Thresholds

Jinyoung Park

Stanford University

University of Oxford
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New result

Conjecture [Kahn-Kalai '06]; proved by P.-Pham ('22).

There exists a universal $K > 0$ such that for every finite set X and increasing property $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq K p_{\mathbf{E}}(\mathcal{F}) \log |X|$$

- $p_c(\mathcal{F})$: *threshold* for $\mathcal{F}$
- $p_{\mathbf{E}}(\mathcal{F})$: *expectation threshold* for $\mathcal{F}$

Basic definitions

* K_n: the complete graph on n vertices

- X: finite set; $2^X = \{\text{subsets of } X\}$
- μ_p: p-biased product probability measure on 2^X

$$\mu_p(A) = p^{|A|}(1-p)^{|X \setminus A|} \quad A \subseteq X$$

- $X_p \sim \mu_p$ "p-**random**" subset of X

  e.g.1. $X = \binom{[n]}{2} = E(K_n)$

  $\rightarrow X_p = G_{n,p}$ Erdős-Rényi random graph

  e.g.2. $X = \{k\text{-clauses from } \{x_1, \dots, x_n\}\}$

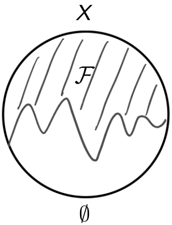
  $\rightarrow X_p$: random CNF formula

- $\mathcal{F} \subseteq 2^X$ is an **increasing property** if

$$B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$$

  e.g.1. $\mathcal{F} = \{\text{connected}\}$; $\mathcal{F} = \{\text{contain a triangle}\}$

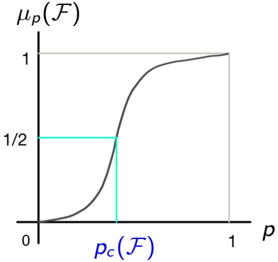
  e.g.2. $\mathcal{F} = \{\text{not satisfiable}\}$



Thresholds

Fact.

For any increasing property $\mathcal{F}$ $(\neq \emptyset, 2^X)$, $\mu_p(\mathcal{F})$ $(= \mathbb{P}(X_p \in \mathcal{F}))$ is continuous and strictly increasing in p.



- $p_c(\mathcal{F})$ is called **the threshold** for $\mathcal{F}$.

- cf. Erdős-Rényi: $p_0 = p_0(n)$ is **a threshold function** for $\mathcal{F}_n$ if

$$\mu_p(\mathcal{F}_n) \to \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases}$$

* $p_c(\mathcal{F}_n)$ is always an Erdős-Rényi threshold (Bollobás-Thomason '87).

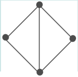
The Kahn–Kalai Conjecture

"It would probably be more sensible to conjecture that it is **not** *true."*

\- Kahn and Kalai (2006)

Question.

What drives $p_c(\mathcal{F})$?

Example 1. Containing a copy of H  $\asymp$: same order

- $X = \binom{[n]}{2}$ (so $X_p = G_{n,p}$); $\mathcal{F}_H$: contain a copy of H

Example 1.

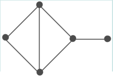
What's the threshold for $G_{n,p}$ to contain a copy of H?

- Usual suspect: expectation calculation

$$\mathbb{E}[\#\ H\text{'s in } G_{n,p}] \asymp n^4 p^5 \to \begin{cases} 0 & \text{if} \quad p \ll n^{-4/5} \\ \infty & \text{if} \quad p \gg n^{-4/5} \end{cases}$$

"threshold for $\mathbb{E}$" $\asymp n^{-4/5}$

- triv. $p_c(\mathcal{F}_H) \gtrsim n^{-4/5}$ $\quad (\because \mathbb{E}X \to 0 \Rightarrow X = 0$ with high probability$)$
- truth: $p_c(\mathcal{F}_H) \asymp n^{-4/5}$

Example 2. Containing a copy of K 

- $X = \binom{[n]}{2}$ (so $X_p = G_{n,p}$); $\mathcal{F}_K$: contain a copy of K

Example 2.

What's the threshold for $G_{n,p}$ to contain a copy of K?

$$\mathbb{E}[\#\ K\text{'s in } G_{n,p}] \asymp n^5 p^6 \to \begin{cases} 0 & \text{if} \quad p \ll n^{-5/6} \\ \infty & \text{if} \quad p \gg n^{-5/6} \end{cases}$$

"threshold for $\mathbb{E}$" $\asymp n^{-5/6}$

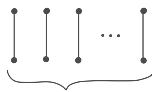
- Q. $p_c(\mathcal{F}_K) \asymp n^{-5/6}$? (triv. $p_c(\mathcal{F}_K) \gtrsim n^{-5/6}$)
- truth: $p_c(\mathcal{F}_K) \asymp n^{-4/5}$

Erdős-Rényi ('60), Bollobás ('81)

(Rough:) For **fixed** graph K,

$$p_c(\mathcal{F}_K) \asymp \text{"threshold for } \mathbb{E}\text{" of the "densest" subgraph of K}$$

Example 3. Containing a perfect matching



- $X = \binom{[n]}{2}$ (so $X_p = G_{n,p}$); $\mathcal{F}$: contain a perfect matching

Example 3.

What's the threshold for $G_{n,p}$ to contain a perfect matching? $(2|n)$

$$\mathbb{E}[\#\text{ Perfact matchings in } G_{n,p}] \approx \left(\frac{np}{e}\right)^{n/2} \to \begin{cases} 0 & \text{if} \quad p \ll 1/n \\ \infty & \text{if} \quad p \gg 1/n \end{cases}$$

"threshold for $\mathbb{E}$" $\asymp 1/n$

- Q. $p_c(\mathcal{F}) \asymp 1/n$? (triv. $p_c(\mathcal{F}) \gtrsim 1/n$)
- truth: $p_c(\mathcal{F}) \asymp \log n/n$

Fact. $p \ll \log n/n \Rightarrow G_{n,p}$ has an isolated vertex w.h.p.

One more example: perfect hypergraph matchings

- Now, $X = \binom{[n]}{r}$
- $X_p =$ random r-uniform hypergraph $\mathcal{H}^r_{n,p}$

Example 3$'$. (Shamir's Problem ('80s))

For $r \geq 3$, what's the threshold for $\mathcal{H}^r_{n,p}$ to contain a perfect matching? $(r|n)$

- cf. $r = 2$: Erdős-Rényi ('66) $\quad$ $r \geq 3$ much harder
- e.g. $r = 3$:
  - $\mathbb{E}[\#$ perfect matchings in $\mathcal{H}^r_{n,p}] \asymp (n^2 p)^{n/3} \to$ "threshold for $\mathbb{E}$" $\asymp n^{-2}$
  - Lower bound from coupon-collector:

$$p_c(\mathcal{F}) \gtrsim \log n / n^2$$

- $p_c(\mathcal{F}) \asymp \log n / n^2$ $\quad$ (Johansson-Kahn-Vu '08) $\quad$ * $\log n$ **gap again**

What drives $p_c(\mathcal{F})$?

- We have some **trivial lower bounds** on p_c:
  - Ex 1, 2 (contain H/K): "threshold for $\mathbb{E}$"
  - Ex 3, $3'$ (contain a PM): coupon collector-ish behavior (log n gap)
- Historically, in many interesting cases, the main task is to find **a matching upper bound**.

The Kahn-Kalai Conjecture ('06): rough statement

For any increasing property, the threshold is at most $\log |X|$ times the "expectation threshold".

- This is a VERY strong conjecture: immediately implies (e.g.)
  - threshold for perfect hypergraph matchings (Johansson-Kahn-Vu '08)

$$p_{\mathbb{E}} \asymp n^{-(r-1)} \xRightarrow{\text{KKC}} p_c \lesssim \log n / n^{r-1}$$

  - threshold for bounded degree spanning trees ("tree conjecture"; Montgomery '19)

$p_{\mathsf{E}}(\mathcal{F})$: the expectation threshold

- For abstract $\mathcal{F}$, it's unclear whose expectation we want to compute, so need a careful definition for the "threshold for $\mathbb{E}$."

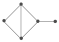
$p_{\mathrm{E}}(\mathcal{F})$: the expectation threshold

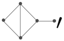
Observation

$p_c(\mathcal{F}) \geq q$ if $\exists\, \mathcal{G} \subseteq 2^X$ such that

1. "$\mathcal{G}$ covers $\mathcal{F}$": $\forall A \in \mathcal{F}\ \ \exists B \in \mathcal{G}$ such that $A \supseteq B$ $\quad (\mathcal{F} \subseteq \langle \mathcal{G} \rangle)$
2. $\sum_{S \in \mathcal{G}} q^{|S|} \leq \frac{1}{2}$ $\quad$ ("q-cheap")

the *upset* *generated by* $\mathcal{G}$

e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of K 

- $\mathcal{G}_1 = \{$all (labeled) copies of K  $'s\}$

  $\rightarrow \sum_{S \in \mathcal{G}_1} q^{|S|} \leq 1/2$ for $q \lesssim n^{-5/6}$ $\quad \rightarrow n^{-5/6} \lesssim p_c(\mathcal{F})$

- $\mathcal{G}_2 = \{$all (labeled) copies of H  $'s\}$

  $\rightarrow \sum_{S \in \mathcal{G}_2} q^{|S|} \leq 1/2$ for $q \lesssim n^{-4/5}$ $\quad \rightarrow n^{-4/5} \lesssim p_c(\mathcal{F})$

$p_{\mathsf{E}}(\mathcal{F})$: the expectation threshold

Observation

$p_c(\mathcal{F}) \geq q$ if $\exists\, \mathcal{G} \subseteq 2^X$ such that

1. "$\mathcal{G}$ covers $\mathcal{F}$": $\forall A \in \mathcal{F} \;\; \exists B \in \mathcal{G}$ such that $A \supseteq B \qquad (\mathcal{F} \subseteq \langle \mathcal{G} \rangle)$
2. $\sum_{S \in \mathcal{G}} q^{|S|} \leq \frac{1}{2}$

the *upset generated by* $\mathcal{G}$

- $p_{\mathsf{E}}(\mathcal{F}) := \max\{q : \exists\, \mathcal{G}\} \quad \to$ a trivial lower bound on $p_c(\mathcal{F})$

The Kahn-Kalai Conjecture ('06)

There exists a universal $K > 0$ such that for every finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$(p_{\mathsf{E}}(\mathcal{F}) \leq)\; p_c(\mathcal{F}) \leq K p_{\mathsf{E}}(\mathcal{F}) \log |X|$$

Results and Proof Sketch

Conj of Talagrand: fractional version of Kahn-Kalai Conj

- $p^*_{\mathsf{E}}(\mathcal{F})$: the **fractional expectation threshold** for $\mathcal{F}$
  - skip def: roughly, replace cover $\mathcal{G}$ by "fractional cover"
- Easy. $p_{\mathsf{E}}(\mathcal{F}) \leq p^*_{\mathsf{E}}(\mathcal{F}) \leq p_c(\mathcal{F})$

Conj (Talagrand '10); proved by Frankston-Kahn-Narayanan-P. ('19).

There exists a universal $K > 0$ such that for every finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq K p^*_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F}).$$

* $\ell(\mathcal{F})$: the size of a largest minimal element of $\mathcal{F}$

- Weaker than KKC, but in all known applications, $p_{\mathsf{E}}(\mathcal{F}) \asymp p^*_{\mathsf{E}}(\mathcal{F})$
- Proof inspired by Alweiss-Lovett-Wu-Zhang

"Erdős-Rado Sunflower Conjecture"

$p_{\mathsf{E}}(\mathcal{F})$ vs. $p^*_{\mathsf{E}}(\mathcal{F})$

FKNP (19') $p_c(\mathcal{F}) \le K p^*_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$

- Recall. In all known applications, $p_{\mathsf{E}}(\mathcal{F}) \asymp p^*_{\mathsf{E}}(\mathcal{F})$

Conjecture (Talagrand '10) $p_{\mathsf{E}}(\mathcal{F}) \asymp p^*_{\mathsf{E}}(\mathcal{F})$

There exists a universal K such that for every finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$(p_{\mathsf{E}}(\mathcal{F}) \le)\ p^*_{\mathsf{E}}(\mathcal{F}) \le K p_{\mathsf{E}}(\mathcal{F})$$

- Implies equivalence of KKC and fractional KKC
  – the most likely way to prove KKC?
- Even simple instances of the conjecture are not easy to establish; Talagrand suggested two test cases, proved by (respectively) DeMarco-Kahn ('15) and Frankston-Kahn-P. ('21)

New result

Conjecture (Kahn-Kalai '06); proved by P.-Pham ('22)

There exists a universal $K > 0$ such that for every finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq K p_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$$

* $\ell(\mathcal{F})$: the size of a largest minimal element of $\mathcal{F}$

- Proofs inspired by ALWZ (sunflower) and FKNP (fractional Kahn-Kalai) but implementation very different
- Reformulation – think: $\mathcal{H} = \{\text{minimal elements of } \mathcal{F}\}$

Theorem (P.-Pham '22)

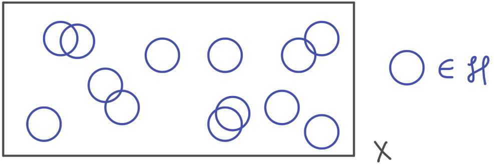
$\exists L > 0$ such that $\forall \ell$-bdd $\mathcal{H}$, if $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$, then, with $m = Lp \log \ell |X|$,

$$\mathbb{P}(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1)$$

Proof sketch

$\exists L > 0$ such that $\forall \ell$-bdd $\mathcal{H}$, if $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$, then, with $m = Lp \log \ell |X|$,

$$\mathbb{P}(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1)$$



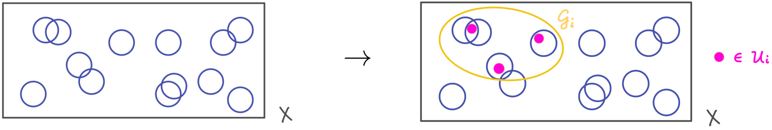
- Choose $W (= X_m)$ little by little: $W = W_1 \sqcup W_2 \sqcup \ldots$
- At the end, want $W \supseteq S \in \mathcal{H}$ whp.
- Run algorithm: no assump $\rightarrow$ two possible outputs
- (Recall) $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$ means:

$\langle \mathcal{H} \rangle$ does not admit a p-cheap cover.

$\exists L > 0$ such that $\forall \ell$-bdd $\mathcal{H}$, if $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$, then, with $m = Lp \log \ell |X|$,

$$\mathbb{P}(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1)$$

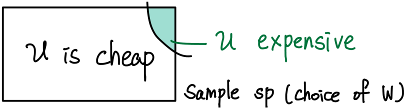
- $W = W_1 \sqcup W_2 \sqcup \ldots$
- At ith step: choose W_i of size $\approx Lp|X|$ at random
  $\rightarrow$ **Construct** cover $\mathcal{U}_i = \mathcal{U}_i(W_i)$ of some $\mathcal{G}_i = \mathcal{G}_i(W_i) \subseteq \mathcal{H}_{i-1}$



- When **terminates**, with $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots$ ("partial cover") either

  (1) $\mathcal{U}$ covers $\mathcal{H}$; or (2) $W \in \langle \mathcal{H} \rangle$

- [Main Point] Typically, $\mathcal{U}$ is "p-cheap."



$\xrightarrow{\text{Ass. } p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)}$ (1) is unlikely $\square$

Open Questions

Gap between $p_{\mathsf{E}}(\mathcal{F})$ and $p_c(\mathcal{F})$

Theorem (P.-Pham '22)

$$(p_{\mathsf{E}}(\mathcal{F}) \leq)\ p_c(\mathcal{F}) \lesssim p_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$$

Question

What characterizes the gap between $p_{\mathsf{E}}(\mathcal{F})$ and $p_c(\mathcal{F})$?

- In many cases the $\log \ell(\mathcal{F})$ gap is tight:

  e.g. perfect hypergraph matchings, spanning trees with bounded degree, Hamiltonian cycle, fixed subgraphs...

- There are some cases for which $\log \ell(\mathcal{F})$ is not tight:

  e.g. clique factors, the k-th power of a Hamilton cycle, non-spanning large graphs... $\rightarrow$ good test cases!

Test cases: gaps smaller than $\log \ell(\mathcal{F})$ Thm. $p_c(\mathcal{F}) \le K p_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$

First successful test case

$\mathcal{F}$: contain **the square of a Hamilton cycle (HC^2)**

Conjecture (Kühn-Osthus '12)

$$p_c(\mathcal{F}) \asymp n^{-1/2}$$

- $p_{\mathsf{E}}(\mathcal{F})(\asymp p^*_{\mathsf{E}}(\mathcal{F})) \asymp n^{-1/2}$ $\quad \rightarrow$ no gap!
  - Kühn-Osthus ('12) $p^* \lesssim n^{-1/2+o(1)}$
  - Nenadov-Škorić ('16) $p^* \lesssim n^{-1/2} \log^4 n$
  - Fischer-Škorić-Steger-Trujić ('18) $p^* \lesssim n^{-1/2} \log^3 n$
  - Montgomery $p^* \lesssim n^{-1/2} \log^2 n$
  - Frankston-Kahn-Narayanan-P. $p^* \lesssim n^{-1/2} \log n$

Kahn-Narayanan-P. ('20)

$$p_c(\mathcal{F}) \asymp n^{-1/2}$$

Good test cases: gaps smaller than $\log \ell(\mathcal{F})$

[Ex 1] $\mathcal{F}$: contain a **triangle-factor** (or a H-factor for fixed H)

Johansson-Kahn-Vu ('08)

$$p_c(\mathcal{F}) \asymp n^{-2/3}(\log n)^{1/3}$$

[Ex 2] Perfect matchings in the "k-out model"

Frieze ('86)

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbb{P}(G_{k\text{-out}} \text{ has a perfect matching}) = \begin{cases} 0 & \text{if } k = 1 \\ 1 & \text{if } k \geq 2 \end{cases}$$

Thank you!