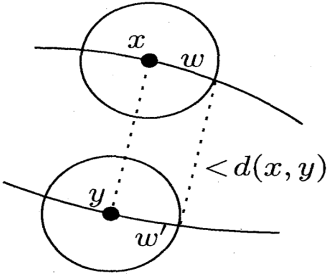
Sparse expanders have negative curvature

Justin Salez



Université Paris-Dauphine & PSL

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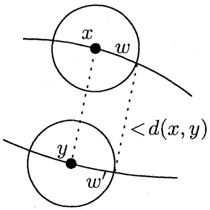

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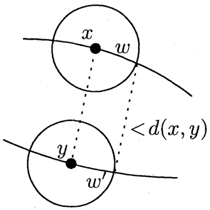

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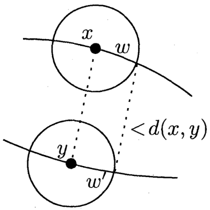

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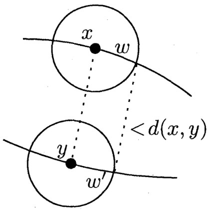

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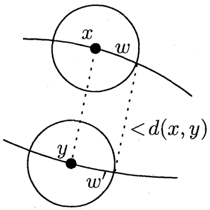

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Question: can expanders have non-negative curvature?

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- P is the transition matrix of lazy simple random walk:

$$P(x,y) \;:=\; \begin{cases} \frac{1}{2\deg(x)} & \text{if } \{x,y\} \in E; \\ \frac{1}{2} & \text{if } x = y; \\ 0 & \text{else.} \end{cases}$$

Online Graph Curvature Calculator (Stagg-Cushing)

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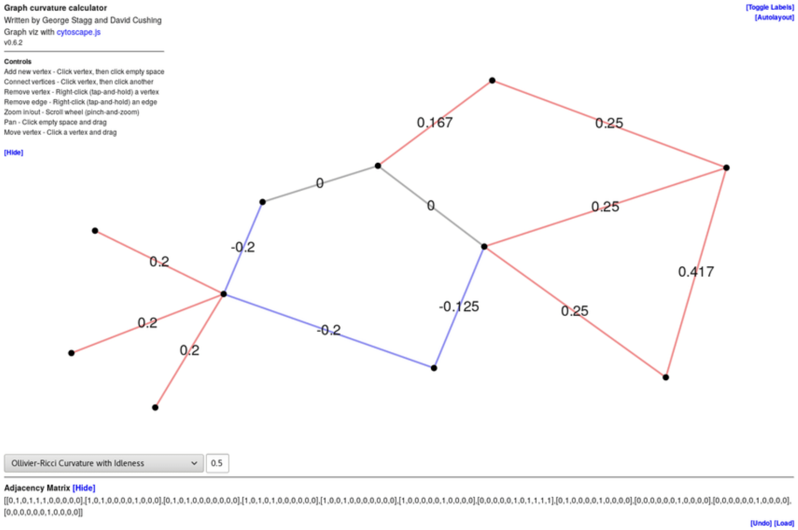

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- Remarkable consequences on geometry and concentration (Ollivier'09, Joulin-Ollivier'10, Lin-Lu-Yau'11, Eldan-Lee-Lehec'17, Jost-Münch-Rose '19, Münch'19, Cushing-Kamtue-Koolen-Liu-Münch-Peyerimhoff'20).

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- Intimately related to the cutoff phenomenon (S.'21)

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Far-reaching applications... (Hoory-Linial-Wigderson'06)

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$\triangleright$ Sparse graphs either have a macroscopic fraction of edges with negative curvature or a macroscopic fraction of eigenvalues near 1

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- curvature and expansion (this talk !)

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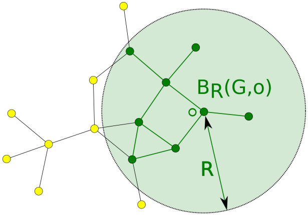

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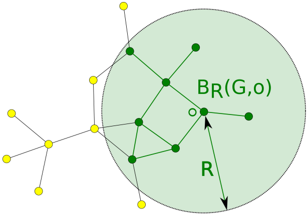

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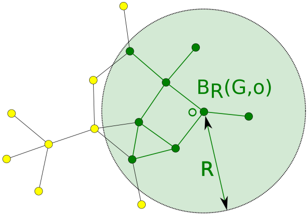

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▷ $\mathcal{G}_\bullet := \{\text{loc. finite, connected rooted graphs}\}$ is a Polish space.

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Intuition: $(\mathbb{G}, o)$ describes how G_n looks from a random vertex.

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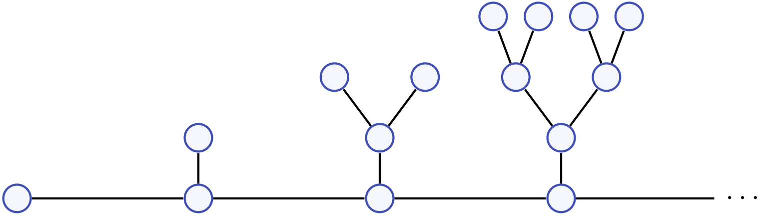

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Theorem (Benjamini-Lyons-Schramm'15) For a sequence (G_n) to admit subsequential limits, it is enough that it satisfies

$$\sup_{n\geq 1}\left\{\frac{1}{|V_n|}\sum_{x\in V_n}\deg(x)1_{\deg(x)>\Delta}\right\} \xrightarrow[\Delta\to\infty]{} 0.$$

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Theorem (S.'21). No limit $(\mathbb{G}, o)$ can satisfy these 3 properties.

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Warning: false without unimodularity... (Benjamini-Kozma'10)

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- What about mixing times, or functional-analytic constants?

Thanks!