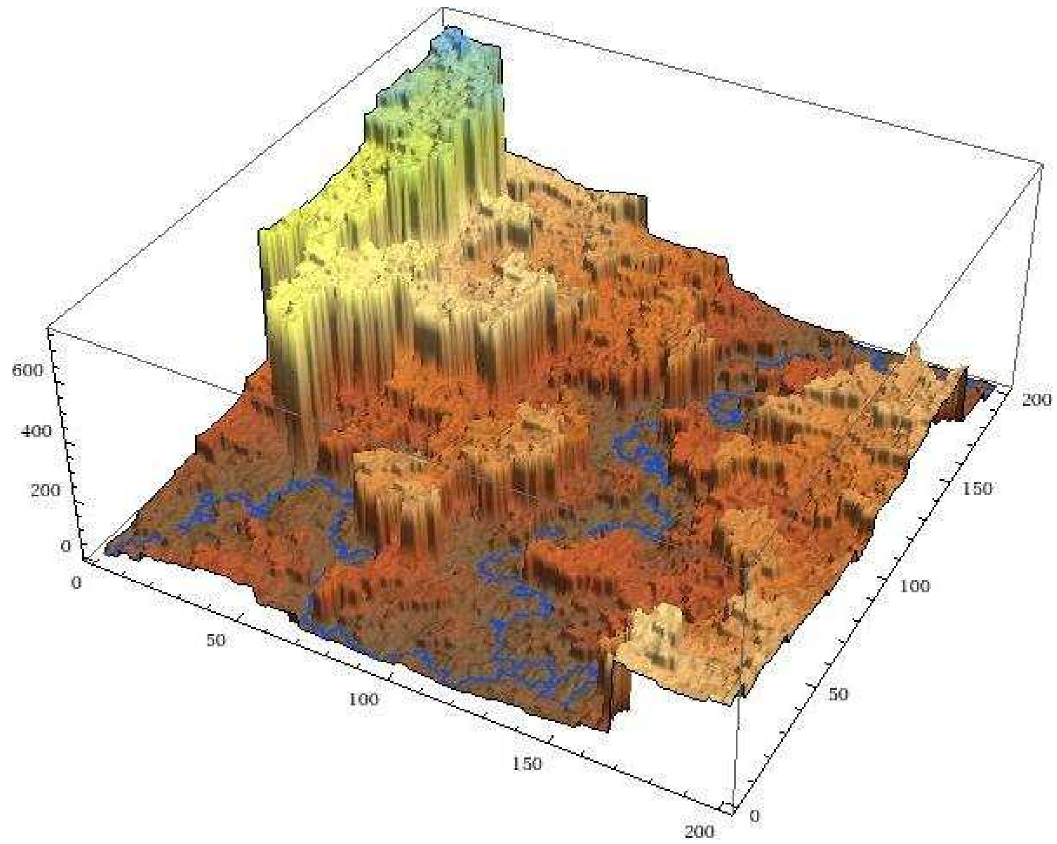


The distances in the tree to the path between opposite corners in a uniform spanning tree in a 200×200 grid.

Picture: Lyons/Peres: Probability on trees and networks

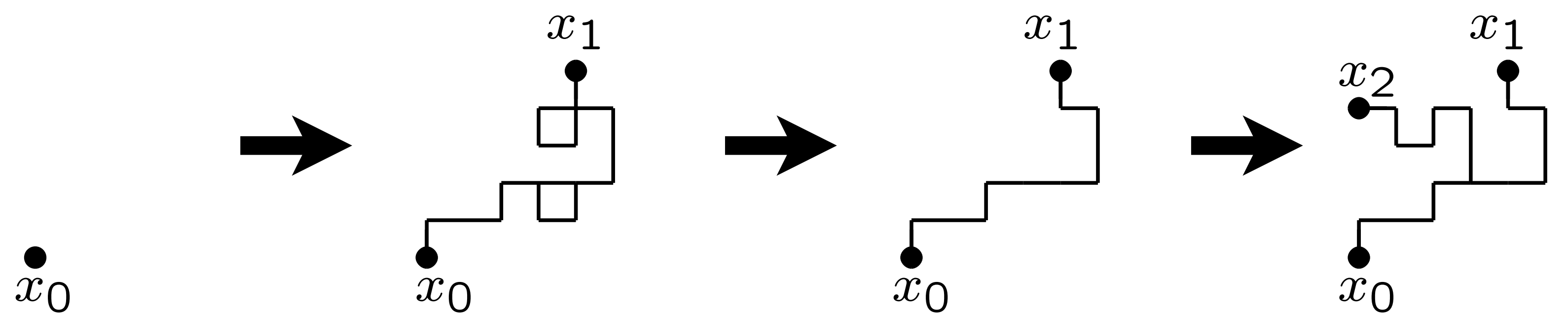
WILSON'S ALGORITHM ON $\mathbb{Z}^2$

Let $x_0 = 0, x_1, x_2, \ldots$ be an enumeration of $\mathbb{Z}^2$.

Let $\mathcal{U}(0)$ be the graph tree consisting of the single vertex x_0.

Given $\mathcal{U}(k-1)$ for some $k \geq 1$, define $\mathcal{U}(k)$ to be the union of $\mathcal{U}(k-1)$ and the loop-erased random walk (LERW) path run from x_k to $\mathcal{U}(k-1)$.

The UST $\mathcal{U}$ is then the local limit of $\mathcal{U}(k)$.

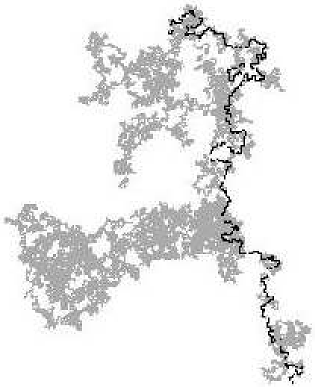


LERW SCALING IN $\mathbb{Z}^d$

Consider LERW as a process $(L_n)_{n\geq 0}$ (assume original random walk is transient).

In $\mathbb{Z}^d$, $d \geq 5$, L rescales diffusively to Brownian motion [Lawler].

In $\mathbb{Z}^4$, with logarithmic corrections rescales to Brownian motion [Lawler].



Picture: Ariel Yadin

In $\mathbb{Z}^3$, $\{L_n : n \in [0,\tau]\}$ has a scaling limit [Kozma, Li/Shiraishi]. Growth exponent $\beta \approx 1.62$.

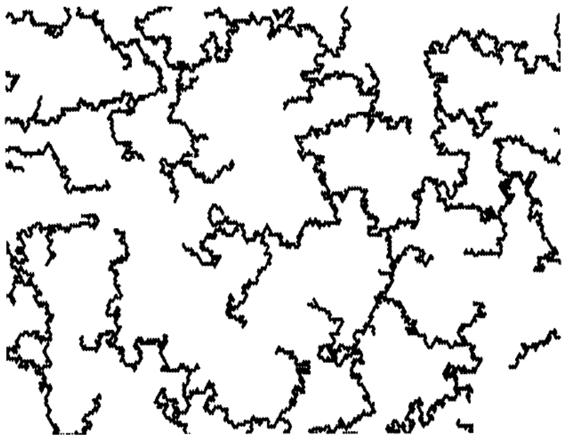
In $\mathbb{Z}^2$, $\{L_n : n \in [0,\tau]\}$ has SLE(2) scaling limit [Lawler/Schramm/Werner]. Growth exponent is 5/4 [Kenyon, Masson, Lawler, Lawler/Viklund].

UST SCALING [SCHRAMM]

Consider $\mathcal{U}$ as an ensemble of paths:

$$\mathfrak{U} = \left\{ (a, b, \pi_{ab}) :\ a, b \in \mathbb{Z}^2 \right\},$$

where π_{ab} is the unique arc connecting a and b in $\mathcal{U}$, as an element of the compact space $\mathcal{H}(\dot{\mathbb{R}}^2 \times \dot{\mathbb{R}}^2 \times \mathcal{H}(\dot{\mathbb{R}}^2))$, cf. [Aizenman/Burchard/Newman/Wilson]. Also SLE(8) scaling limit of [Lawler/Schramm/Werner].



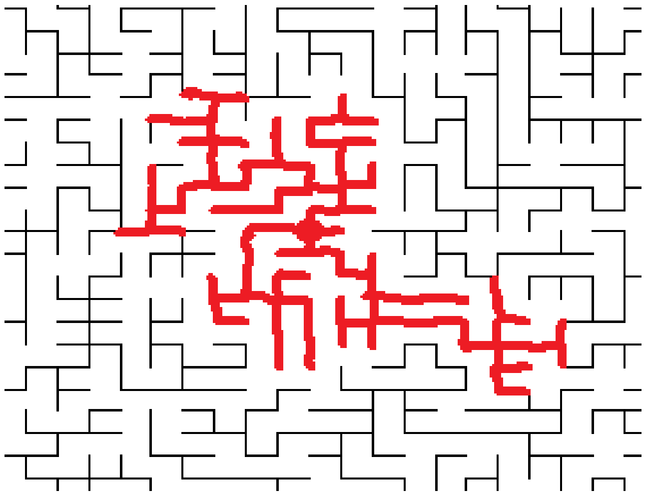
Picture: Oded Schramm

Scaling limit $\mathfrak{T}$ almost-surely satisfies:

- each pair $a, b \in \dot{\mathbb{R}}^2$ connected by a path;
- if $a \neq b$, then this path is simple;
- if $a = b$, then this path is a point or a simple loop;
- the trunk, $\cup_{\mathfrak{T}} \pi_{ab} \backslash \{a, b\}$, is a dense topological tree with degree at most 3.

ISSUE: This topology does not carry information about intrinsic distance or volume.

VOLUME ESTIMATES [BARLOW/MASSON]



With high probability,

$$B_E(x, \lambda^{-1} R) \subseteq B_{\mathcal{U}}(x, R^{5/4}) \subseteq B_E(x, \lambda R),$$

as $R \to \infty$ then $\lambda \to \infty$. In particular,

$$\mathbf{P}\left(R^{-8/5} |B_{\mathcal{U}}(x, R)| \notin [\lambda^{-1}, \lambda] \right) \leq c_1 e^{-c_2 \lambda^{1/9}}.$$

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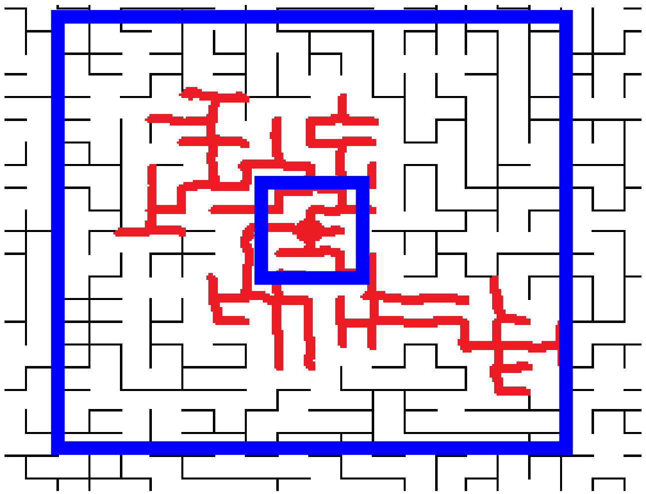

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ABSTRACT FRAMEWORK FOR CONVERGENCE

Define $\mathbb{T}$ to be the collection of measured, rooted, spatial trees, i.e.

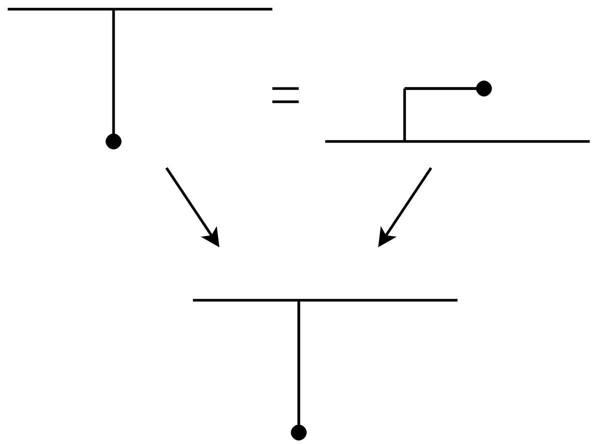
$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}),$$

where:

- $(\mathcal{T}, d_{\mathcal{T}})$ is a complete and locally compact real tree;
- $\mu_{\mathcal{T}}$ is a locally finite Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$;
- $\phi_{\mathcal{T}}$ is a continuous map from $(\mathcal{T}, d_{\mathcal{T}})$ into $\mathbb{R}^2$;
- $\rho_{\mathcal{T}}$ is a distinguished vertex in $\mathcal{T}$.

Equip this space with a generalised Gromov-Hausdorff topology.

BRIEF INTRODUCTION TO GH TOPOLOGY



The (pointed) Gromov-Hausdorff distance

$$d_{GH}\left((\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}}), (\mathcal{T}', d_{\mathcal{T}'}, \rho_{\mathcal{T}'})\right)$$

is given by

$$\inf_{\psi,\psi'} d_H\left(\psi(\mathcal{T}), \psi'(\mathcal{T}')\right).$$

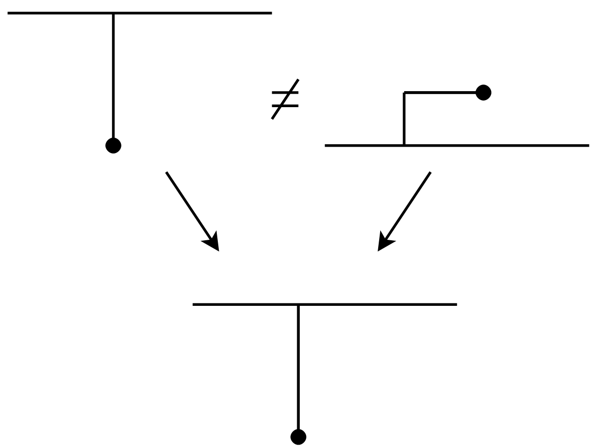
This is equal to

$$\frac{1}{2}\inf_{\mathcal{C}} \mathrm{dis}(\mathcal{C}),$$

where the infimum is taken over correspondences $\mathcal{C} \subseteq \mathcal{T} \times \mathcal{T}'$ containing $(\rho_{\mathcal{T}}, \rho_{\mathcal{T}'})$, and the distortion $\mathrm{dis}(\mathcal{C})$ of a correspondence is given by

$$\sup\left\{\left|d_{\mathcal{T}}(x,y) - d_{\mathcal{T}'}(x',y')\right| : (x,x'),(y,y') \in \mathcal{C}\right\}.$$

MEASURED, SPATIAL GH TOPOLOGY



We refine d_{GH} to $\Delta((\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}), (\mathcal{T}', d_{\mathcal{T}'}, \mu_{\mathcal{T}'}, \phi_{\mathcal{T}'}, \rho_{\mathcal{T}'}))$, via the expression

$$\inf_{\psi,\psi',\mathcal{C}} \left(d_H\left(\psi(\mathcal{T}), \psi'(\mathcal{T}')\right) + d_P\left(\mu_{\mathcal{T}} \circ \psi_{\mathcal{T}}^{-1}, \mu_{\mathcal{T}'} \circ \psi_{\mathcal{T}'}^{-1}\right) + \sup_{(x,x')\in\mathcal{C}} \left|\phi_{\mathcal{T}}(x) - \phi_{\mathcal{T}'}(x')\right|\right).$$

2d-UST SCALING LIMIT

Theorem [Barlow/C/Kumagai, Holden/Sun]. If $\mathbf{P}_\delta$ is the law of the measured, rooted spatial tree

$$\left(\mathcal{U}, \delta^{5/4} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}(\cdot), \delta\phi_{\mathcal{U}}, 0\right)$$

under $\mathbf{P}$, then $\mathbf{P}_\delta$ converges in $\mathcal{M}_1(\mathbb{T})$ as $\delta \to 0$.

Proof involves:

- establishing tightness/convergence of trees spanning a finite number of points, cf. [Lawler/Viklund] for a single LERW path;
- strengthening estimates of [Barlow/Masson] to show everything else is close.

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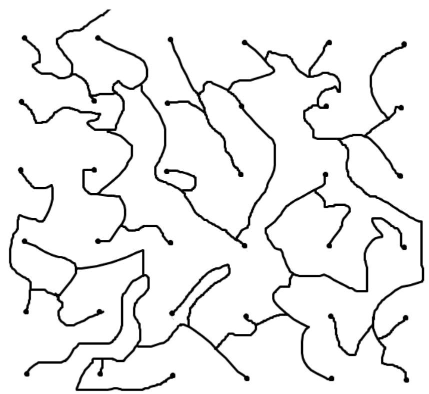

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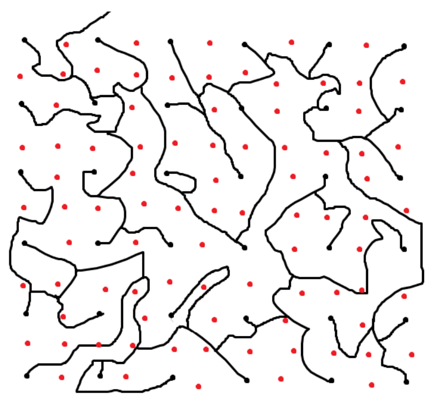

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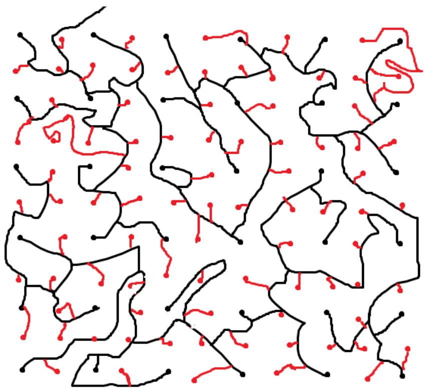

2d-UST LIMIT PROPERTIES [BARLOW/C/KUMAGAI, cf. SCHRAMM]

If $\tilde{\mathbf{P}} := \lim_{\delta\to 0} \mathbf{P}_\delta$, then for $\tilde{\mathbf{P}}$-a.e. $(\mathcal{T}, d_\mathcal{T}, \mu_\mathcal{T}, \phi_\mathcal{T}, \rho_\mathcal{T})$ it holds that:

(a) the Hausdorff dimension of $(\mathcal{T}, d_\mathcal{T})$ is given by $d_f := \frac{8}{5}$;

(b) $\mu_\mathcal{T}$ is non-atomic and supported on the leaves of $\mathcal{T}$, and satisfies

$$\mu_\mathcal{T}(B_\mathcal{T}(x,r)) \approx r^{8/5}$$

(loglog errors pointwise, log errors uniform on compacts);

(c) the restriction of the continuous map $\phi_\mathcal{T} : \mathcal{T} \to \mathbb{R}^2$ to $\mathcal{T}^o$ is a homeomorphism between this set and its image $\phi_\mathcal{T}(\mathcal{T}^o)$, which is dense in $\mathbb{R}^2$;

(d) $(\mathcal{T}, d_\mathcal{T})$ has precisely one end at infinity;

(e) $\max_{x\in\mathcal{T}} \deg_\mathcal{T}(x) = 3 = \max_{x\in\mathbb{R}^2} |\phi_\mathcal{T}^{-1}(x)|$.

SCALING LIMIT OF 3d UST [ANGEL/C/HERNANDEZ-TORRES/SHIRAISHI]

As measured, rooted spatial trees

$$\left(\mathcal{U}, \delta^{\beta} d_{\mathcal{U}}, \delta^{3} \mu_{\mathcal{U}}, \delta \phi_{\mathcal{U}}, 0\right),$$

where $\beta \approx 1.62\dots$, converge in distribution along the subsequence $\delta_n = 2^{-n}$.

Key issues (as compared to 2d approach):

- scaling limit of LERW in irregular domains not understood;
- SRW does not hit arbitrary paths quickly!

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Proof involves:

- extending LERW convergence of [Li/Shiraishi],
- showing SRW hits LERW quickly, cf. [Sapozhnikov/Shiraishi].

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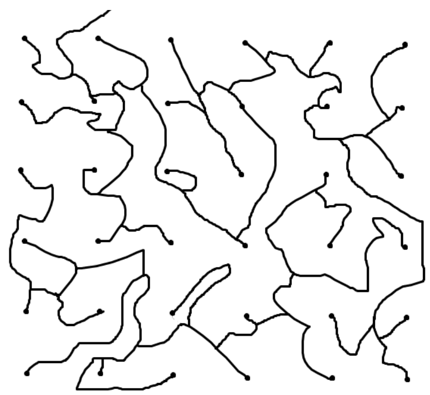

SCALING LIMIT OF 3d UST [ANGEL/C/HERNANDEZ-TORRES/SHIRAISHI]

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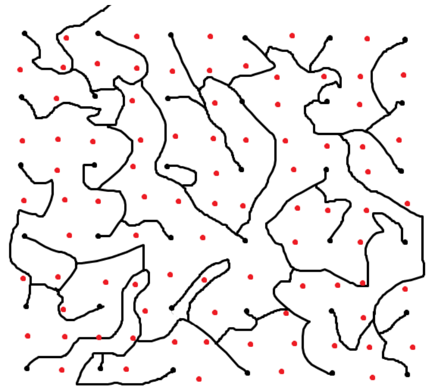

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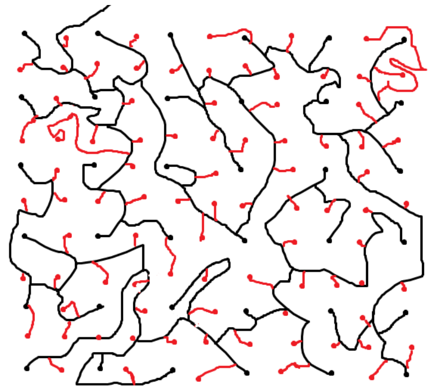

3d-UST LIMIT PROPERTIES

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$$\mu_\mathcal{T}\left(B_\mathcal{T}(x,r)\right) \approx r^{d_f}$$

(loglog errors pointwise, log errors uniform on compacts);

(c) $(\mathcal{T}, d_\mathcal{T})$ has precisely one end at infinity;

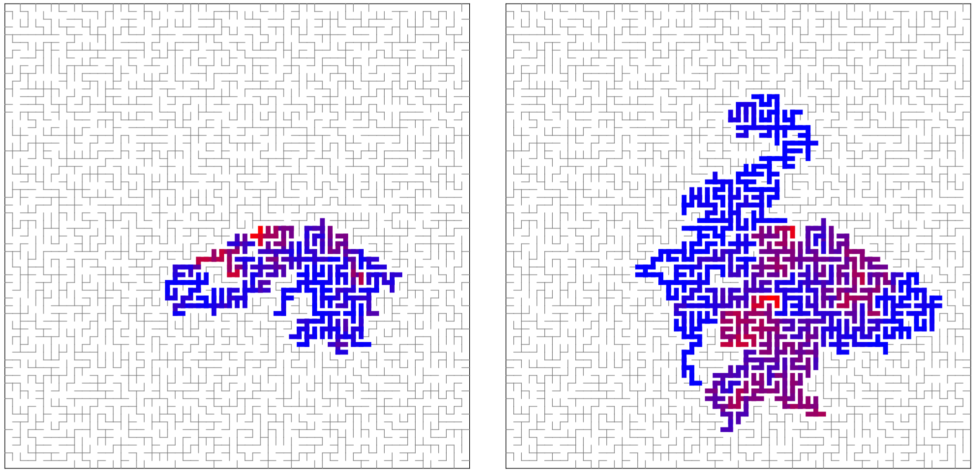
(d) $\max_{x\in\mathbb{R}^3} |\phi_\mathcal{T}^{-1}(x)| \leq M$.

Conjecture maximum degree is 3, and trunk is not a tree.

3. SIMPLE RANDOM WALK

SIMPLE RANDOM WALK ON 2d-UST

Let $X^{\mathcal{U}} = (X_n^{\mathcal{U}})_{n\geq 0}$ be simple random walk on $\mathcal{U}$. After 5,000 and 50,000 steps (picture: Sunil Chhita):



We will write $(p_n^{\mathcal{U}}(x,y))_{x,y\in\mathcal{U},\, n\geq 0}$ for the (smoothed) **quenched** heat kernel on $\mathcal{U}$, as defined by

$$p_n^{\mathcal{U}}(x,y) = \frac{P_x^{\mathcal{U}}\left(X_n^{\mathcal{U}} = y\right) + P_x^{\mathcal{U}}\left(X_{n+1}^{\mathcal{U}} = y\right)}{2\mathrm{deg}_{\mathcal{U}}(y)}.$$

The **annealed/averaged** heat kernel is $\mathbf{E}p_n^{\mathcal{U}}(x,y)$.

(SUB-)GAUSSIAN HEAT KERNELS ON TREES

Suppose T is a graph tree with **fractal dimension** d_f, i.e. such that

$$|B_T(x,r)| \asymp r^{d_f},$$

then (cf. [Barlow, Bass, Coulhon, Grigor'yan, Jones, Kumagai, Perkins, Telcs])

$$p_n^T(x,y) \asymp c_1 n^{-d_s/2} \exp\left\{-c_2 \left(\frac{d_T(x,y)^{d_w}}{n}\right)^{\frac{1}{d_w-1}}\right\},$$

where:

walk dimension $d_w = d_f + 1$, **spectral dimension** $d_s = \frac{2d_f}{d_w}$.

This talk, if time permits, will address:

- **exponents** for $\mathcal{U}$ (2d/3d);
- **scaling limit** for $X^{\mathcal{U}}$ (2d/3d);
- **fluctuations** around polynomial terms (2d);
- **quenched vs. averaged** heat kernel (2d).

EXPONENTS

	General form	$d = 2$	$d = 3$
LERW growth exponent	α	5/4 = 1.25	1.62
Hausdorff dimension of $\mathcal{U}$	$d_f = d/\alpha$	8/5 = 1.60	1.85
Intrinsic walk dimension	$d_w = 1 + d_f$	13/5 = 2.60	2.85
Extrinsic walk dimension	αd_w	13/4 = 3.25	4.62
Spectral dimension of $\mathcal{U}$	$2d_f/d_w$	16/13 = 1.23	1.30

Exponents for 2d case established in [Barlow/Masson].

Exponents for 3d case based on results of [A/C/H-T/S] and numerical simulation for β of Wilson.

Both depend on general estimates of [Kumagai/Misumi].

RANDOM WALKS ON GRAPHS

Let $G = (V, E)$ be a finite, connected graph, equipped with (strictly positive, symmetric) edge conductances $(c(x,y))_{\{x,y\}\in E}$. Let μ be a finite measure on V (of full-support).

Let X be the continuous time Markov chain with generator Δ, as defined by:

$$(\Delta f)(x) := \frac{1}{\mu(\{x\})} \sum_{y:\, y\sim x} c(x,y)(f(y) - f(x)).$$

NB. Common choices for μ are:
- $\mu(\{x\}) := \sum_{y:\, y\sim x} c(x,y)$, the **constant speed random walk (CSRW)**;
- $\mu(\{x\}) := 1$, **the variable speed random walk (VSRW)**.

DIRICHLET FORM AND RESISTANCE METRIC

Define a quadratic form on G by setting

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x,y:x\sim y} c(x,y)\,(f(x)-f(y))\,(g(x)-g(y))\,.$$

Note that (regardless of the particular choice of μ,) $\mathcal{E}$ is a **Dirichlet form** on $L^2(\mu)$, and

$$\mathcal{E}(f,g) = -\sum_{x\in V} (\Delta f)(x)g(x)\mu(\{x\}).$$

Suppose we view G as an electrical network with edges assigned conductances according to $(c(x,y))_{\{x,y\}\in E}$. Then the **effective resistance** between x and y is given by

$$R(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f):\ f(x)=1, f(y)=0\right\}.$$

R is a metric on V, e.g. [Tetali 1991], and characterises the weights (and therefore the Dirichlet form) uniquely [Kigami 1995].

SUMMARY

RANDOM WALK X WITH GENERATOR Δ

$$\Updownarrow$$

DIRICHLET FORM $\mathcal{E}$ on $L^2(\mu)$

$$\Updownarrow$$

RESISTANCE METRIC R AND MEASURE μ

RESISTANCE METRIC, e.g. [KIGAMI 2001]

Let F be a set. A function $R : F \times F \to \mathbb{R}$ is a **resistance metric** if, for every finite $V \subseteq F$, one can find a weighted (i.e. equipped with conductances) graph with vertex set V for which $R|_{V \times V}$ is the associated effective resistance.

EXAMPLES

- Effective resistance metric on a graph;

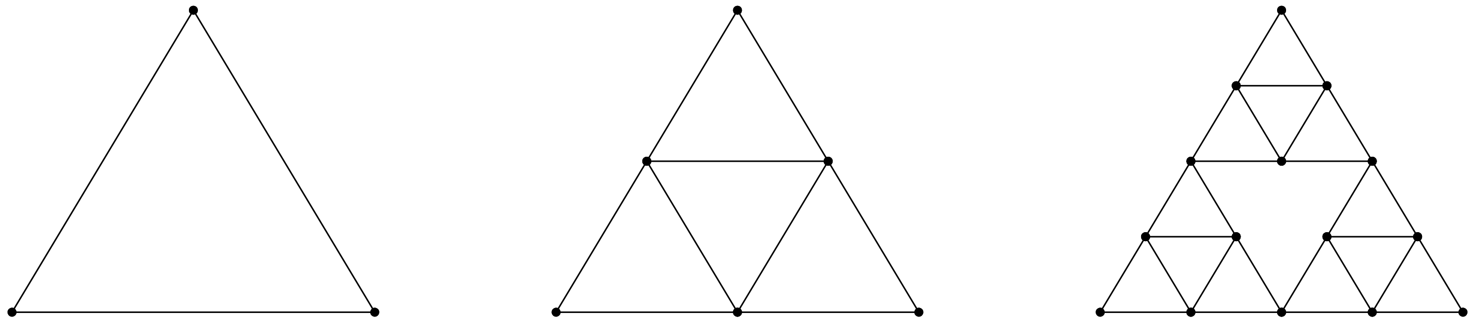
- One-dimensional Euclidean (not true for higher dimensions);

- Any shortest path metric on a tree;

- Resistance metric on a Sierpinski gasket, where for 'vertices' of limiting fractal, we set

$$R(x, y) = (3/5)^n R_n(x, y),$$

then use continuity to extend to whole space.



RESISTANCE AND DIRICHLET FORMS

Theorem (e.g. [Kigami 2001]) There is a one-to-one correspondence between resistance metrics and a class of quadratic forms called **resistance forms**.

The relationship between a resistance metric R and resistance form $(\mathcal{E},\mathcal{F})$ is characterised by

$$R(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f):\ f\in\mathcal{F},\ f(x)=1,\ f(y)=0\right\}.$$

Moreover, if (F,R) is compact, then $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$ for any finite Borel measure μ of full support. (Version of the statement also hold for locally compact spaces.)

A FIRST EXAMPLE

Let $F = [0,1]$, $R =$ Euclidean, and μ be a finite Borel measure of full support on $[0,1]$.

Associated resistance form:

$$\mathcal{E}(f,g) = \int_0^1 f'(x)g'(x)dx, \qquad \forall f,g \in \mathcal{F},$$

where $\mathcal{F} = \{f \in C([0,1]) : f \text{ is abs. cont. and } f' \in L^2(dx)\}$.

Moreover, integration by parts gives:

$$\mathcal{E}(f,g) = -\int_0^1 (\Delta f)(x)g(x)\mu(dx).$$

where $\Delta f = \frac{d}{d\mu}\frac{df}{dx}$.

If $\mu(dx) = dx$, then the Markov process naturally associated with Δ is reflected Brownian motion on $[0,1]$.

SUMMARY

RESISTANCE METRIC R AND MEASURE μ

$$\updownarrow$$

RESISTANCE FORM $(\mathcal{E}, \mathcal{F})$, DIRICHLET FORM on $L^2(\mu)$

$$\updownarrow$$

STRONG MARKOV PROCESS X WITH GENERATOR Δ,
where

$$\mathcal{E}(f,g) = -\int_F (\Delta f) g d\mu.$$

GENERAL SCALING RESULT [C. 2016]
See also [ATHREYA/LOHR/WINTER] for trees

Write $\mathbb{F}_c$ for the space of marked compact resistance metric spaces, equipped with finite Borel measures of full support. Suppose that the sequence $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1}$ in $\mathbb{F}_c$ satisfies

$$(F_n, R_n, \mu_n, \rho_n) \to (F, R, \mu, \rho)$$

in the (marked) Gromov-Hausdorff-Prohorov topology for some $(F, R, \mu, \rho) \in \mathbb{F}_c$.

It is then possible to isometrically embed $(F_n, R_n)_{n\geq 1}$ and (F, R) into a common metric space (M, d_M) in such a way that

$$P^n_{\rho_n}\left((X^n_t)_{t\geq 0} \in \cdot\right) \to P_\rho\left((X_t)_{t\geq 0} \in \cdot\right)$$

weakly as probability measures on $D(\mathbb{R}_+, M)$.

Holds for locally compact spaces if $\liminf_{n\to\infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c)$ diverges as $r \to \infty$. (Can also include 'spatial embeddings'.)

COROLLARY: SRW SCALING LIMIT

Fix $d=2$ or $d=3$, let $\mathbb{P}_\delta$ be the annealed law of

$$\left(\delta X^{\mathcal{U}}_{\delta^{-\alpha d_w} t}\right)_{t\geq 0}.$$

NB. $\alpha d_w = 3.25, 4.62$ is the extrinsic walk dimension in the relevant dimension.

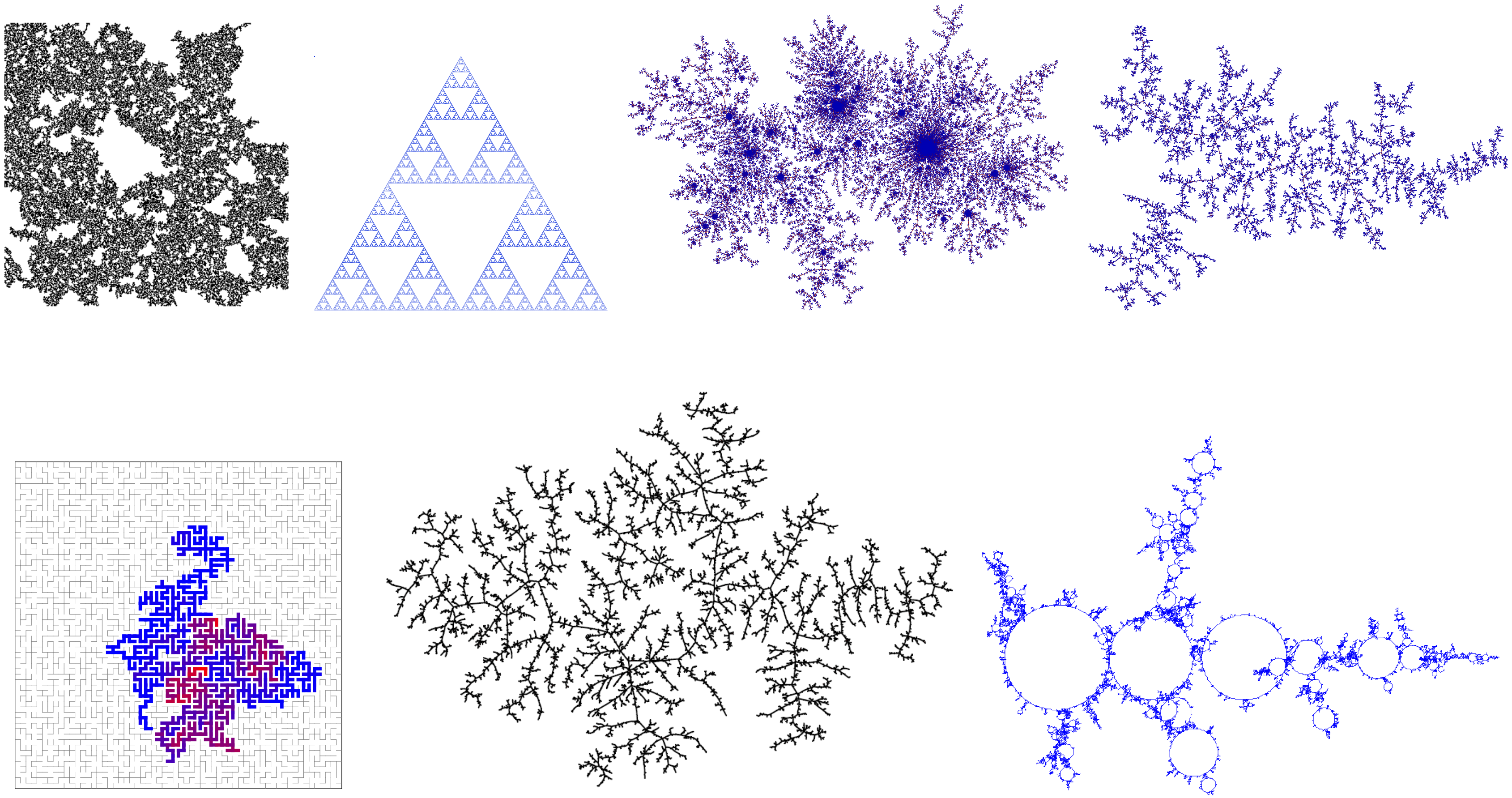
It then holds that $\mathbb{P}_\delta \to \tilde{\mathbb{P}}$ (subsequentially in 3d), where $\tilde{\mathbb{P}}$ is the annealed law of

$$\left(\phi_{\mathcal{T}}(X^{\mathcal{T}}_t)\right)_{t\geq 0},$$

as probability measures on $C(\mathbb{R}_+, \mathbb{R}^d)$.

Proof. Apply general results concerning convergence of random walks on trees [Barlow/C/Kumagai, Athreya/Lohr/Winter], or resistance spaces [C].

OTHER MOTIVATING EXAMPLES



Sources: Ben Avraham/Havlin, Kortchemski, Chhita, Broutin.

PROOF IDEA 1: RESOLVENTS

For $(F, R, \mu, \rho) \in \mathbb{F}_c$, let

$$G_x f(y) = E_y \int_0^{\sigma_x} f(X_s) ds$$

be the resolvent of X killed on hitting x. NB. Processes associated with resistance forms hit points.

We have [Kigami 2012] that

$$G_x f(y) = \int_F g_x(y, z) f(z) \mu(dz),$$

where

$$g_x(y, z) = \frac{R(x, y) + R(x, z) - R(y, z)}{2}.$$

Metric measure convergence $\Rightarrow$ resolvent convergence $\Rightarrow$ semigroup convergence $\Rightarrow$ finite dimensional distribution convergence.

PROOF IDEA 2: TIGHTNESS

Using that X has local times $(L_t(x))_{x\in F, t\geq 0}$, and

$$E_y L_{\sigma_A}(z) = g_A(y,z) = \frac{R(y,A)+R(z,A)-R_A(y,z)}{2},$$

can establish via Markov's inequality a general estimate of the form:

$$\sup_{x\in F} P_x\left(\sup_{s\leq t} R(x,X_s)\geq\varepsilon\right)\leq\frac{32N(F,\varepsilon/4)}{\varepsilon}\left(\delta+\frac{t}{\inf_{x\in F}\mu(B_R(x,\delta))}\right),$$

where $N(F,\varepsilon)$ is the minimal size of an ε cover of F.

Metric measure convergence $\Rightarrow$ estimate holds uniformly in n $\Rightarrow$ tightness (application of Aldous' tightness criterion).

Similar estimate also gives non-explosion in locally compact case.