Geodesics in random geometry

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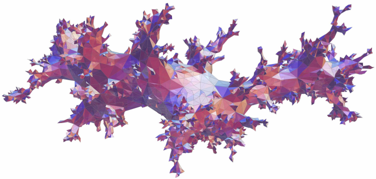
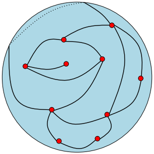
Oxford Discrete Mathematics and Probability seminar



Canonical random geometry in two dimensions

- Replace the sphere $\mathbb{S}^2$ by a discretization, namely a graph drawn on the sphere (= planar map).
- Choose such a planar map uniformly at random in a suitable class (triangulations,...) and equip its vertex set with the graph distance.



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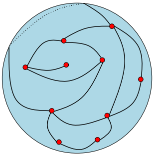
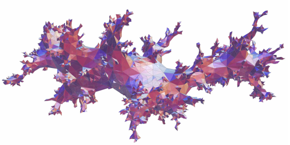


- Choose such a planar map uniformly at random in a suitable class (triangulations,...) and equip its vertex set with the graph distance.
- Let the size of the graph tend to infinity and pass to the limit after rescaling to get a random metric space: the Brownian sphere.



- This convergence holds independently of the class of planar maps (even if edges are assigned random lengths): Universality of Brownian sphere.

Goal of the lecture: Discuss remarkable properties of geodesics in the Brownian sphere (Miller-Qian 2020, LG 2021)

1. Random planar maps and the Brownian sphere

Definition

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere). Loops and multiple edges allowed.

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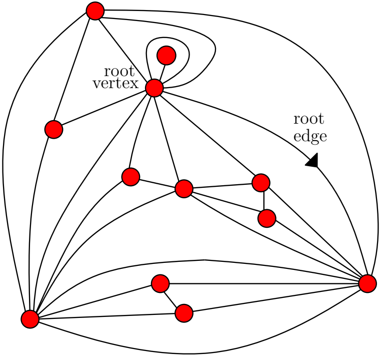


A rooted triangulation with 20 faces

Faces = connected components of the complement of edges

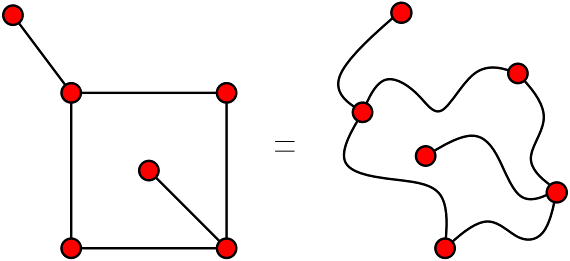
p-angulation:

- each face is incident to p edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

The same planar map:



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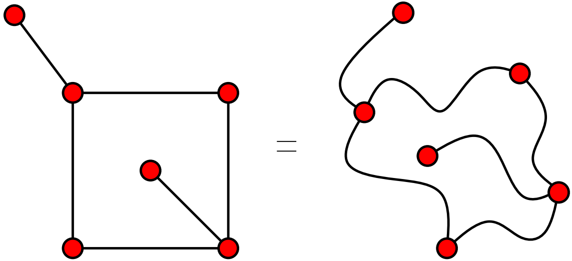
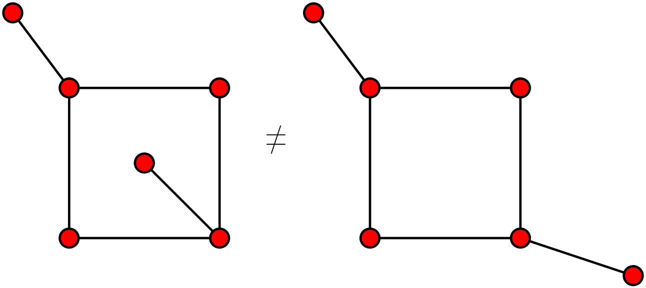
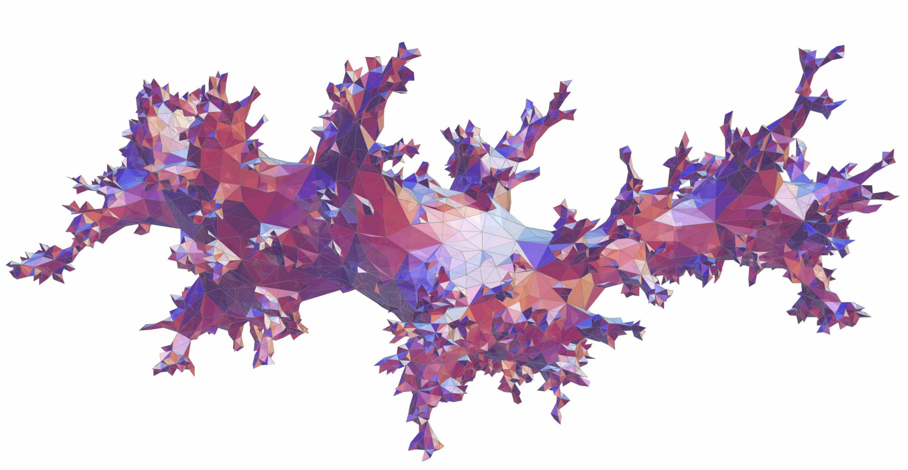


Two different planar maps:



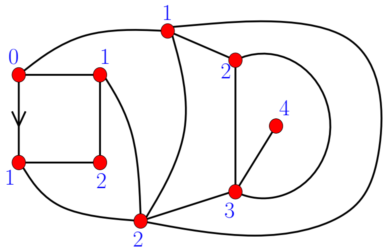
A large triangulation of the sphere
Can we get a continuous model out of this ?



Planar maps as metric spaces

M planar map

- $V(M)$ = set of vertices of M
- d_{gr} graph distance on $V(M)$
- $(V(M), d_{\text{gr}})$ is a (finite) metric space



In blue : distances from the root vertex

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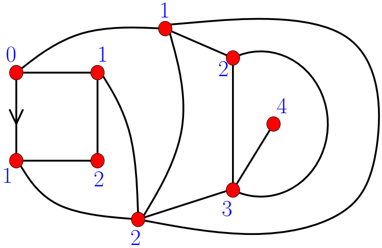

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$\mathbb{M}_n^p = \{$rooted $p-$angulations with n faces$\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$. We want to let $n \to \infty$ (p fixed)

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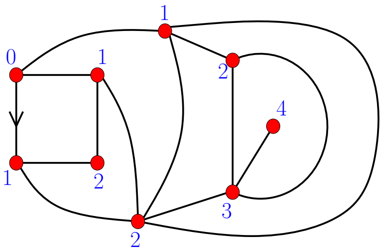

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Choose M_n uniformly at random in $\mathbb{M}_n^p$. We want to let $n \to \infty$ (p fixed)

View $(V(M_n), d_{\rm gr})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

($U_\varepsilon(K_1)$ is the ε-enlargement of K_1)

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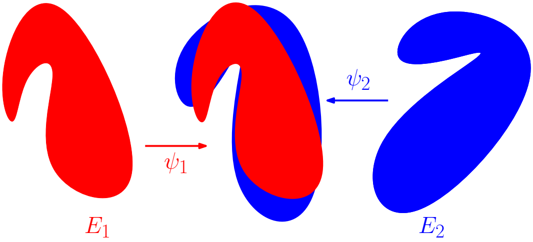
Definition (Gromov-Hausdorff distance)

If (E_1, d_1) and (E_2, d_2) are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1, E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1), \psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$, *then*

$(\mathbb{K}, d_{\mathrm{GH}})$ *is a separable complete metric space (Polish space)*

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$\rightarrow$ If M_n is uniformly distributed over $\{p-$ angulations with n faces$\}$, it makes sense to study the convergence in distribution as $n \to \infty$ of

$$(V(M_n), n^{-a} d_{\rm gr})$$

as random variables with values in $\mathbb{K}$.

(Problem stated for triangulations by O. Schramm [ICM, 2006])

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Choice of the rescaling factor n^{-a} : $a > 0$ is chosen so that $\mathrm{diam}(V(M_n)) \approx n^a$.

$\Rightarrow a = \frac{1}{4}$ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]

The Brownian sphere

$\mathbb{M}_n^p = \{\text{rooted } p-\text{angulations with } n \text{ faces}\}$
M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

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Theorem (LG 2013, Miermont 2013 for p=4)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \left(\frac{9}{p(p-2)}\right)^{1/4} \quad \textit{if } p \textit{ is even.}$$

Then,

$$(V(M_n), c_p\, n^{-1/4}\, d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{\mathrm{(d)}} (\mathbf{m}_\infty, D)$$

in the Gromov-Hausdorff sense. The limit $(\mathbf{m}_\infty, D)$ is a random compact metric space that does not depend on p (universality) and is called the Brownian sphere (or Brownian map).

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Remarks • $p = 3$ (triangulations) solves Schramm's problem.
• Extensions to other random planar maps: Abraham, Addario-Berry-Albenque (case of odd p), Beltran-LG, Bettinelli-Jacob-Miermont, etc.

Properties of the Brownian sphere

The Brownian sphere is a geodesic space: any pair of points is connected by a (possibly not unique) geodesic. (A Gromov-Hausdorff limit of geodesic spaces is a geodesic space.)

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Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D) = 4 \qquad \textit{a.s.}$$

(Already "known" in the physics literature.)

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Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D) = 4 \qquad a.s.$$

(Already "known" in the physics literature.)

Theorem (topological type)

Almost surely, $(\mathbf{m}_\infty, D)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

Connections with Liouville quantum gravity

Miller, Sheffield (2015-2016) have developed a program aiming to relate the Brownian sphere with Liouville quantum gravity:

- new construction of the Brownian sphere using the Gaussian free field and the random growth process called Quantum Loewner Evolution (an analog of the celebrated SLE processes studied by Lawler, Schramm and Werner)
- this construction makes it possible to equip the Brownian sphere with a conformal structure, and in fact to show that this conformal structure is determined by the Brownian sphere.

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More recently: the Miller-Sheffield construction has been simplified by a direct construction of the Liouville quantum gravity metric from the Gaussian free field (Gwynne-Miller 2019 after the work of several authors).

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The Brownian sphere $(\mathbf{m}_\infty, D)$ is constructed by identifying certain pairs of points in Aldous' Brownian continuum random tree (CRT).

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Constructions of the CRT (Aldous, 1991-1993):

- As the scaling limit of many classes of discrete trees
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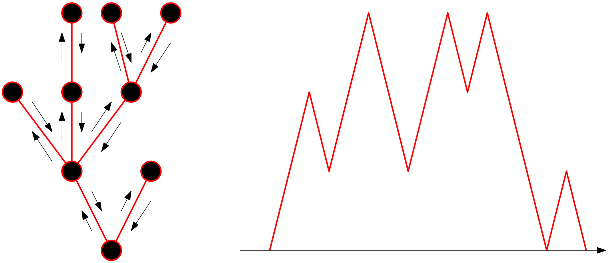
Coding a (discrete) plane tree by its contour function (or Dyck path):



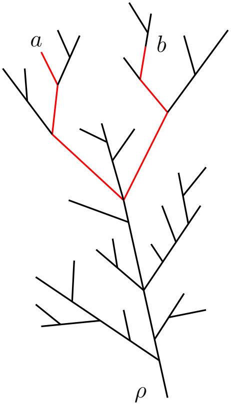
The notion of a real tree

Definition

A real tree, or $\mathbb{R}$-tree, is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique continuous and injective path (up to re-parametrization)
- this path is isometric to a line segment

$\mathcal{T}$ is a rooted real tree if there is a distinguished point ρ, called the root.



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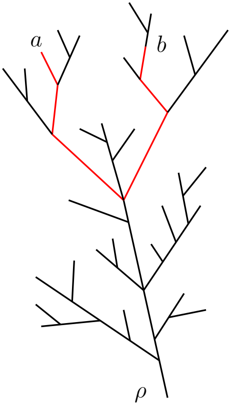

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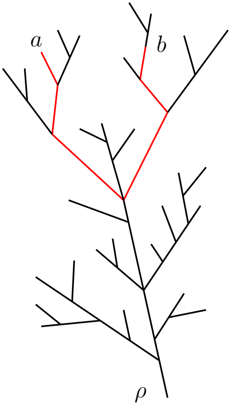

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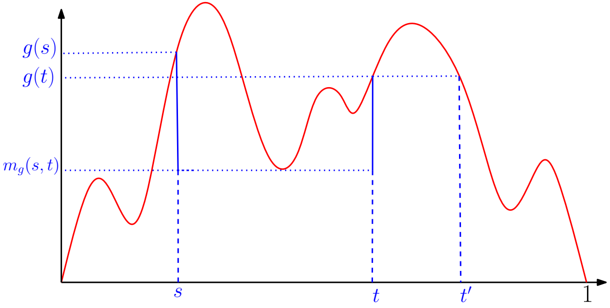
Fact. The coding of discrete trees by contour functions can be extended to real trees: also gives a cyclic ordering on the tree.

The real tree coded by a function g

$g : [0,1] \longrightarrow [0,\infty)$
continuous,

$g(0) = g(1) = 0$

$m_g(s,t) = \min_{[s\wedge t, s\vee t]} g$



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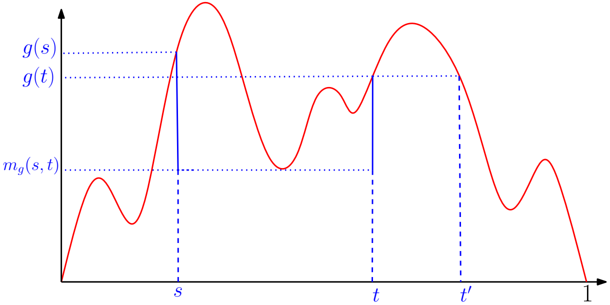

$d_g(s,t) = g(s) + g(t) - 2\, m_g(s,t)$ pseudo-metric on $[0,1]$

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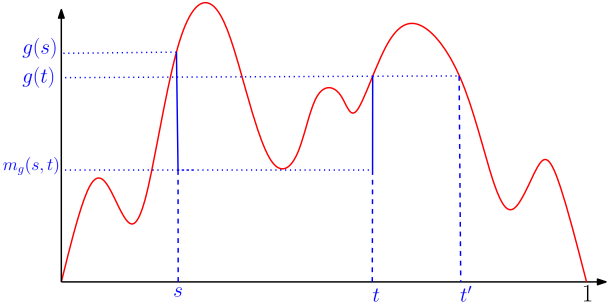

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Proposition

$\mathcal{T}_g := [0,1]/\sim$ equipped with d_g is a real tree, called the tree coded by g. It is rooted at $\rho = 0$.

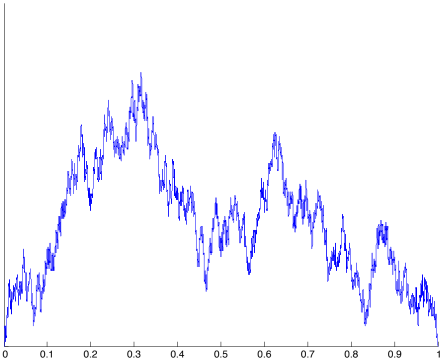
The canonical projection $[0,1] \to \mathcal{T}_g$ induces a cyclic ordering on $\mathcal{T}_g$

Definition of the CRT

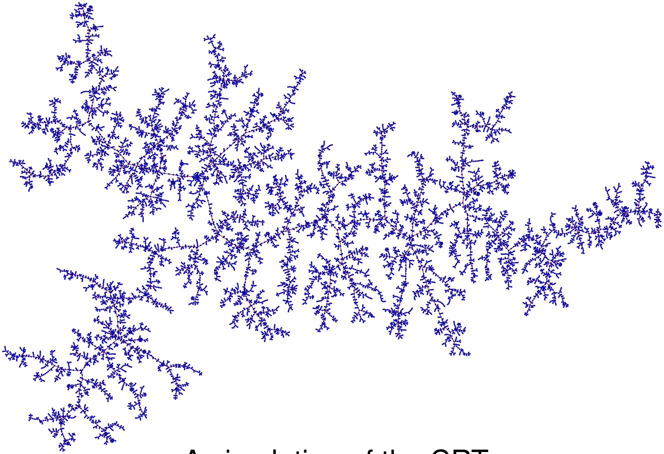
Let $(\mathbf{e}_t)_{0\leq t\leq 1}$ be a Brownian excursion with duration 1 (= Brownian motion started from 0 conditioned to be at 0 at time 1 and to stay ≥ 0)

Definition

The CRT $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by the Brownian excursion $\mathbf{e}$.



Simulation of a Brownian excursion



A simulation of the CRT

(simulation: I. Kortchemski)

Assigning Brownian labels to a real tree

Let $(\mathcal{T}, d)$ be a real tree with root ρ.

$(Z_a)_{a\in\mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0$
- $E[(Z_a - Z_b)^2] = d(a, b), \qquad a, b \in \mathcal{T}$

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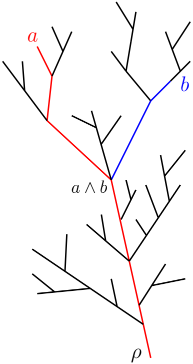

We view Z_a as a label assigned to $a \in \mathcal{T}$.
Labels evolve like Brownian motion along the branches of the tree:

- The label Z_a is the value at time $d(\rho, a)$ of a standard Brownian motion
- Similar property for Z_b, but one uses
  - the same BM between 0 and $d(\rho, a \wedge b)$
  - an independent BM between $d(\rho, a \wedge b)$ and $d(\rho, b)$

The definition of the Brownian sphere

$(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT, $(Z_a)_{a \in \mathcal{T}_{\mathbf{e}}}$ Brownian motion indexed by the CRT (Two levels of randomness!).

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$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a,b]$ is the "interval" from a to b corresponding to the cyclic ordering on $\mathcal{T}_{\mathbf{e}}$ (vertices visited when going from a to b in clockwise order around the tree).

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Then set

$$D(a,b) = \inf_{a_0=a,a_1,\ldots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^0(a_{i-1},a_i),$$

$$a \approx b \text{ if and only if } D(a,b)=0 \quad (\text{equivalent to } D^0(a,b)=0).$$

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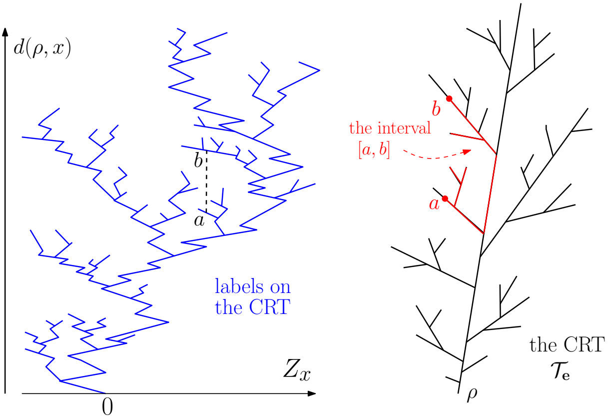
Definition

The Brownian sphere $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_{\mathbf{e}}/\approx$, which is equipped with the distance induced by D.

Summary and interpretation

Starting from the CRT $\mathcal{T}_{\mathbf{e}}$, with Brownian labels Z_a, $a \in \mathcal{T}_{\mathbf{e}}$,
$\rightarrow$ The two vertices $a, b \in \mathcal{T}_{\mathbf{e}}$ are glued ($a \approx b$) if:

- they have the same label $Z_a = Z_b$,
- one can go from a to b around the tree visiting only vertices with label greater than or equal to $Z_a = Z_b$.



A key property of distances in the Brownian sphere

Notation:

- Π is the canonical projection from the CRT $\mathcal{T}_{\mathbf{e}}$ onto $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}} / \approx$
- For $x = \Pi(a)$, $Z_x := Z_a$ (does not depend on choice of a).

Fact

Let a_* be the (unique) point of the CRT $\mathcal{T}_{\mathbf{e}}$ with minimal label, and $x_* = \Pi(a_*)$. Then, for every $x \in \mathbf{m}_\infty$,

$$D(x_*, x) = Z_x - \min Z$$

("labels" exactly correspond, up to a shift, to distances from x_*).

A key property of distances in the Brownian sphere

Notation:

- Π is the canonical projection from the CRT $\mathcal{T}_{\mathbf{e}}$ onto $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}} / \approx$
- For $x = \Pi(a)$, $Z_x := Z_a$ (does not depend on choice of a).

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The Brownian sphere comes with two distinguished points, namely x_* and $x_0 = \Pi(\rho)$ (ρ is the root of $\mathcal{T}_{\mathbf{e}}$)

$\longrightarrow$ x_0 and x_* are independently uniformly distributed over $\mathbf{m}_\infty$ (in a sense that can be made precise)

$\longrightarrow$ in particular, x_* is a "typical point" of $\mathbf{m}_\infty$

3. Geodesics from the “typical” point x_*

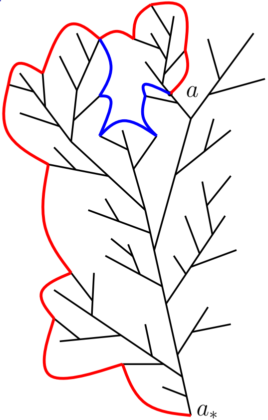
$x_* = \Pi(a_*)$ unique point of $\mathbf{m}_\infty$ s.t. $Z_{x_*} = \min Z$ then for every $x \in \mathbf{m}_\infty$,

$$D(x_*, x) = Z_x - \min Z \stackrel{\text{(notation)}}{=} \widetilde{Z}_x.$$

Let $x = \Pi(a)$, $a \in \mathcal{T}_{\mathbf{e}}$ be any point of $\mathbf{m}_\infty$. Can construct a “simple geodesic” from x_* to x by setting for $t \in [0, \widetilde{Z}_a]$

$$\varphi_a(t) = \Pi\Big(\text{last vertex } b \text{ before } a \text{ s.t. } \widetilde{Z}_b = t\Big)$$

(“last” and “before” refer to cyclic order on $\mathcal{T}_{\mathbf{e}}$)



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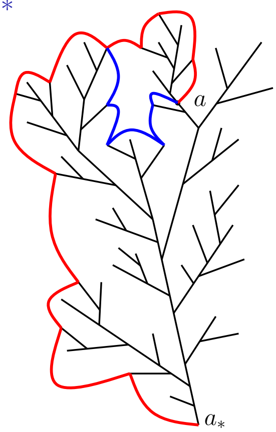

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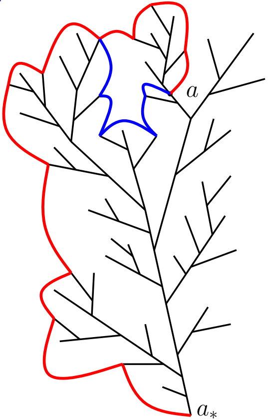

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All geodesics from x_* are simple geodesics.

Remark. If a is not a leaf, there are several possible choices, depending on which side of a one starts.

The main result about geodesics to a typical point

Define the skeleton of $\mathcal{T}_{\mathbf{e}}$ by $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \backslash \{\text{leaves of } \mathcal{T}_{\mathbf{e}}\}$ and set

$\mathrm{Skel} = \Pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}))$, where $\Pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\approx\, = \mathbf{m}_\infty$ canonical projection

Then

- the restriction of Π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
- $\dim(\mathrm{Skel}) = 2$ (recall $\dim(\mathbf{m}_\infty) = 4$)

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Theorem (Geodesics from the root)

Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from x_* to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from x_* to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) \leq 3$).*

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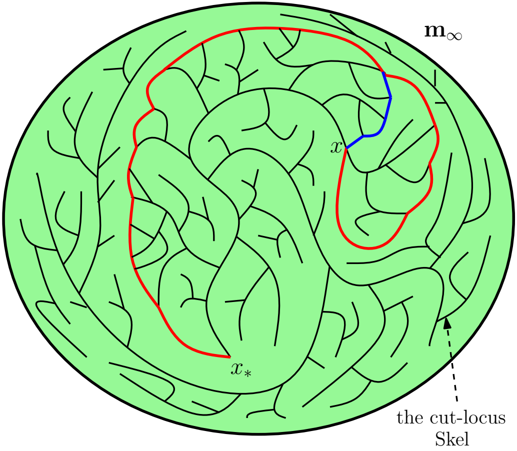
Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to x_*: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if x_* replaced by a point chosen "at random" in $\mathbf{m}_\infty$.

Illustration of the cut-locus



The cut-locus Skel
is homeomorphic to
a non-compact real tree
and is dense in $\mathbf{m}_\infty$

Geodesics to x_*
do not visit Skel
(except possibly
at their starting point)
but “move around” Skel.

Confluence property of geodesics

Fact: Two geodesics to x_* coincide near x_*.
(easy from the description of these geodesics)

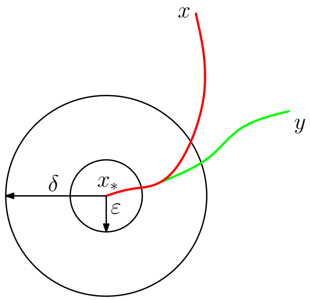
Corollary

Given $\delta > 0$*, there exists* $\varepsilon > 0$ *s.t.*

- *if* $D(x_*, x) \geq \delta$, $D(x_*, y) \geq \delta$
- *if* γ *is any geodesic from* x_* *to* x
- *if* γ' *is any geodesic from* x_* *to* y

then

$$\gamma(t) = \gamma'(t) \quad \text{for all } t \leq \varepsilon$$

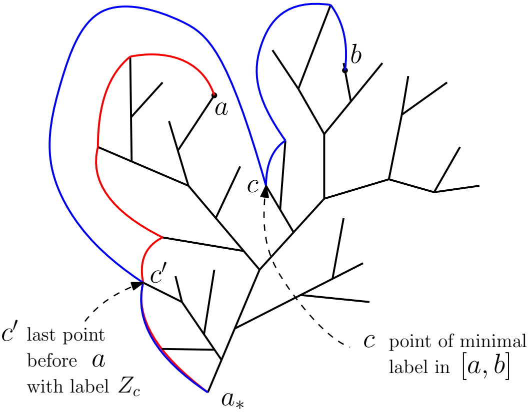


"Only one way" of leaving x_* along a geodesic.
(also true if x_* is replaced by a typical point of $\mathbf{m}_\infty$)

Why the confluence property

Let $a, b \in \mathcal{T}_{\mathbf{e}}$ such that $a_* \notin [a, b]$ (otherwise interchange a and b). Recall the simple geodesics φ_a and φ_b (from x_* to $x = \Pi(a)$ and to $x = \Pi(b)$ respectively). Then

$$\varphi_a(t) = \varphi_b(t) \text{ for every } 0 \leq t \leq \min_{d \in [a,b]} Z_d - \min Z (> 0).$$



4. Geodesics between exceptional points

If x and y are typical points of the Brownian sphere (chosen according to the volume measure)

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If x and y are both exceptional:

- There can be up to 9 geodesics from y to x (Miller-Qian (2020), following earlier work of Angel, Kolesnik, Miermont (2017)).
- Miller-Qian (2020) even compute the Hausdorff dimension of the set of pairs (x, y) such that there are exactly k geodesics from y to x.

Geodesic stars

A point x of the Brownian sphere $\mathbf{m}_\infty$ is called a geodesic star with n arms ($n \geq 2$) if it is the endpoint of n geodesics that are disjoint (except for their terminal point)

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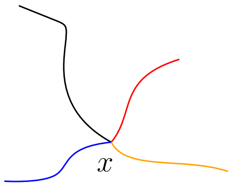

A geodesic star with 4 arms

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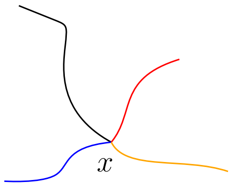

A geodesic star with 4 arms

Theorem (Miller-Qian 2020, LG 2021)

Let $\mathbf{S}_n$ be the set of all geodesic arms with n arms. Then, for $n = 2, 3, 4$,

$$\dim(\mathbf{S}_n) = 5 - n$$

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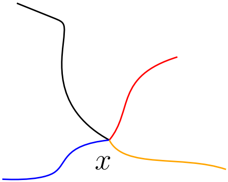

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Open problem: Is $\mathbf{S}_5$ not empty ?

Remark. Miller and Qian proved that the set of all interior points of geodesics has dimension 1. An interior point of a geodesic is a geodesic star with 2 arms, but not a typical one!

Results on geodesics in the Brownian sphere

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- Related results in Liouville quantum gravity: Gwynne-Miller (2019) on the confluence property, Gwynne (2020) on geodesic networks.

Sketch of proof of the lower bound $\dim(\mathbf{S}_n) \geq 5 - n$

From now on, consider the "free Brownian sphere" (with a random volume) under the σ-finite measure $\mathbb{N}_0$.

Define a notion of ε-approximate geodesic stars: for $\varepsilon > 0$, $x \in \mathbf{m}_\infty$ belongs to $\mathbf{S}_n^\varepsilon$ if there are n geodesics to x starting from the boundary of the ball of radius 1 centered at x that are disjoint up to the time when they arrive at distance ε from x.

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Then, if $\mathrm{Vol}(\cdot)$ is the volume measure on $\mathbf{m}_\infty$, for $n = 2, 3, 4$,

$$\mathbb{E}_{\mathbb{N}_0}\Big(\mathrm{Vol}(\mathbf{S}_n^\varepsilon)\Big) \geq c\,\varepsilon^{n-1}$$

and for every $\delta > 0$,

$$\mathbb{E}_{\mathbb{N}_0}\left(\iint \mathbf{1}_{\mathbf{S}_n^\varepsilon \times \mathbf{S}_n^\varepsilon}(x,y)\, D(x,y)^{-(5-m-\delta)}\, \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\right) \leq c_\delta\, \varepsilon^{2(n-1)}$$

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$\longrightarrow$ Standard techniques (extraction of convergent subsequence from $\varepsilon^{-(n-1)}\mathrm{Vol}_{|\mathbf{S}_n^\varepsilon}$, Frostman lemma) then show that $\mathbf{S}_n = \bigcap_{\varepsilon>0} \mathbf{S}_n^\varepsilon$ has dimension $\geq 5 - n$ on an event of positive $\mathbb{N}_0$-measure.

A useful tool: hulls

Let $x, y \in \mathbf{m}_\infty$ and $r > 0$. Write $B_r(x)$ for the closed ball of radius r centered at x. On the event $\{D(x,y) > r\}$, one can define the hull of radius r centered at x relative to y, denoted by $B_r^{\bullet,y}(x)$:

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The complement of $B_r^{\bullet,y}(x)$ is the connected component of the complement of $B_r(x)$ that contains y.

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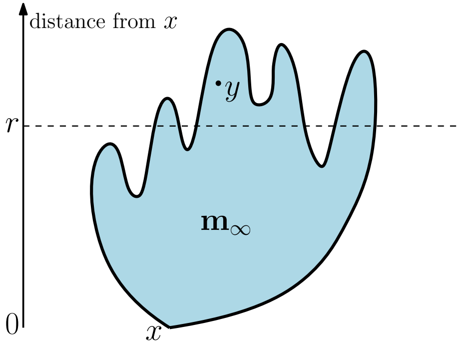

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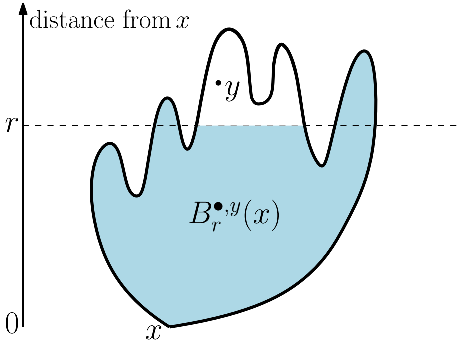

Cactus representation of the Brownian sphere (the vertical coordinate here is the distance from x)

One can make sense of the boundary size $|\partial B_x^{\bullet,y}|$ of the hull (at least when x and y are "typical")

Forest representation of a hull

Consider the hull $B_r^{\bullet,x_0}(x_*)$ and its boundary size $|\partial B_r^{\bullet,x_0}(x_*)|$.
Then conditionally on $|\partial B_r^{\bullet,x_0}(x_*)| = u$, one can represent the hull in terms of a Poisson forest of real trees equipped with Brownian labels:

$$(\mathcal{T}_{e_i}, (Z^i_a)_{a\in\mathcal{T}_{e_i}}), \quad i \in I$$

where $\sum_{i\in I} \delta_{e_i}$ is Poisson with intensity $u\,\mathbf{n}(\mathrm{d}e)$ (here $\mathbf{n}$ is the Itô excursion measure). Consider the trees as planted uniformly over $[0, u]$, and identify 0 with u. Furthermore condition the minimal label of the forest to be equal to $-r$.

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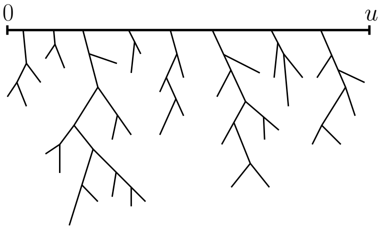
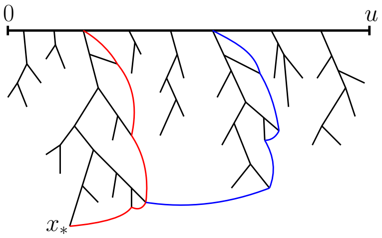


Then the hull $B_r^{\bullet,x_0}(x_*)$ equipped with its intrinsic distance is obtained from the labeled forest by exactly the same construction as the Brownian sphere from $(\mathcal{T}_{\mathbf{e}}, (Z_a)_{a\in\mathcal{T}_{\mathbf{e}}})$.
Labels shifted by $+r$ again correspond to distances from the point x_*, which is the point with minimal label.

The one-point estimate



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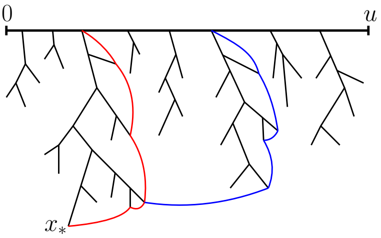

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In particular, geodesics to x_* are constructed in the same manner (going backward, or forward, in the forest in order to meet points with smaller and smaller label until reaching x_*):

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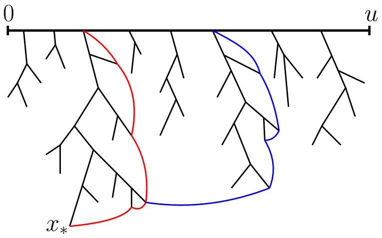

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In particular, geodesics to x_* are constructed in the same manner (going backward, or forward, in the forest in order to meet points with smaller and smaller label until reaching x_*):

$\longrightarrow$ The event that x_* is an ε-approximate geodesic star with m arms occurs if and only if in addition to the tree carrying x_* there are $m-1$ trees in the forest carrying vertices with label $< -r+\varepsilon$.

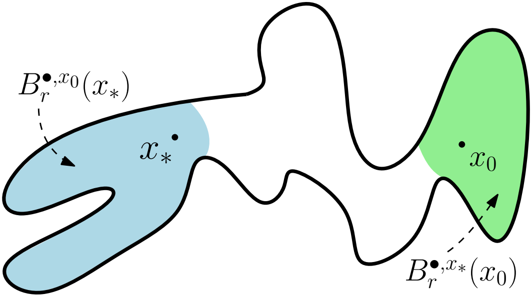
$\longrightarrow$ The probability of this event is $\approx \varepsilon^{m-1}$.

An ingredient for the two-point estimate

Recall that x_* and x_0 are the two distinguished points of $\mathbf{m}_\infty$ (distributed independently and uniformly).

Theorem

Let $r > 0$. Conditionally on the event $\{D(x_*, x_0) > 2r\}$, the hulls $B_r^{\bullet,x_0}(x_*)$ and $B_r^{\bullet,x_*}(x_0)$ viewed as (measure) metric spaces for their intrinsic distances are independent conditionally on their boundary sizes, and their conditional distribution can be described as before from a Poisson labeled forest.



This is a kind of spatial Markov property of the Brownian sphere (valid only for the free Brownian sphere!).