Slow bootstrap percolation

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Joint with József Balogh, Alexey Pokrovskiy, and Tibor Szabó

In the spirit of these days…
(a first example)

r-neighbour bootstrap percolation on G:

- Host graph - G
- Initial set of "infected" vertices - $A = A_0 \subseteq V(G)$
- Infection rule:

$$A_{i+1} = A_i \cup \{v \in V(G) \mid |N(v) \cap A_i| \geq r\}$$

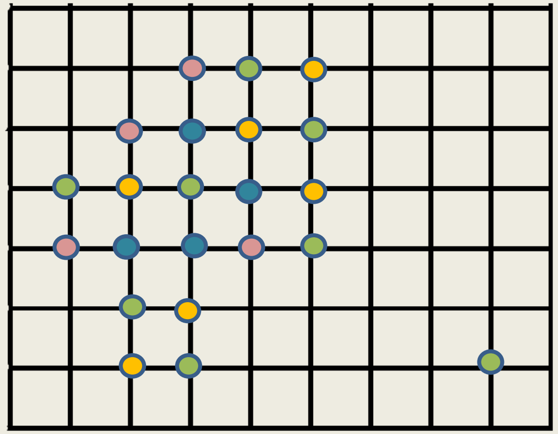
2-neighbour bootstrap percolation

A_0

A_1

A_2

A_3



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- Infection rule:

$$A_{i+1} = A_i \cup \{v \in V(G) \mid |N(v) \cap A_i| \geq r\}$$

- When do we have $\cup A_i = V(G)$?
- If $A_0 \sim Bin(V(G), p)$, when do we have $\cup A_i = V(G)$ whp?
- **How many steps until the infection stop spreading?**

Graph bootstrap percolation (our setting)

***H*-bootstrap percolation on *n* vertices.**

- Host graph - usually K_n
- Target graph - H
- Starting graph - $G = G_0 \subseteq K_n$ (initial "infected" set of edges)
- Infection rule:

$$G_{i+1} = G_i \cup \{e \subseteq E(K_n) \mid e \text{ creates a new copy of } H \text{ in } G_i\}$$

? For which starting graphs G_0 we have $G_{\text{final}} = K_n$?

? If $G_0 = G(n,p)$, when do we have $G_{\text{final}} = K_n$ whp?

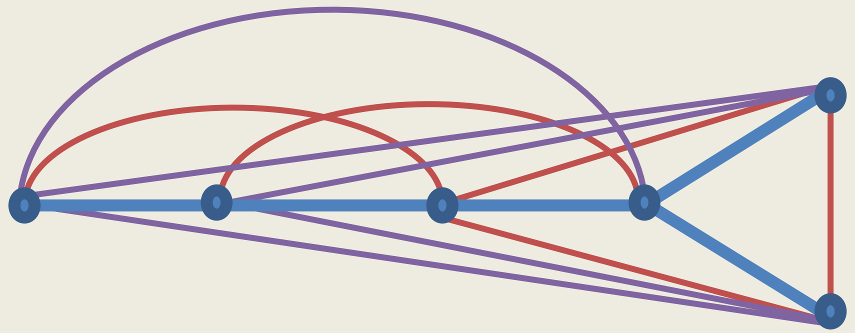
? **How many steps until the infection stop spreading?**

Some history – weak saturation

Definition - weak saturation (Bollobás 1968):

A graph G on n vertices is **weakly $\boldsymbol{H}$-saturated** if G is H-free and ∃ an ordering of $E(K_n) \setminus E(G) = \{e_1, e_2, \dots, e_t\}$ such that the addition of e_i to $G_{i-1} = G \cup \{e_1, \dots, e_{i-1}\}$ will create a new copy of H, for every $i \in [t]$.

Example: G is weakly K_3-saturated:



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Definition - the weak saturation number:

$$wsat(n, H) = \min\{e(G) \mid G\ is\ weakly\ H - saturated\}$$

Example: $wsat(n, K_3) = n - 1$

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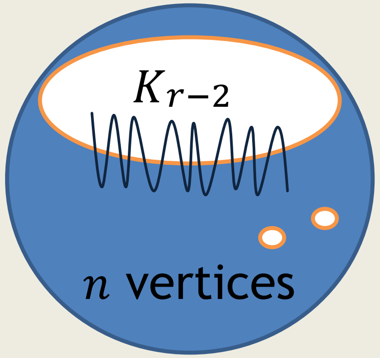
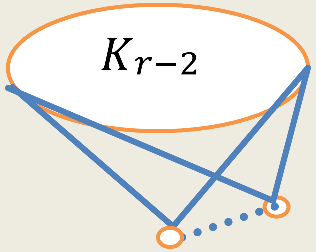
Example: $H = K_r$

$G = K_n \setminus K_{n-r+2}$
is weakly K_r-saturated





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Theorem (Lovász / Alon / Frankl / Kalai) :

$$wsat(n, K_r) = \binom{n}{2} - \binom{n-r+2}{2}$$

(Conjectured by Bollobás)

Some history – weak saturation of K_r

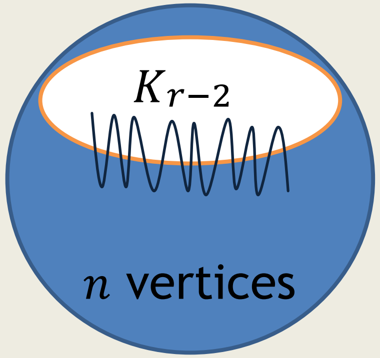
Definition - the weak saturation number:

$$wsat(n, H) = \min\{e(G) \mid G \text{ is weakly } H - \text{saturated}\}$$

Theorem (Lovász / Alon / Frankl / Kalai) :

$$wsat(n, K_r) = \binom{n}{2} - \binom{n-r+2}{2} = \Theta(n)$$

(**Conjectured by Bollobás**)



$G = K_n \setminus K_{n-r+2}$

Note: all the missing edges can be added "simultaneously".

Question (Bollobás): Determine the maximum possible number of steps.

Graph bootstrap percolation

H - fixed size graph

Definition: $\boldsymbol{H}$-bootstrap percolation on $\boldsymbol{n}$ vertices.

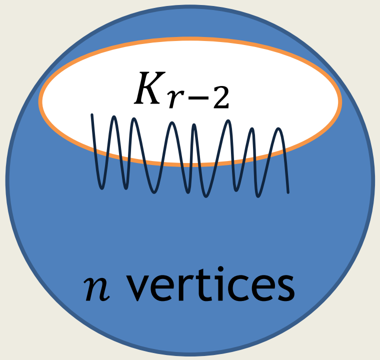
$G_0 \subseteq K_n$- starting graph

$$G_{i+1} = G_i \cup \{e \subseteq E(K_n) \mid e \text{ creates a new copy of } H \text{ in } G_i\}$$

Stop: when $G_{i+1} = G_i$.

Example: $H = K_r$, $G_0 = K_n \setminus K_{n-r+2}$

$G_1 = K_n$



Graph bootstrap percolation

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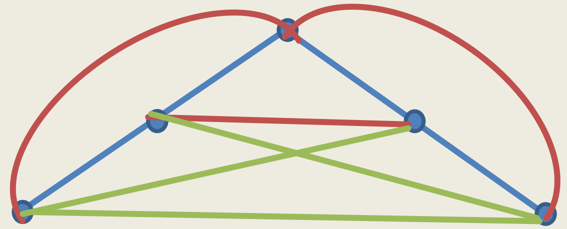

G_0 G_1 $\boldsymbol{G_2 = K_5}$

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Stop: when $G_{i+1} = G_i$.

Length (running time): $\ell(H, G_0) = \min\{i \mid G_i = G_{i+1}\}$

$$L(H, n) = \max_{G_0 \subseteq K_n} \ell(H, G_0)$$

Graph bootstrap percolation

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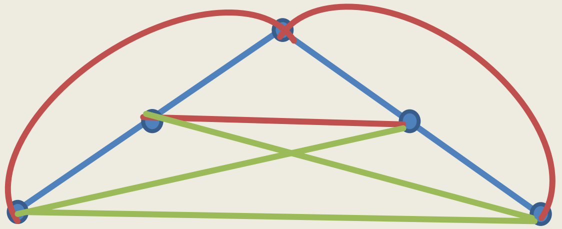

G_0 G_1 $\boldsymbol{G_2 = K_5}$

$\ell(H, G_0) = 2$

Graph bootstrap percolation

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Question (Bollobás): $L(K_r, n)$?

Running time of K_r-bootstrap percolation

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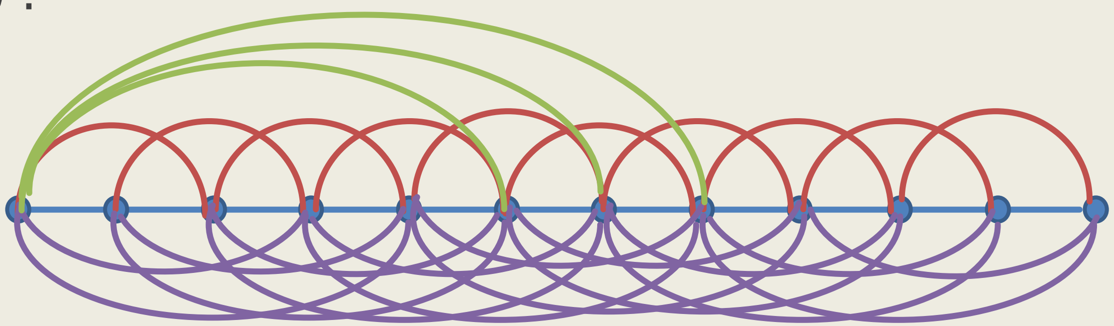
Example: $L(K_3, n)$



Running time of K_r-bootstrap percolation

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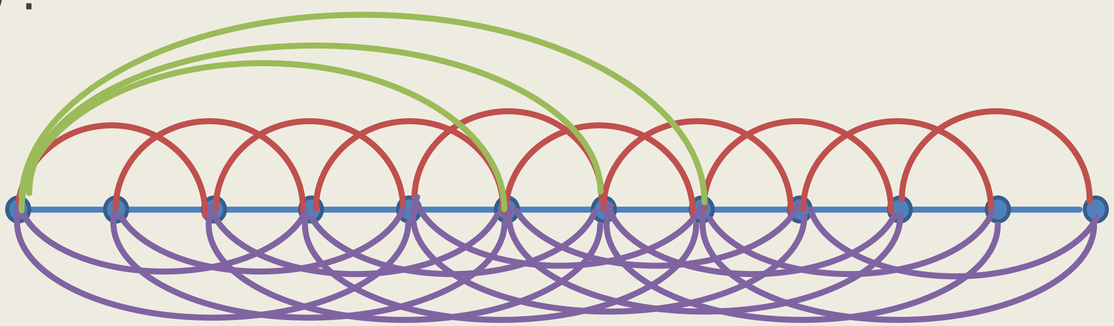
Question (Bollobás): $L(K_r, n)$?

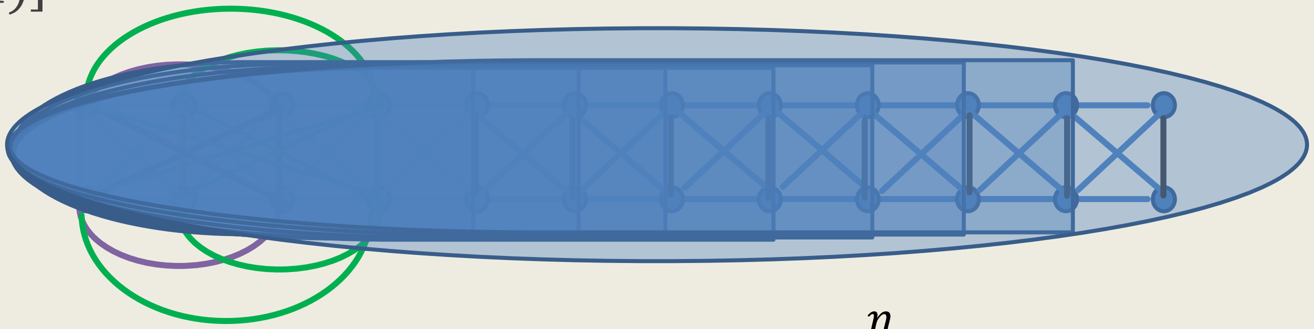
Example: $L(K_3, n) \geq \log n$



Running time of K_r-bootstrap percolation – history

Theorem (Bollobás-Przykucki-Riordan-Sahasrabudhe 2017, Matzke):

- $L(K_3, n) = \lfloor \log_2(n-1) \rfloor$
- $L(K_4, n)$



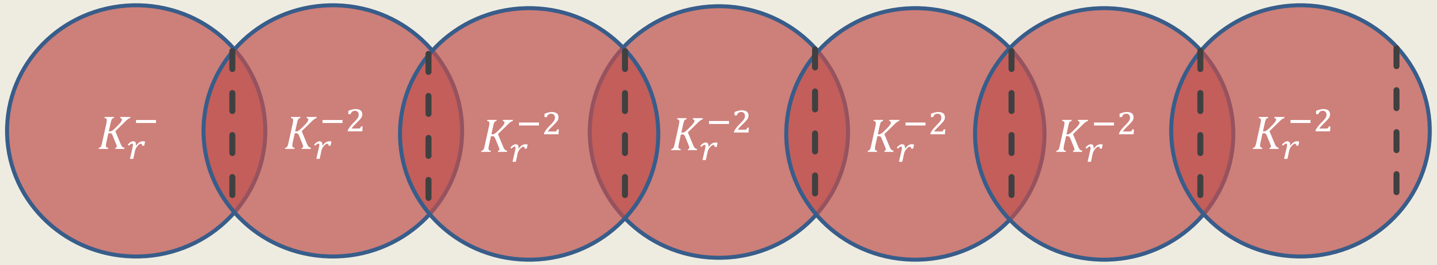
$$L(K_4, n) \geq \frac{n}{2} - 1$$

Running time of K_r-bootstrap percolation – history

Theorem (Bollobás-Przykucki-Riordan-Sahasrabudhe 2017, Matzke):

- $L(K_3, n) = \lfloor \log_2(n-1) \rfloor$
- $L(K_4, n) = n - 3$

The only 2 exact results

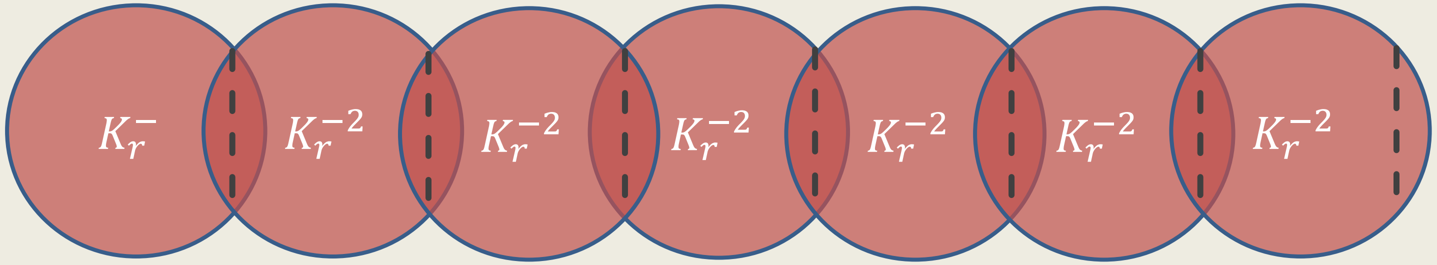


$$L(K_r, n) = \Omega(n)$$

Running time of K_r-bootstrap percolation – history

Theorem (Bollobás-Przykucki-Riordan-Sahasrabudhe 2017, Matzke):

- $L(K_3, n) = \lfloor \log_2(n-1) \rfloor$
- $L(K_4, n) = n - 3$
- $L(K_5, n) \geq n^{\frac{13}{8} - o(1)}$



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- $L(K_4, n) = n - 3$
- $L(K_5, n) \geq n^{\frac{13}{8} - o(1)}$
- (BPRS) for $r \geq 6$, $L(K_r, n) \geq n^{2 - \frac{r-2}{\binom{r}{2}} - o(1)}$

Conjecture (Bollobás-Przykucki-Riordan-Sahasrabudhe 2017):

$$L(K_r, n) = o(n^2)$$

Running time of K_r-bootstrap percolation – our results

Theorem (Bollobás-Przykucki-Riordan-Sahasrabudhe 2017):

for $r \geq 5$, $L(K_r, n) \geq n^{2-\frac{r-2}{\binom{r}{2}}-o(1)}$

Conjecture: $L(K_r, n) = o(n^2)$.

Theorem (Balogh, K., Pokrovskiy, and Szabó, 2020):

- $L(K_5, n) \geq n^{2-\frac{c}{\sqrt{\log n}}}$.
- $L(K_r, n) = \Theta(n^2)$, for $r \geq 6$.

Running time of K_r-bootstrap percolation – our results

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- $L(K_5, n) \geq n^{2-\frac{c}{\sqrt{\log n}}}$.
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Denote: $t_3(n)$ = max 3-AP-free set in $[n]$.

Lemma (Balogh, K., Pokrovskiy, and Szabó, 2020):

$$L(K_5, n) \geq Cn \cdot t_3(n)$$

Theorem (Behrend): $\exists A \subseteq [n]$, $|A| \geq n^{1-\frac{c}{\sqrt{\log n}}}$, and A is 3-AP-free.

Known (Roth): $t_3(n) = o(n)$.

Running time of K_r-bootstrap percolation – our results

Theorem (Balogh, K., Pokrovskiy, and Szabó, 2020):

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Lemma (Balogh, K., Pokrovskiy, and Szabó, 2020):

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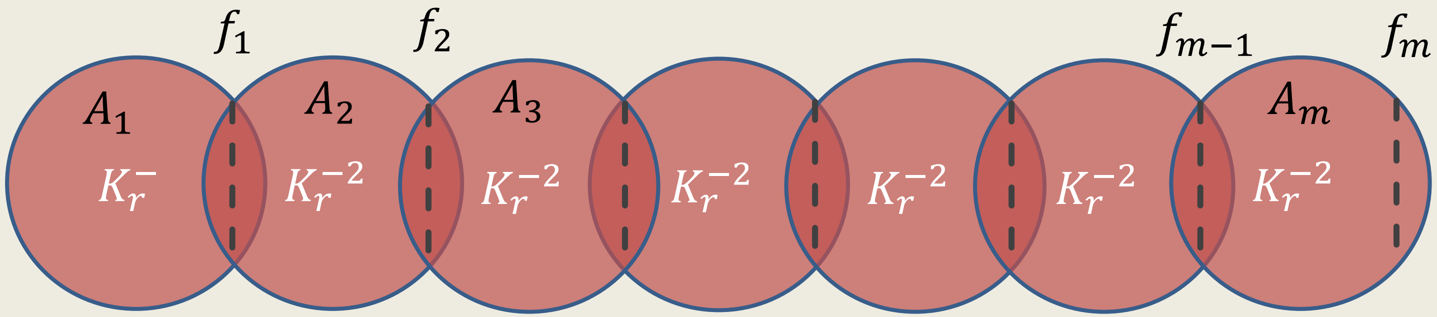
Question: $L(K_5, n) = o(n^2)$?

Proof idea



- $f_i \subseteq A_i \cap A_{i+1}$,
- $f_i \nsubseteq A_j$ for every $j \neq i, i+1$,
- No other edge get infected in the first m steps.

For a graph G, let $\mathcal{H} \coloneqq \mathcal{H}(G)$ be the following hypergraph:

- $V(\mathcal{H}) = V(G)$
- $E(\mathcal{H}) = \{v_1 v_2 \dots v_r \mid K_r^- \subseteq G[v_1 v_2 \dots v_r]\}$

Proof idea

We are looking for:

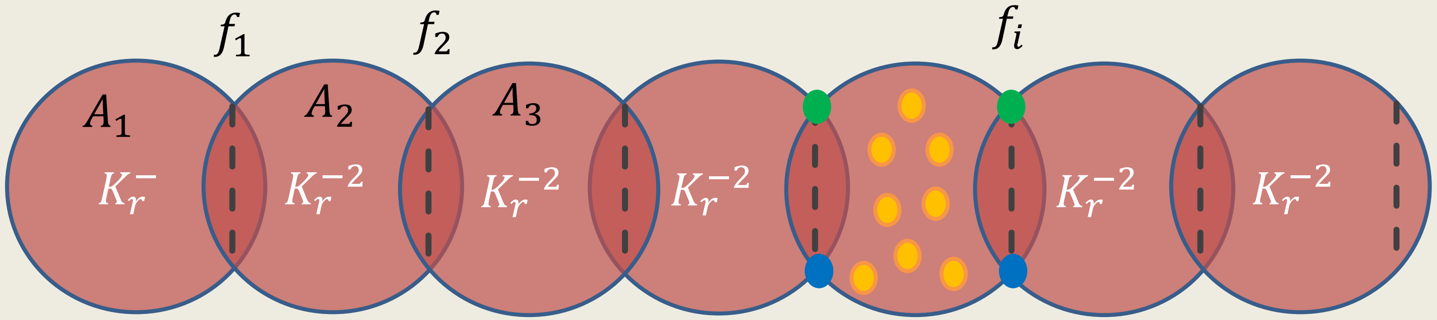
A graph G with the corresponding r-uniform hypergraph $\mathcal{H} := \mathcal{H}(G)$, such that:

1. $E(\mathcal{H}) = \{e_1, \dots, e_m\}$ with $G[e_i] = K_r$, and
2. $\exists f_i \subseteq e_i \cap e_{i+1}$, $|f_i| = 2$ for every $i \in [m-1]$, and
3. $f_i \nsubseteq e_j$ for every $j \neq i, i+1$, and
4. $\exists f_m \subseteq e_m$, $|f_m| = 2$ and $f_m \nsubseteq e_j$ for every $j \neq m$, and
5. $\forall K_r^- \in G$ we have $V(K_r^-) \in E(\mathcal{H})$.

Then for $G_0 = G \setminus \{f_1, \dots, f_m\}$, $\ell(K_r, G_0) \geq m$.

- $L(K_r, n) \geq m$.

Proof idea



$X = \{x_1, \ldots, x_n\}$, $W = W_1 \cup \cdots \cup W_n$, $W_i = \left\{w_{i_1}, \ldots w_{i_{\lceil r/2 \rceil - 2}}\right\}$,

$Y = \{y_1, \ldots, y_\ell\}$, $Z = Z_1 \cup \cdots \cup Z_\ell$, $Z_i = \left\{z_{i_1}, \ldots z_{i_{\lceil r/2 \rceil - 2}}\right\}$. (say, $\ell = n + 20$)

The vertices: $V(G) = V(\mathcal{H}) = X \cup Y \cup W \cup Z$

The hyperedges: $e_t = \left\{x_{t \,(mod\, n)}, x_{t+1 \,(mod\, n)}, y_{t \,(mod\, \ell)}, y_{t+1 \,(mod\, \ell)}\right\} \cup W_{t \,(mod\, n)} \cup Z_{t \,(mod\, \ell)}$

(say, $1 \leq t \leq \frac{n}{100} = m$)

$G[e_i] = K_r$, let $f_i = \left\{x_{i+1 \,(mod\, n)}, y_{i+1 \,(mod\, \ell)}\right\}$.

Take: $G_0 = G \setminus \{f_1, \ldots, f_m\}$,

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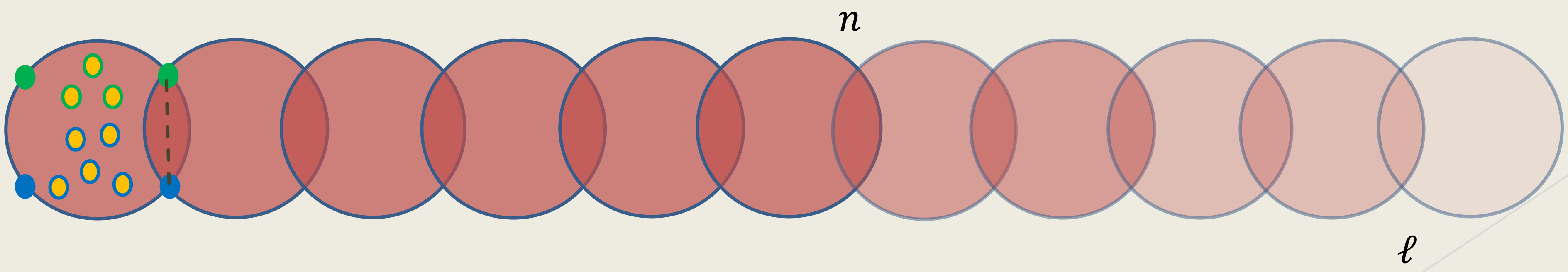

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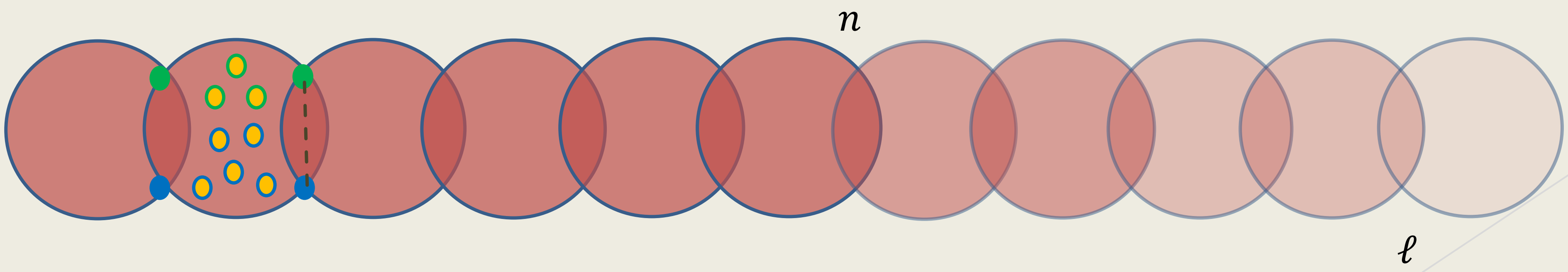

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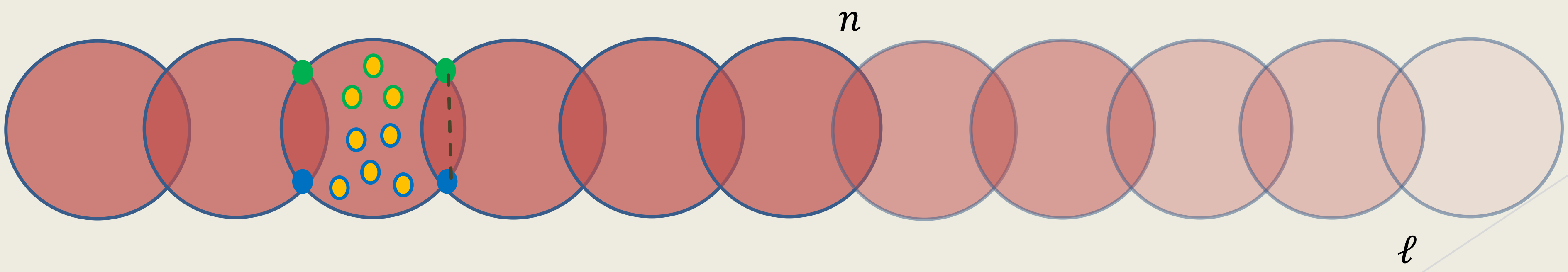

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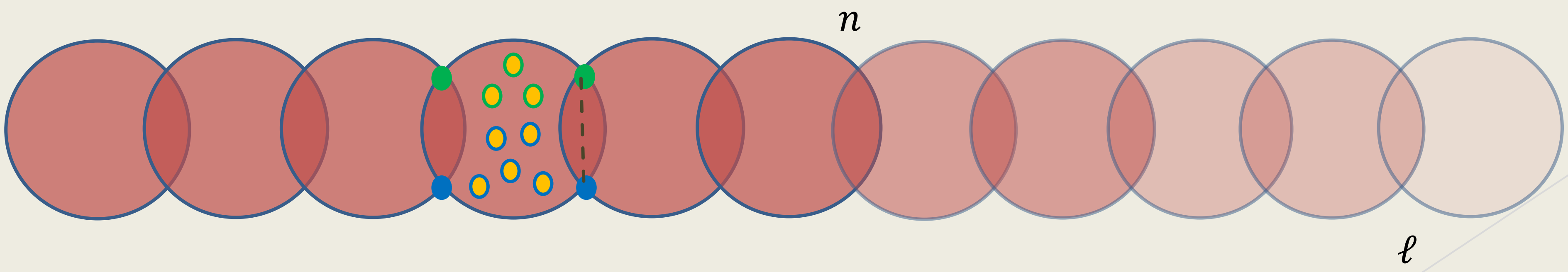

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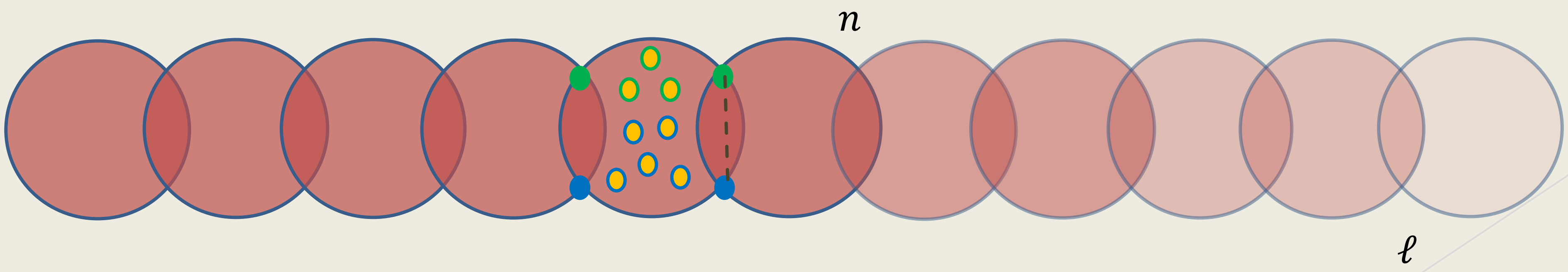


Take: $G_0 = G \setminus \{f_1, \dots, f_m\}$.

Proof idea

$X = \{x_1, \dots, x_n\}$, $W = W_1 \cup \cdots \cup W_n$, $W_i = \left\{w_{i_1}, \dots w_{i_{\lfloor r/2 \rfloor - 2}}\right\}$,

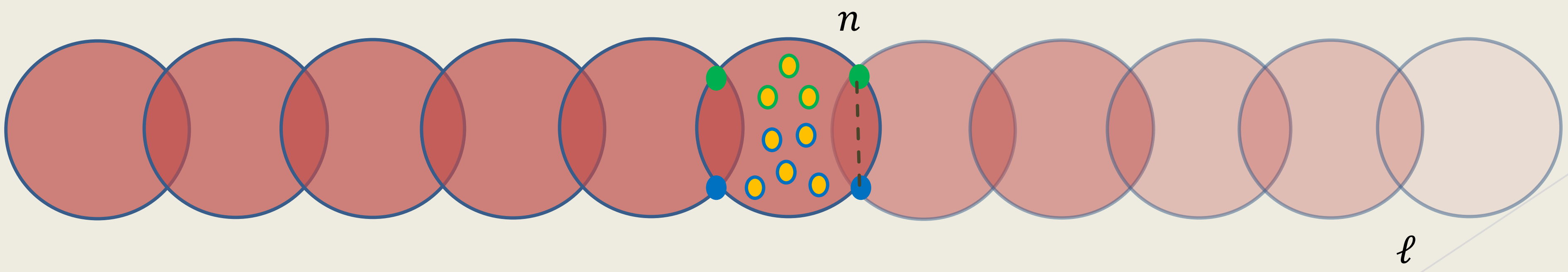
$Y = \{y_1, \dots, y_\ell\}$, $Z = Z_1 \cup \cdots \cup Z_\ell$, $Z_i = \left\{z_{i_1}, \dots z_{i_{\lceil r/2 \rceil - 2}}\right\}$. (say, $\ell = n + 20$)

The vertices: $V(G) = V(\mathcal{H}) = X \cup Y \cup W \cup Z$

The hyperedges: $e_t = \left\{x_{t \ (mod\ n)}, x_{t+1\ (mod\ n)}, y_{t\ (mod\ \ell)}, y_{t+1\ (mod\ \ell)}\right\} \cup W_{t\ (mod\ n)} \cup Z_{t\ (mod\ \ell)}$

(say, $1 \le t \le \frac{n}{100} = m$)

$G[e_i] = K_r$, let $f_i = \left\{x_{i+1\ (mod\ n)}, y_{i+1\ (mod\ \ell)}\right\}$.



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Proof idea

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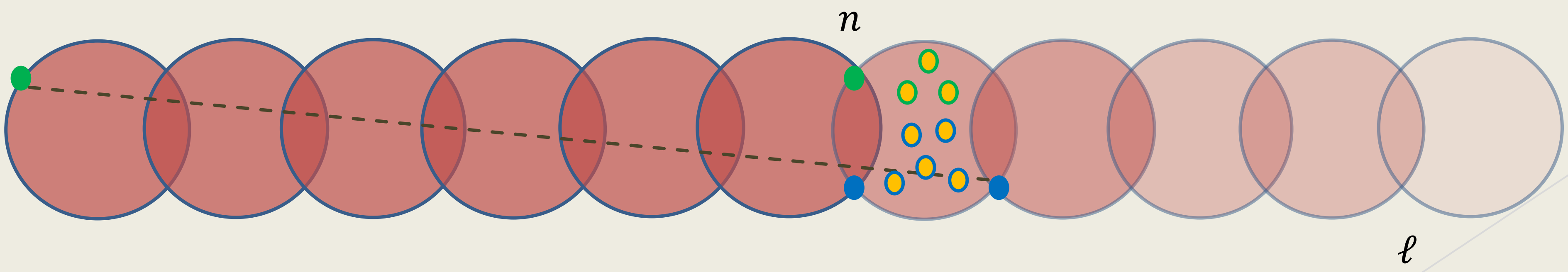

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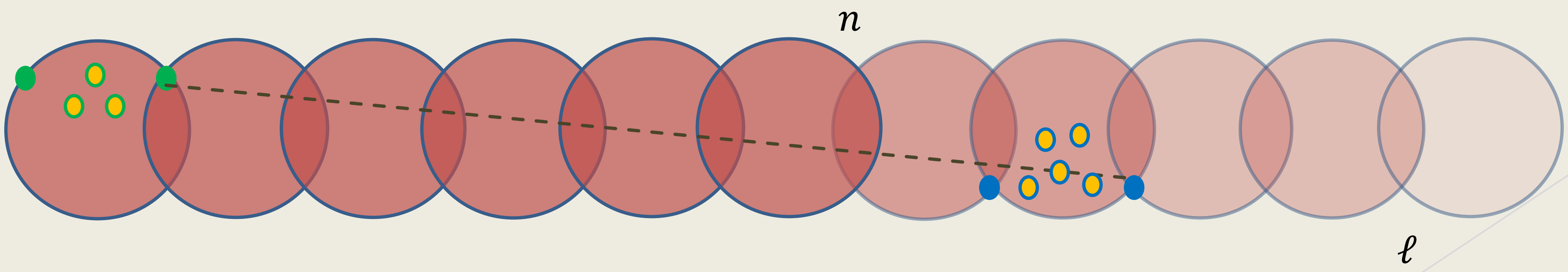

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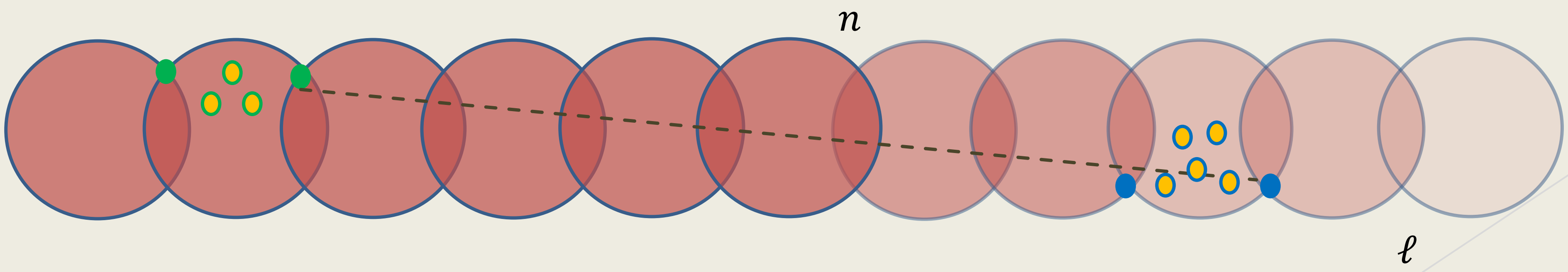

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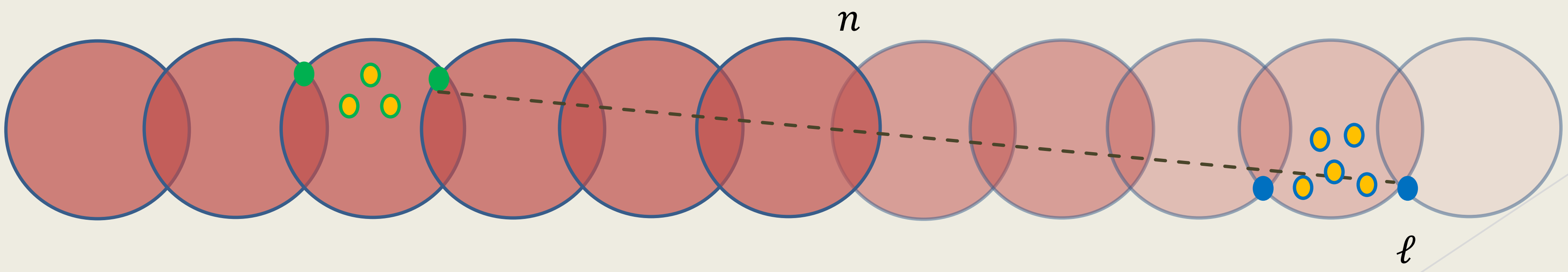

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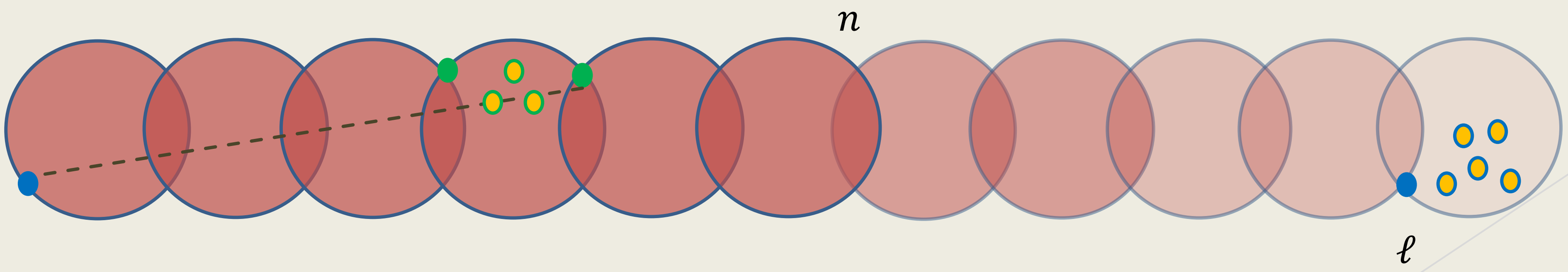

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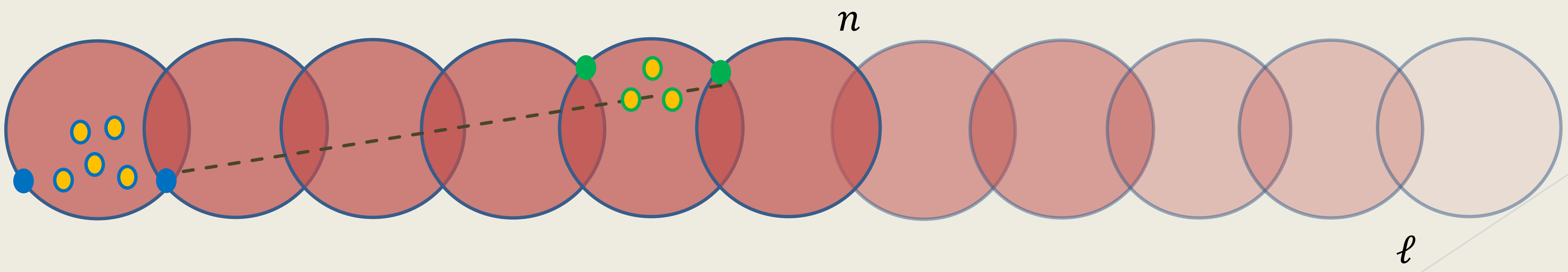

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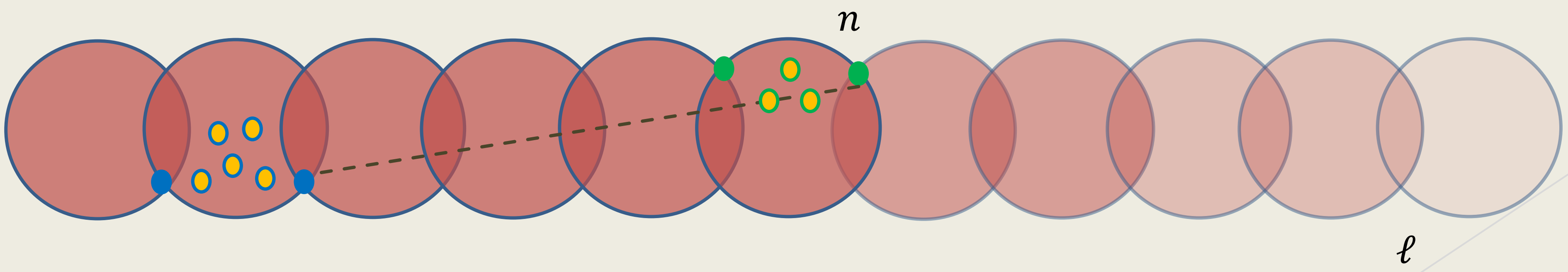

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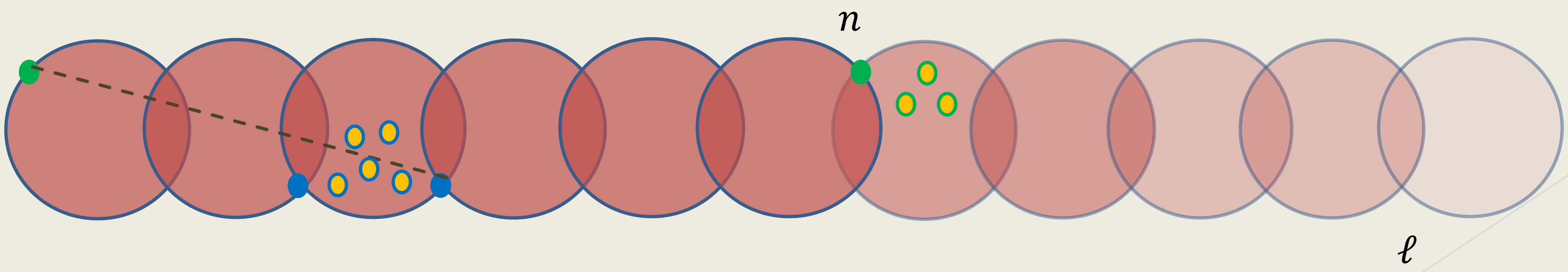

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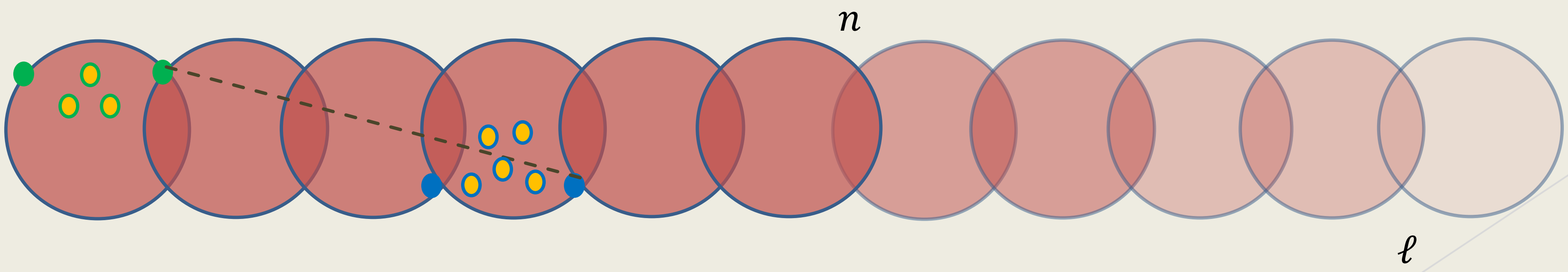
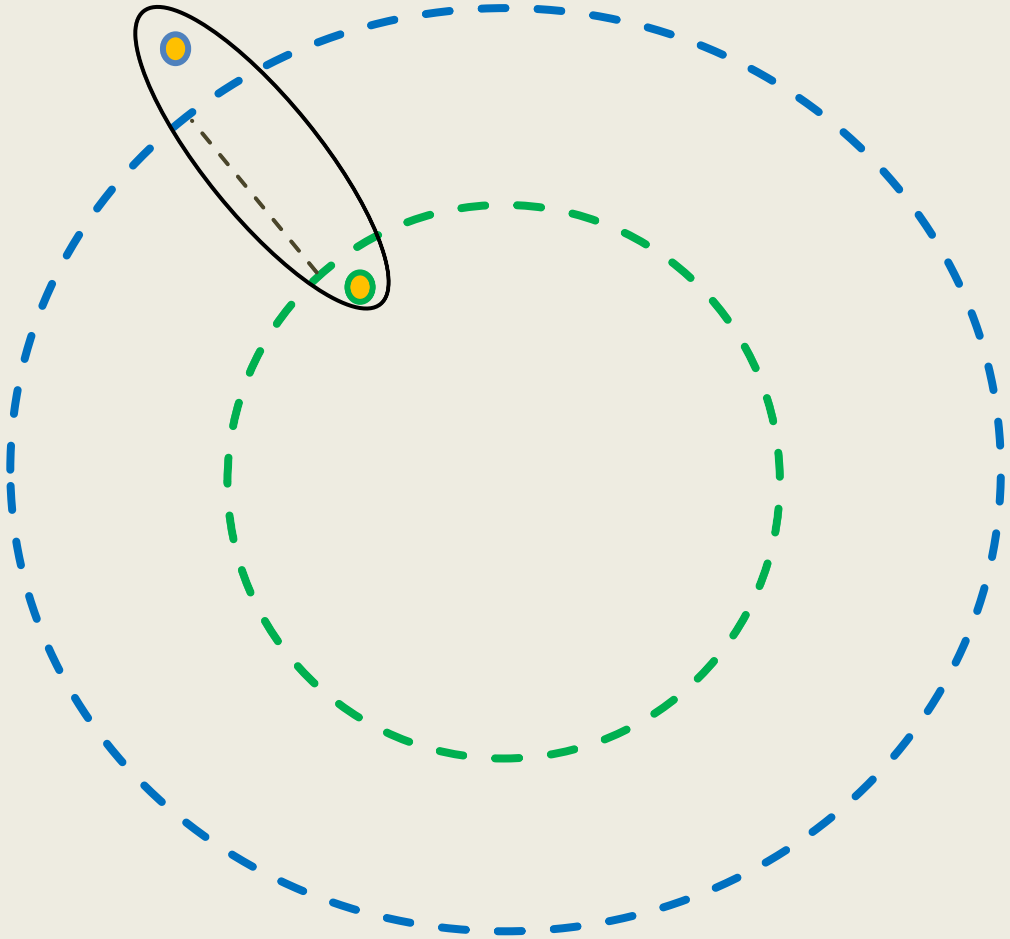
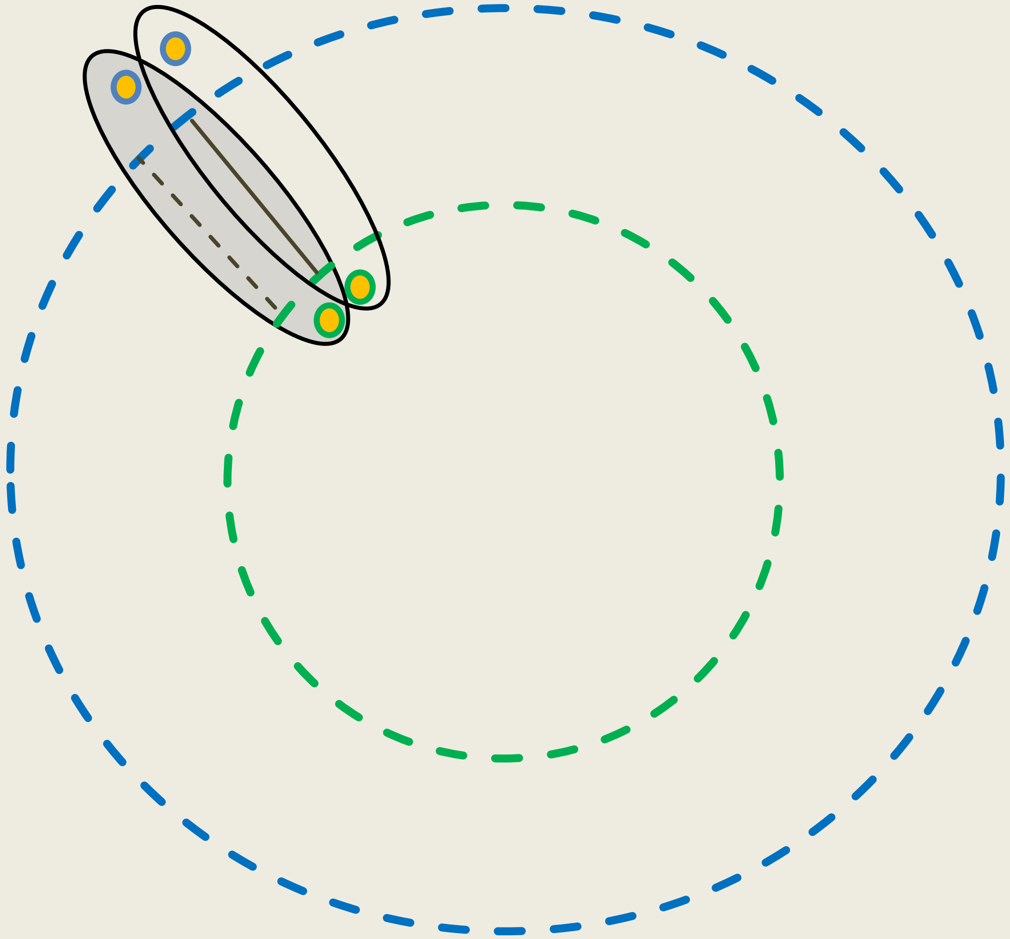


Take: $G_0 = G \setminus \{f_1, \dots, f_m\}$.

Proof idea- K_6-percolation



Proof idea- K_6-percolation



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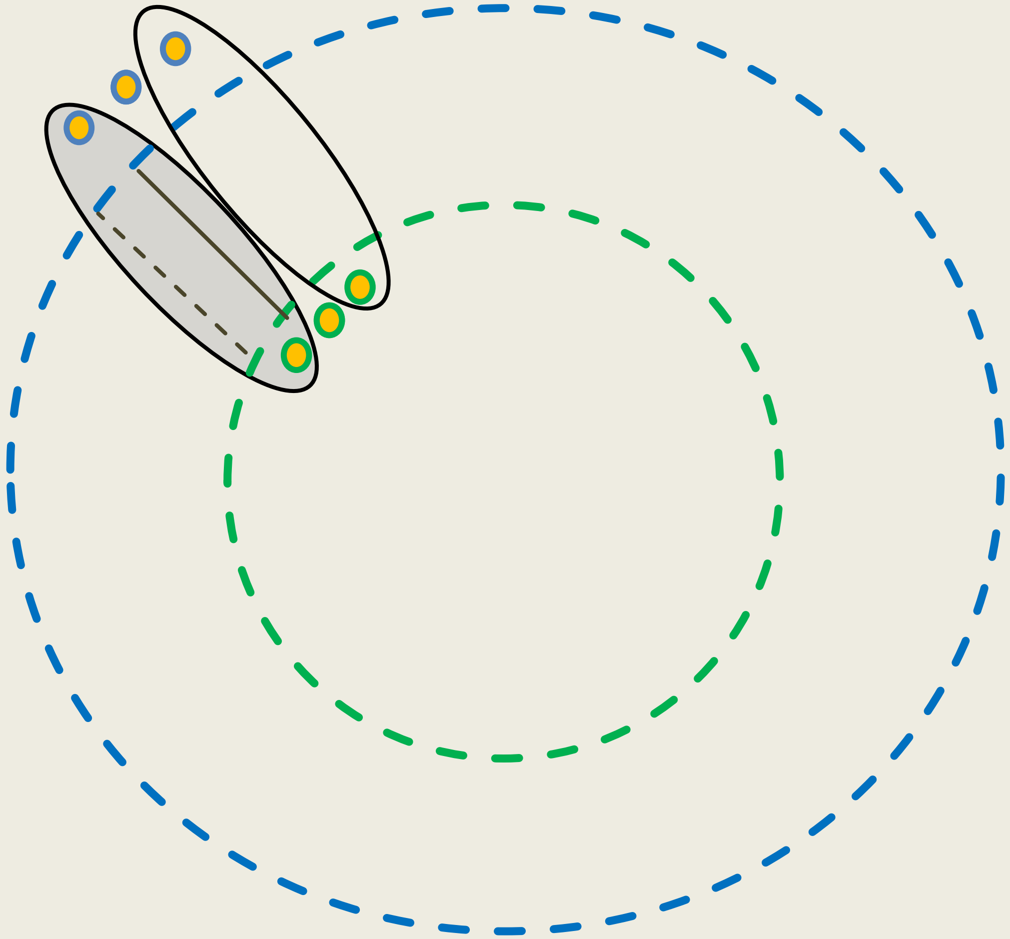

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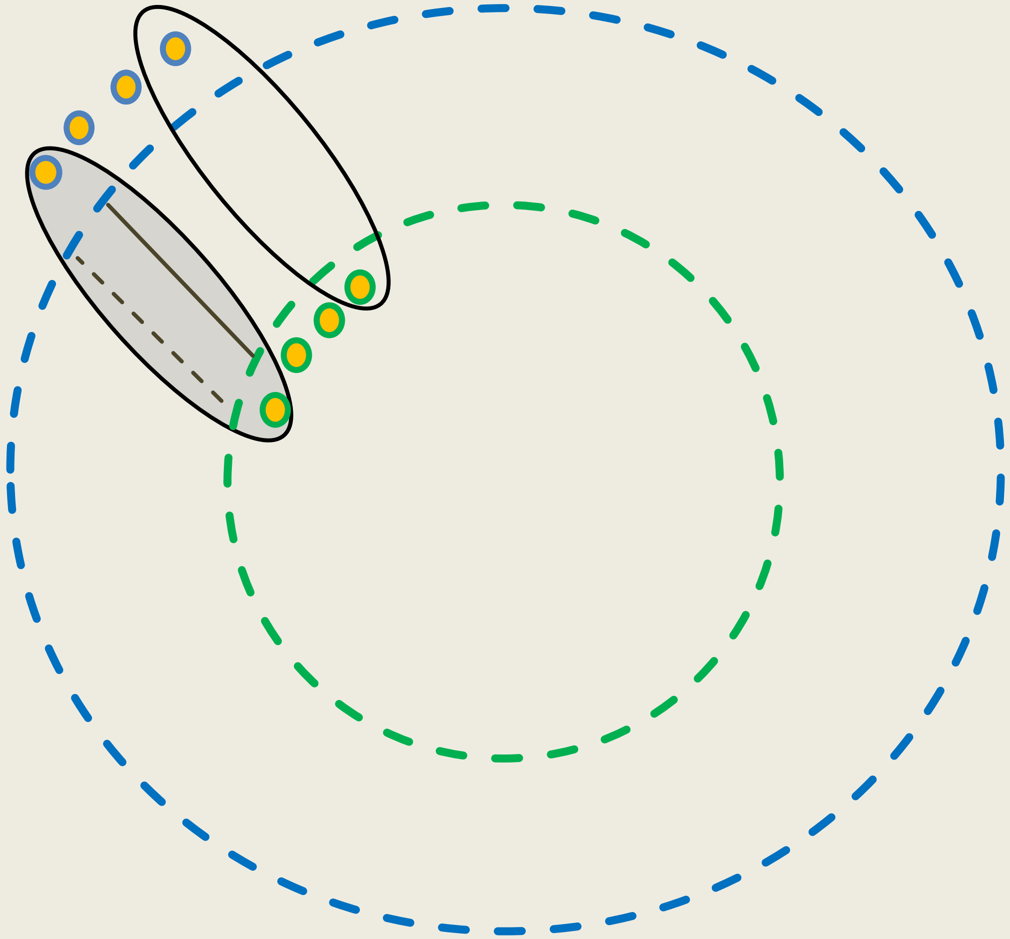

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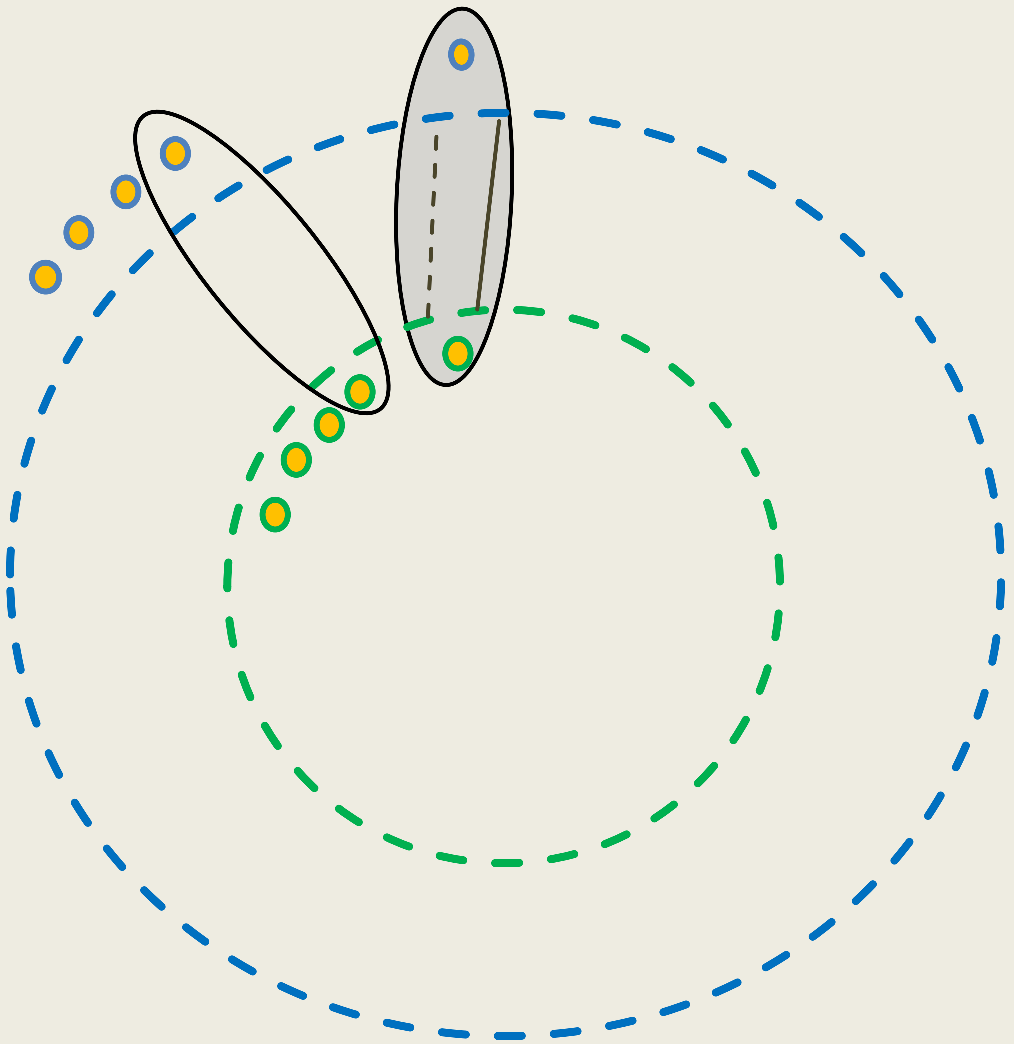

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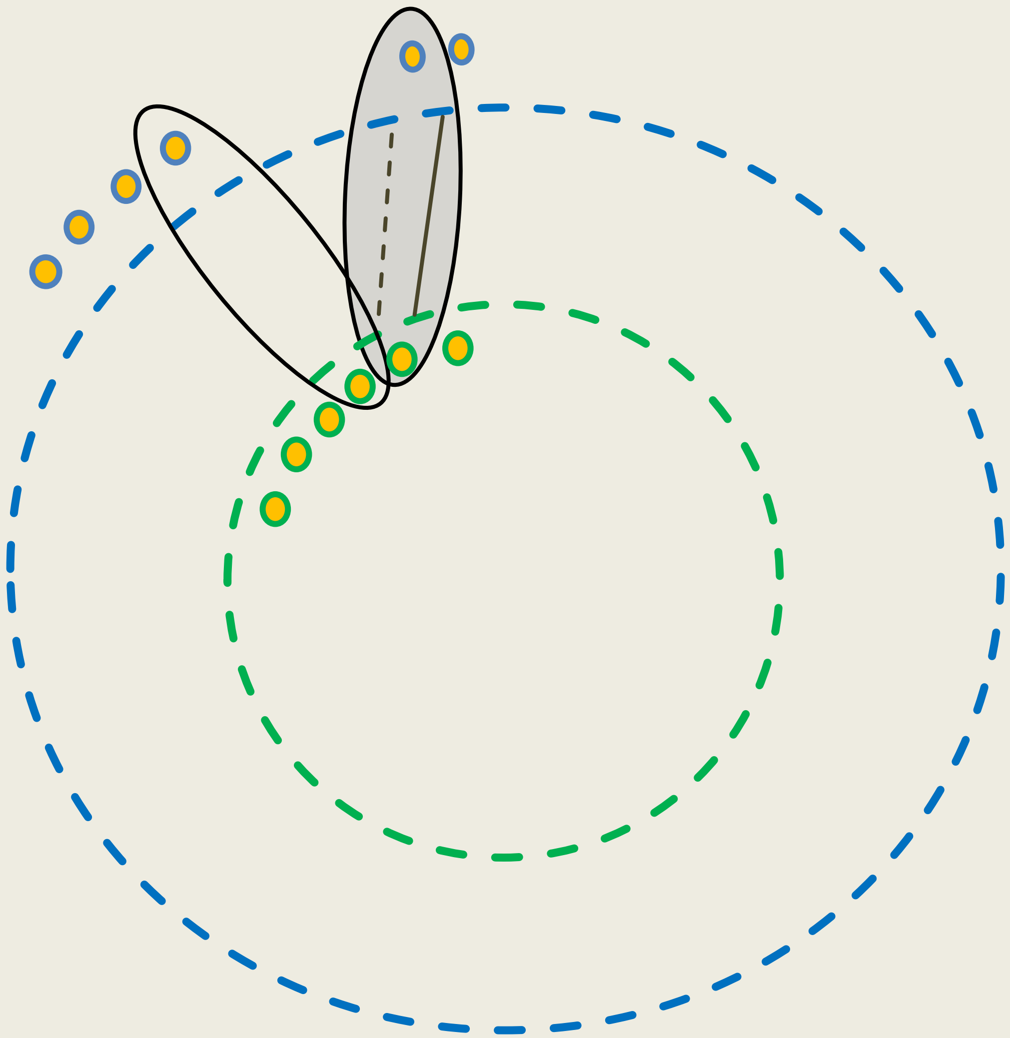

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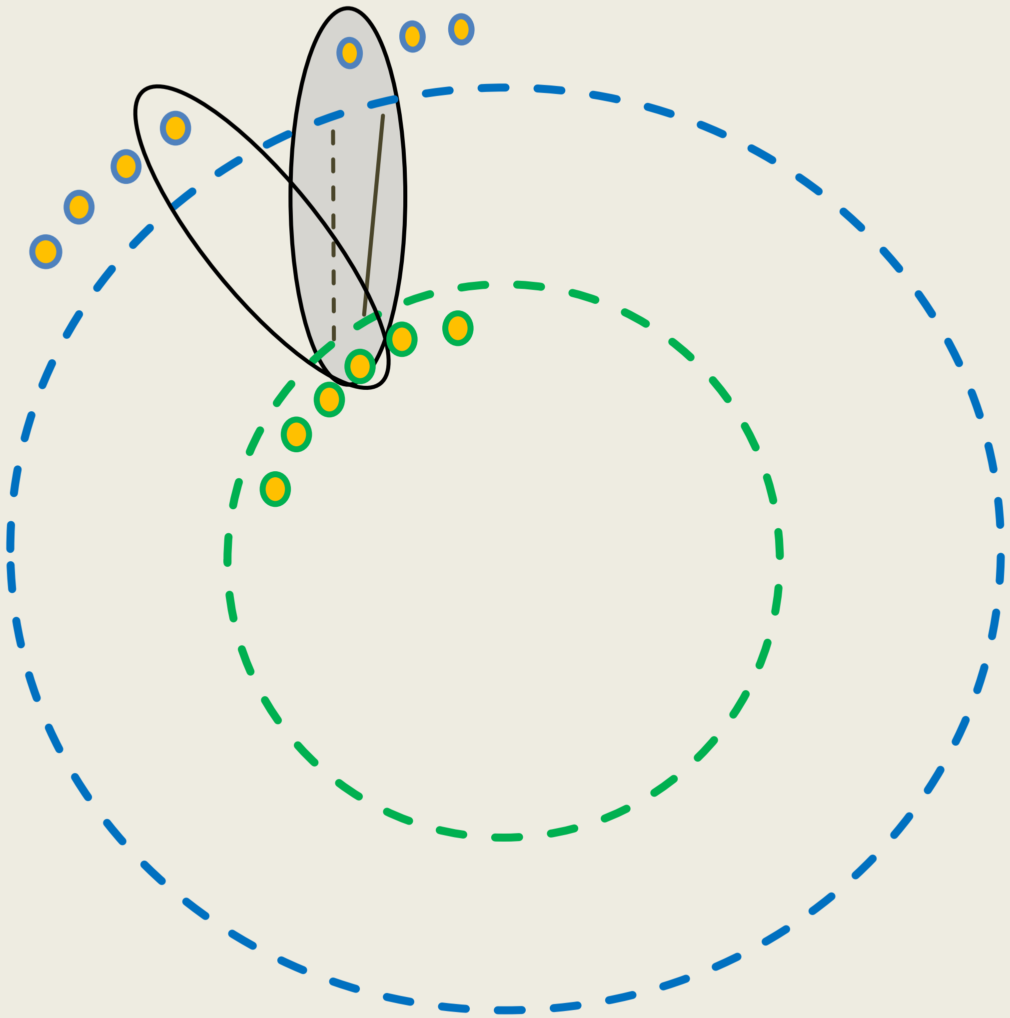

Proof idea- K_6-percolation

$X = \{x_1, \dots, x_n\}$, $W = W_1 \cup \dots \cup W_n$, $W_i = \left\{w_{i_1}, \dots w_{i_{\lfloor r/2 \rfloor - 2}}\right\}$,

$Y = \{y_1, \dots, y_\ell\}$, $Z = Z_1 \cup \dots \cup Z_\ell$, $Z_i = \left\{z_{i_1}, \dots z_{i_{\lfloor r/2 \rfloor - 2}}\right\}$. (say, $\ell = n + 20$)

The vertices: $V(G) = V(\mathcal{H}) = X \cup Y \cup W \cup Z$

The hyperedges: $e_t = \left\{x_{t \,(mod\, n)}, x_{t+1 \,(mod\, n)}, y_{t \,(mod\, \ell)}, y_{t+1 \,(mod\, \ell)}\right\} \cup W_{t \,(mod\, n)} \cup Z_{t \,(mod\, \ell)}$

(say, $1 \leq t \leq \frac{n}{100} = m$)

$G[e_i] = K_r$, let $f_i = \left\{x_{i+1 \,(mod\, n)}, y_{i+1 \,(mod\, \ell)}\right\}$.

- ✓ $E(\mathcal{H}) = \{e_1, \dots, e_m\}$ with $G[e_i] = K_r$,
- ✓ $\exists f_i \subseteq e_i \cap e_{i+1}$, $|f_i| = 2$ for every $i \in [m-1]$,
- ✓ $f_i \nsubseteq e_j$ for every $j \neq i, i+1$,
- ✓ $\exists f_m \subseteq e_m, |f_m| = 2$ and $f_m \nsubseteq e_j$ for every $j \neq m$,
- ✓ $\forall K_r^- \in G$ we have $V(K_r^-) \in E(\mathcal{H})$. (for $r \geq 6$)

Proof idea - K_5-percolation

$X = \{x_1, \dots, x_n\}$, $W = W_1 \cup \cdots \cup W_n$, $W_i = \left\{w_{i_1}, \dots w_{i_{\lfloor r/2 \rfloor - 2}}\right\}$,
$Y = \{y_1, \dots, y_\ell\}$, $Z = Z_1 \cup \cdots \cup Z_\ell$, $Z_i = \left\{z_{i_1}, \dots z_{i_{\lceil r/2 \rceil - 2}}\right\}$. (say, $\ell = n + 20$)

The vertices: $V(G) = V(\mathcal{H}) = X \cup Y \cup W \cup Z$

The hyperedges: $e_t = \left\{x_{t \pmod n}, x_{t+1 \pmod n}, y_{t \pmod \ell}, y_{t+1 \pmod \ell}\right\} \cup W_{t \pmod n} \cup Z_{t \pmod \ell}$

(say, $1 \le t \le \frac{n}{100} = m$)

$G[e_i] = K_r$, let $f_i = \left\{x_{i+1 \pmod n}, y_{i+1 \pmod \ell}\right\}$.

- ✓ $E(\mathcal{H}) = \{e_1, \dots, e_m\}$ with $G[e_i] = K_r$,
- ✓ $\exists f_i \subseteq e_i \cap e_{i+1}$, $|f_i| = 2$ for every $i \in [m-1]$,
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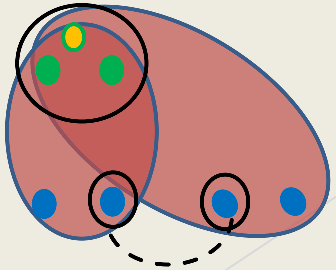
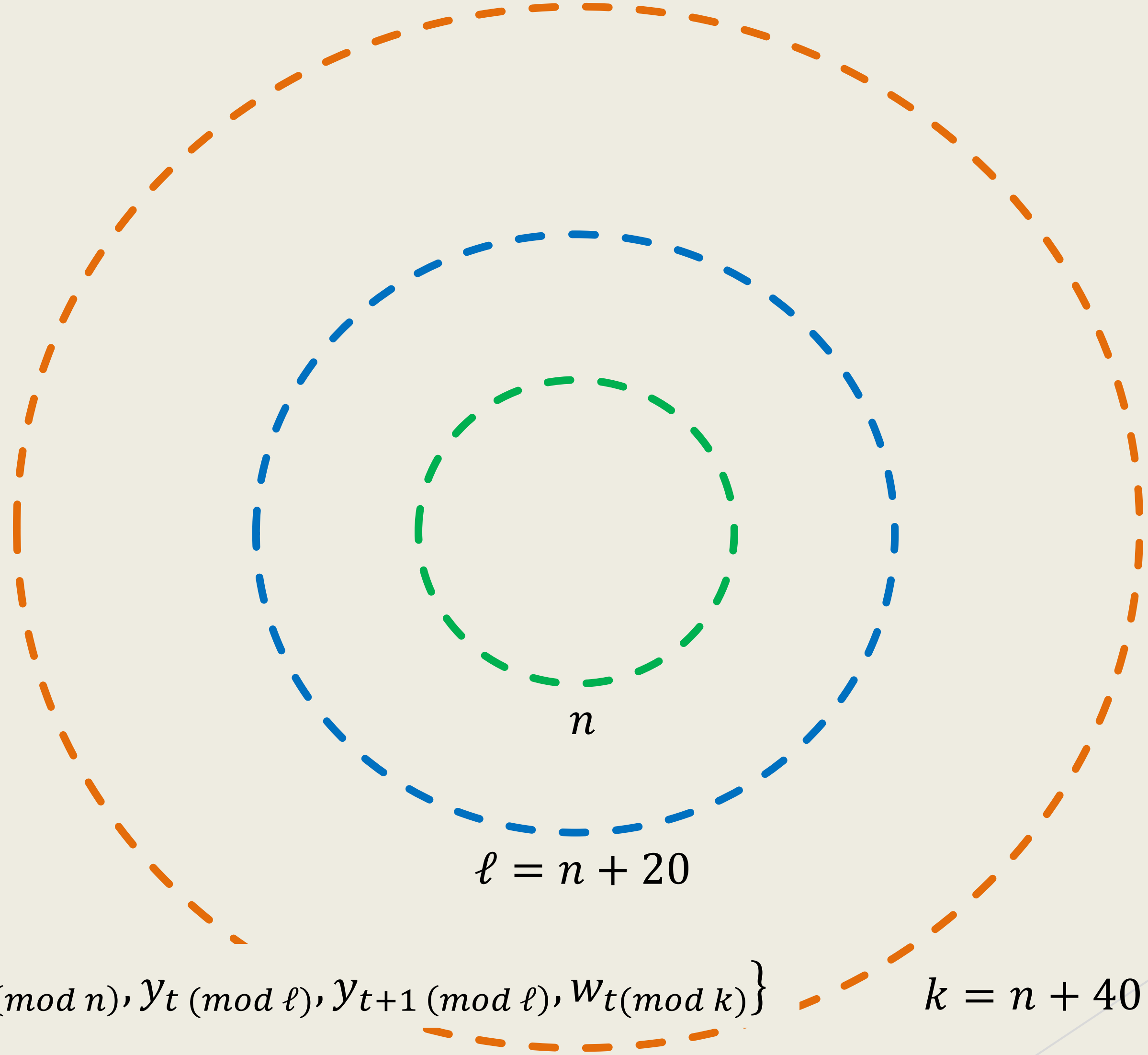
- ✓ $\exists f_m \subseteq e_m, |f_m| = 2$ and $f_m \nsubseteq e_j$ for every $j \neq m$,
- × $\forall K_r^- \in G$ we have $V(K_r^-) \in E(\mathcal{H})$. (for $r = 5$)



Proof idea – K_5-percolation

$X = \{x_1, \dots, x_n\}$,
$Y = \{y_1, \dots, y_\ell\}$,
$W = \{w_1, \dots, w_k\}$.



$$e_t = \{x_{t \pmod n}, x_{t+1 \pmod n}, y_{t \pmod \ell}, y_{t+1 \pmod \ell}, w_{t \pmod k}\}$$

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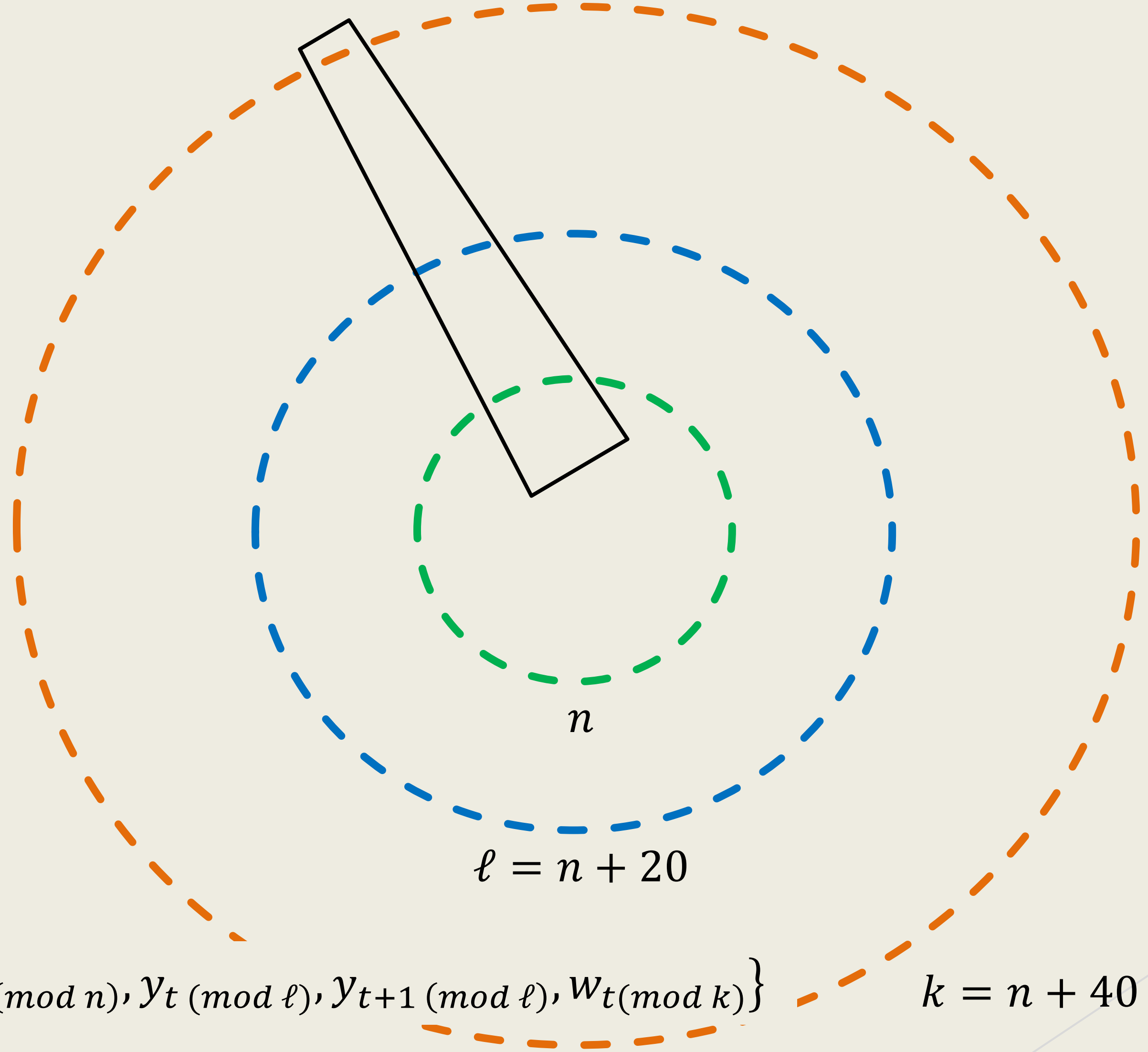

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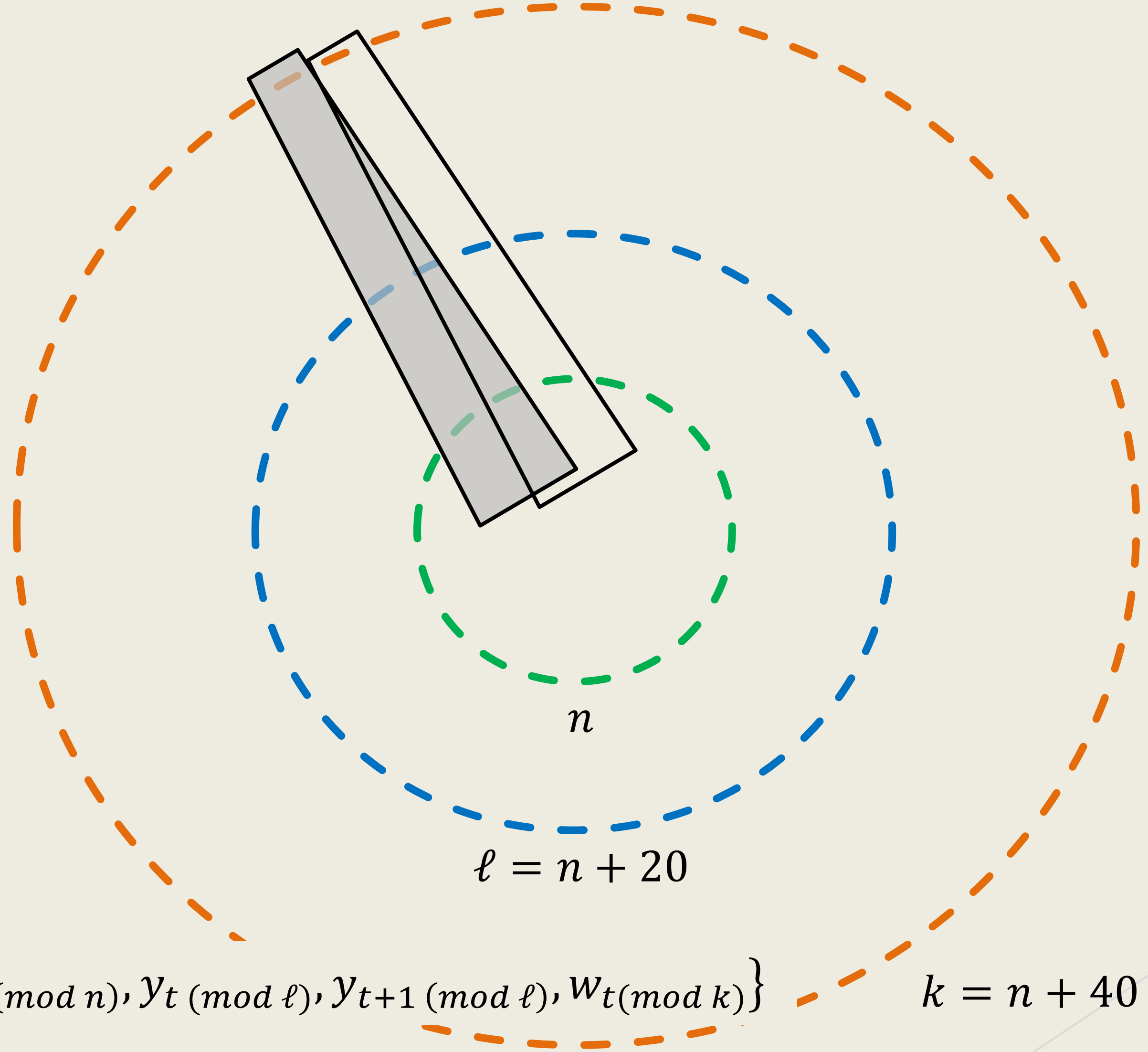

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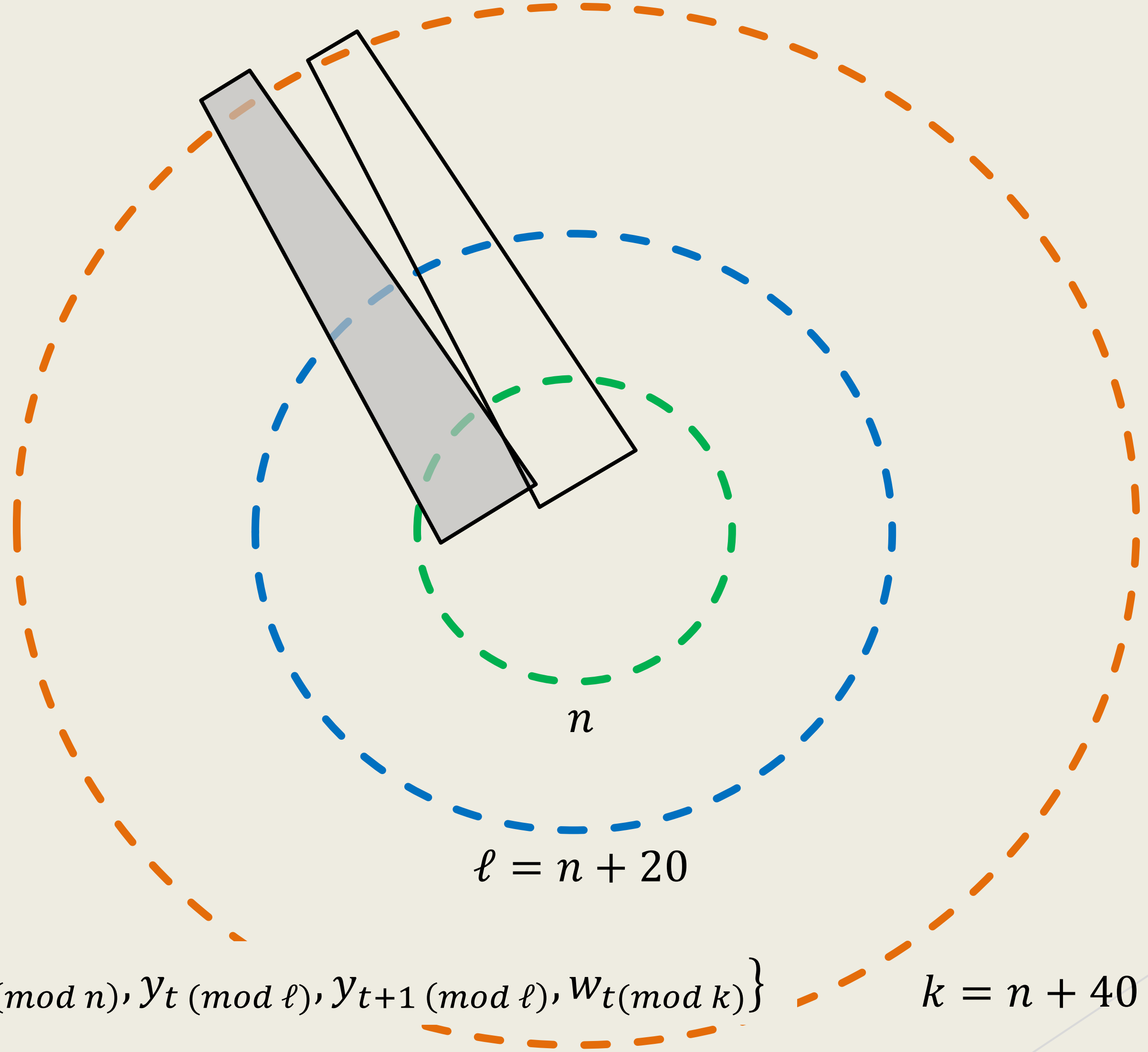

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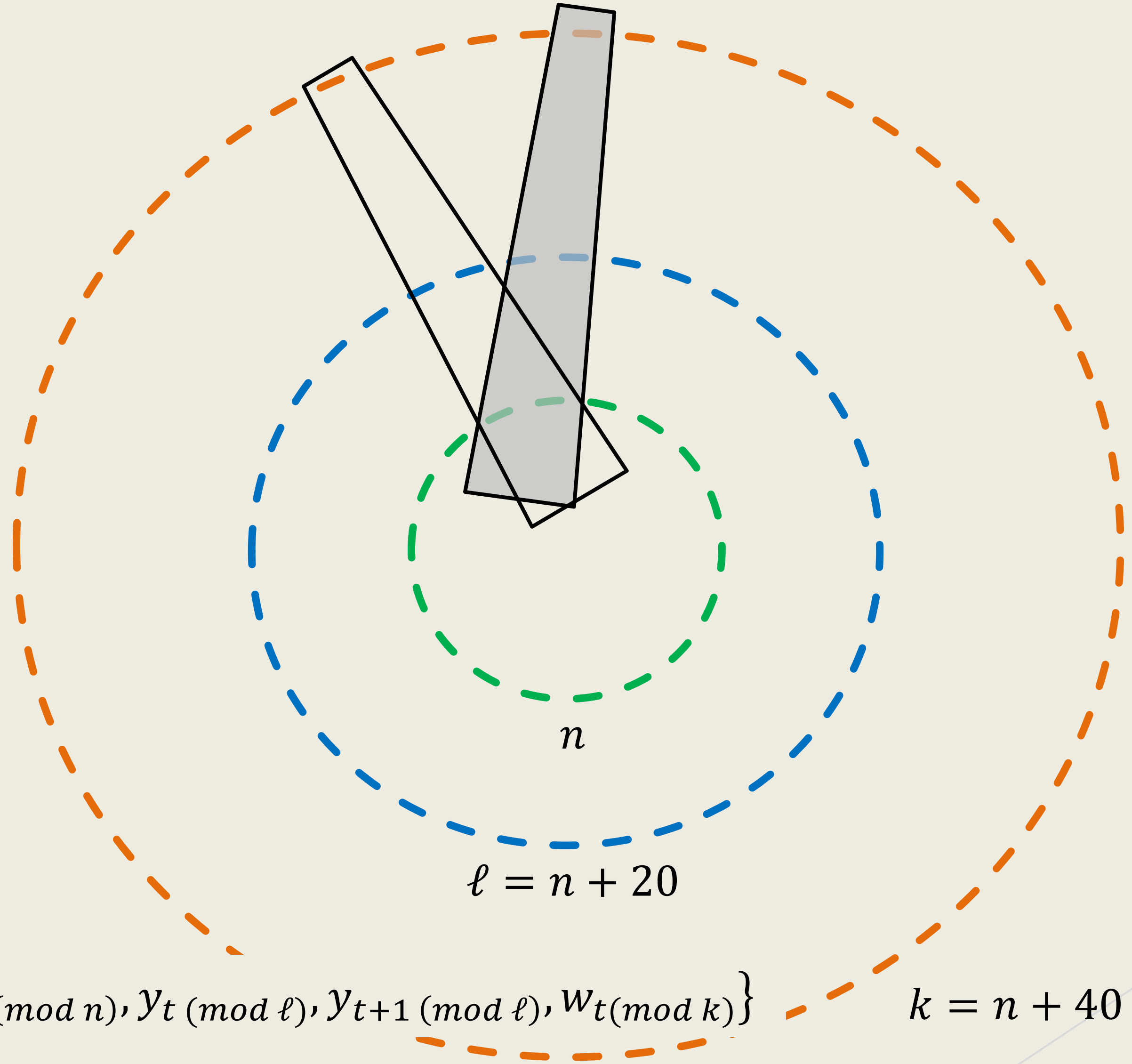

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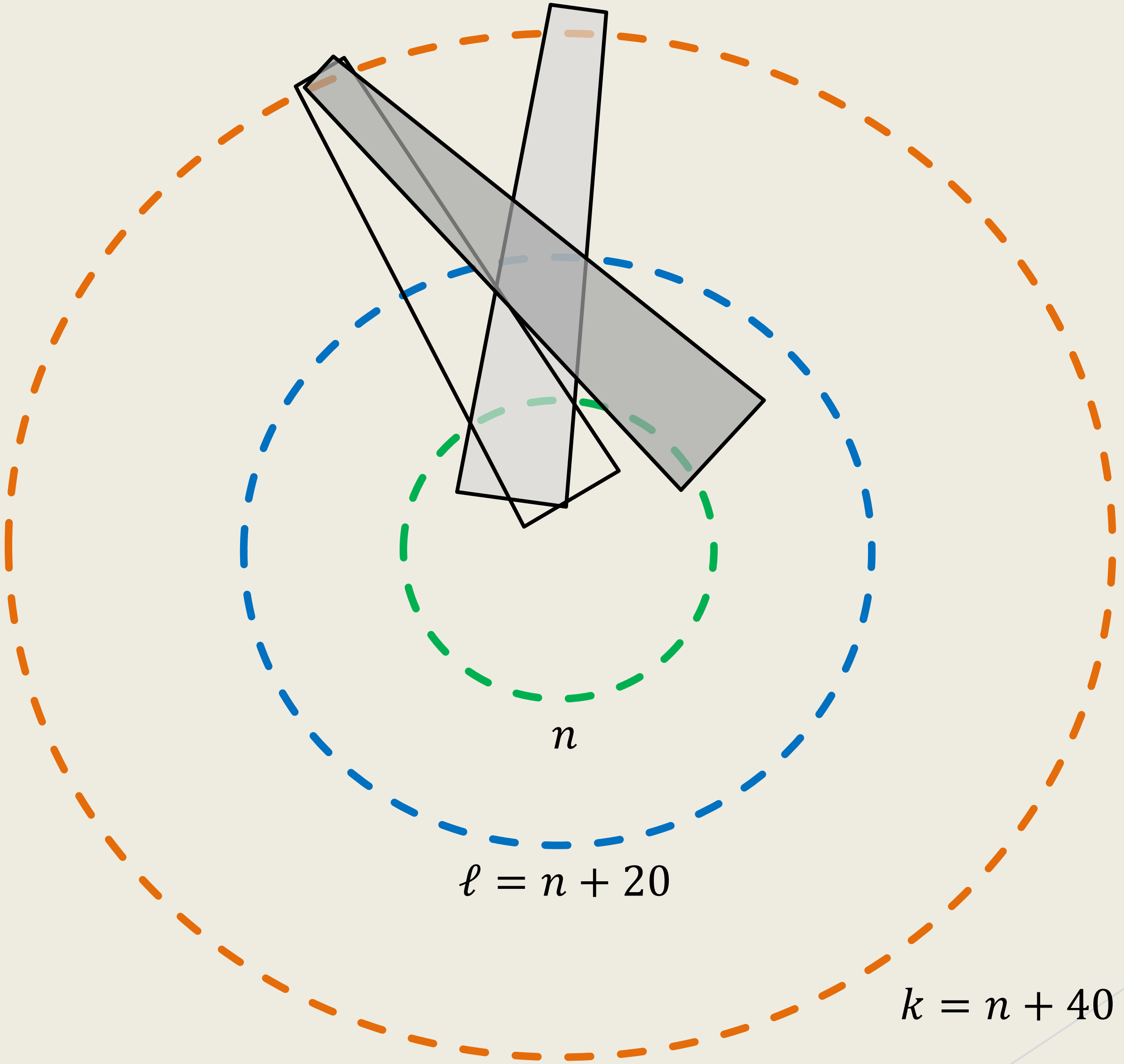

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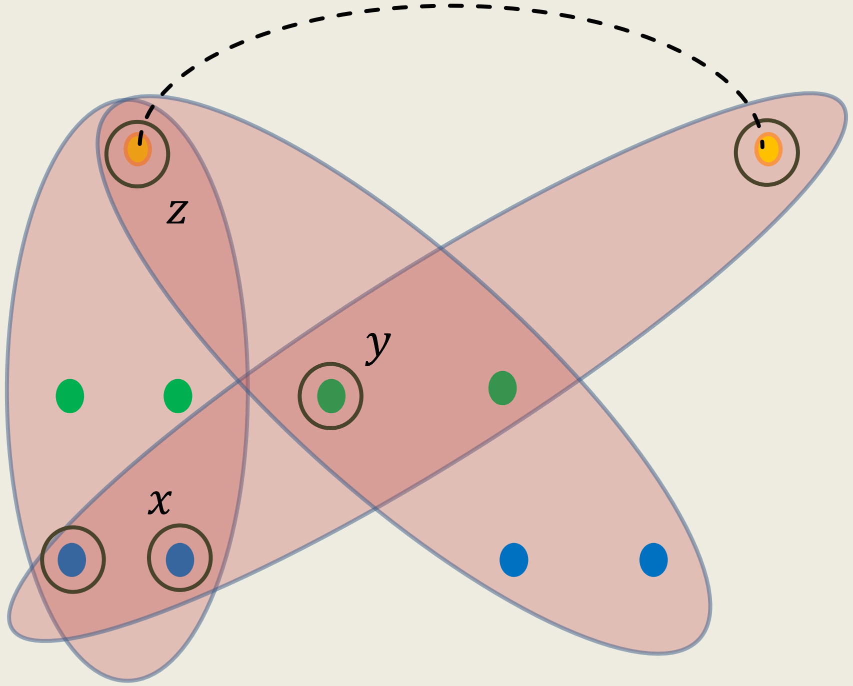

$$y = x + 20a$$
$$z = x + 40b$$
$$z = y + 20c$$

$$\Rightarrow \; 2b = a + c$$

Solution: $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_\ell\}$, $W = \{w_1, \dots, w_k\}$.

$e_{t,b} = \{x_{t \,(\mathrm{mod}\, n)}, x_{t+1 \,(\mathrm{mod}\, n)}, w_{t+b (\mathrm{mod}\, k)}, y_{t+2b \,(\mathrm{mod}\, \ell)}, y_{t+2b+1 \,(\mathrm{mod}\, \ell)}\}$,
($b \in 3AP$-free)

Concluding remarks

Length (running time): $\ell(H, G_0) = \min\{i \mid G_i = G_{i+1}\}$, $L(H,n) = \max_{G_0 \subseteq K_n} \ell(H, G_0)$

Question (Bollobás): $L(K_r, n)$?

Conjecture (Bollobás-Przykucki-Riordan-Sahasrabudhe 2017): $L(K_r, n) = o(n^2)$

Theorem (Balogh, K., Pokrovskiy, and Szabó, 2020):

- $L(K_5, n) \geq n^{2 - \frac{c}{\sqrt{\log n}}}$.
- $L(K_r, n) = \Theta(n^2)$, for $r \geq 6$.

Questions:

- $L(K_5, n) = o(n^2)$?
- Determine the dependency on r in $L(K_r, n) = \Theta(n^2)$, for $r \geq 6$.