Hypergraph Matchings Avoiding Forbidden Submatchings

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Oxford Discrete Mathematics and Probability Seminar

joint work with Luke Postle

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Part I

Avoiding Submatchings

General Question and Key Definition

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*A matching of G is H**-avoiding** if it spans no edge of H.*

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Under what conditions does $L(G) \cup H$ have independence number almost the minimum of the independence numbers of H and $L(G)$?

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More generally, when can we almost decompose the edges of G into large H-avoiding matchings?

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Under what conditions does $L(G) \cup H$ have chromatic number almost the maximum of the chromatic numbers of H and $L(G)$?

Definitions

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*The **codegree** of vertices $u, v \in V(G)$ is the number of edges containing both.*

Definition

*The **girth** of a hypergraph H is the smallest integer $g \geq 2$ for which H has a g-Berge cycle.*

Definition

*The i**-degree** $d_i(v)$ of a vertex $v \in V(H)$ is the number of edges of H of size i containing v.*

Main Result

Theorem (D. and Postle 2022+)

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Recall that χ_ℓ denotes the list chromatic number (generalizing chromatic number).

Part II

Combining Two Streams

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This asymptotically proved the List Coloring Conjecture.

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If H is an r-uniform hypergraph on n vertices of girth at least five and maximum degree Δ, then

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- plus mixed uniformity, Li-Postle 2022+

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Key Ideas:

- for H track a "weighted degree" because H has a mix of uniformities
- nibble calculations for G and nibble calculations for H interweave perfectly
- introduce a new linear Talagrand's Inequality to concentrate and use Lovász Local Lemma to finish

Part III

Application: Steiner Systems

Steiner Systems

Steiner Systems

Definition

For $n \geq q > r \geq 2$, a ***partial*** (n, q, r)***-Steiner system*** *is a set S of q-subsets of an n-set V s.t. every r-subset of V is contained in at most one q-set in S.*

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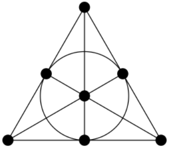

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Existence Conjecture

For sufficiently large n, there exists an (n, q, r)*-Steiner system whenever n is admissible.*

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- $q = 3$ and $r = 2$, Kirkman 1847
- $r = 2$, Wilson 1975
- approximate version - "nibble method", Rödl 1985
- full conjecture - algebraic techniques, Keevash 2014+
- full conjecture - combinatorial techniques, Kühn, Lo, Glock, and Osthus 2016+

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An $(n, 3, 2)$-Steiner system S contains an $(i + 3, i)$-configuration for every fixed i.

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Conjecture (Erdős 1973)

For every integer $g \geq 2$, there exists n_g such that for all admissible $n \geq n_g$, there exists an $(n, 3, 2)$-Steiner system with no $(i + 2, i)$-configuration for all $2 \leq i \leq g$.

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Observation (Glock, Kühn, Lo, and Osthus 2020)

Every (n, q, r)-Steiner system contains an $(i(q - r) + r + 1, i)$-configuration for every fixed i.

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A More General Conjecture

Conjecture (Glock, Kühn, Lo, and Osthus 2020, Keevash and Long 2020)

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Two distinct vertices lie in at most $\binom{n-r-1}{q-r-1} = o\left(\binom{n-r}{q-r}\right) = o(D)$ common edges.

Question

What do the forbidden configurations become?

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A vertex v in H is an edge in G and a q-set of $[n]$.

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Recall we forbid $(i(q-r)+r, i)$-configurations so there are $i(q-r)+r-q = (i-1)(q-r)$ other vertices to choose from.

Thus v is in $O(n^{(i-1)(q-r)}) = O\left(\binom{n-r}{q-r}^{i-1}\right) = O\left(D^{i-1}\right)$ edges of size i.

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This not only implies the theorem but an almost decomposition into approximate high girth Steiner systems!

Part IV

Matchings in Bipartite Hypergraphs

Bipartite Hypergraphs

Definition

*A hypergraph $G = (A, B)$ is **bipartite with parts** A **and** B if $V(G) = A \cup B$ and every edge of G contains exactly one vertex from A.*

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An A_L-perfect matching of G_L is equivalent to an L-coloring of $E(G)$.

Part V

Application: Latin Squares

Latin Squares

Definition

*A **Latin square** is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.*

1	2	3	4	5
2	4	5	1	3
3	5	4	2	1
4	3	1	5	2
5	1	2	3	4

Latin Squares

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For all n does there exist an $n \times n$ Latin square with no intercalate (a 2×2 sub-Latin square)?

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Theorem (D. and Postle 2022+)

Approximate high girth Latin squares exist.

Theorem (D. and Postle 2022+)

Approximate high girth permutations exist.

Part VI

Application: Rainbow Matchings

Rainbow Matchings

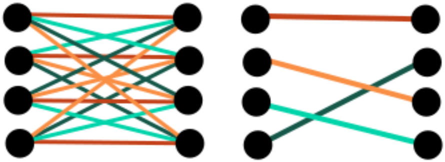
Rainbow Matchings

Definition

A matching M of a (not necessarily properly) edge colored hypergraph G is ***rainbow*** *if every edge of M is colored differently.*

Definition

A rainbow matching is ***full*** *if every color of the coloring appears on some edge of M.*



Aharoni-Berger Conjecture

A typical example of a rainbow matching conjecture:

Conjecture (Aharoni and Berger 2009)

If G is a bipartite multigraph properly edge colored with q colors where every color appears at least $q + 1$ times, then there exists a full rainbow matching.

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A version for non-bipartite graphs:

Conjecture

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Note: 2. and 3. imply 1.

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  number of times a color appears is related to degree of the graph

Sparse Versions of the Aharoni-Berger Conjecture

Sparse setting versions of the previous conjectures:

Conjecture

If G is a bipartite multigraph properly edge colored where every color appears at least $\Delta(G) + 1$ times, then there exists a full rainbow matching.

Conjecture

If G is a multigraph properly edge colored where every color appears at least $\Delta(G) + 2$ times, then there exists a full rainbow matching.

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Indeed there is even a set of D disjoint full rainbow matchings of G.

Alspach's Conjecture

Conjecture (Alspach 1988)

If G is a $2d$-regular graph that is edge colored such that each color class is a spanning subgraph of G in which all vertices have degree two, then G has a full rainbow matching.

- strong asymptotic version, Munhá Correia, Pokrovskiy, and Sudakov 2021
- strong asymptotic version in the sparse setting, D. and Postle 2022+

Grinblat's Conjecture

Originally motivated by equivalence classes in algebras:

Conjecture (Grinblat 2002)

If G is a multigraph that is (not necessarily properly) edge colored with n colors where each color class is the disjoint union of non-trivial complete subgraphs and spans at least $3n-2$ vertices, then G has a rainbow matching of size n.

- strong asymptotic version, Clemens, Ehrenmüller, and Pokrovskiy 2017
- full proof, Munhá Correia and Sudakov 2021
- bounded multiplicity graphs $2n+o(n)$ vertices, Munhá Correia and Yepremyan
- bounded multiplicity strong asymptotic version for hypergraphs in the sparse setting, D. and Postle 2022+

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- **Pippenger's Theorem**
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We derive high girth versions of settings where Rödl's nibble yields approximate decompositions.

Some notable applications include:

- high girth Steiner systems,
- edge coloring and hypergraph coloring,
- rainbow matchings, and
- Latin squares and high dimensional permutations

Conclusion

Thank you for listening!