Free boundary dimers: random walk representation and scaling limit

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(joint work with Marcin Lis and Wei Qian)

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Outline

1) The *dimer model*

- height function
- *Dirichlet Gaussian free field* as scaling limit (Kenyon, ...)
- Kasteleyn theory and random walk representation

2) The *dimer model with free boundary* (or *monomer-dimer model*)

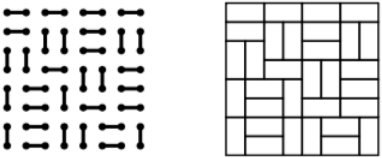
- height function
- *Neumann Gaussian free field* as scaling limit (in the upper half plane) (Berestycki–L.–Qian)
- Kasteleyn theory and random walk representation

1) The dimer model

The dimer model

Let G be a finite, planar, bipartite graph.

A *dimer cover* (or *perfect matching*): a set of edges (=dimers), such that each vertex is incident to exactly one dimer.



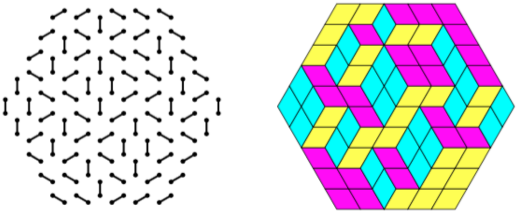
The *dimer model* with edge weights w_e:

$$\mathbb{P}(\mathbf{m}) = \frac{1}{Z} \prod_{e \in \mathbf{m}} w_e.$$

Typically $w_e \equiv 1$.

The dimer model as a random surface

Honeycomb lattice: *lozenge tiling* or a stack of cubes



©Kenyon

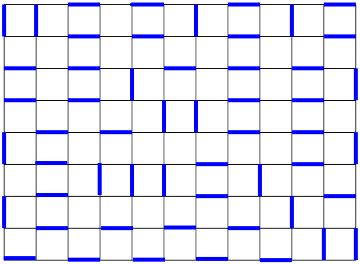
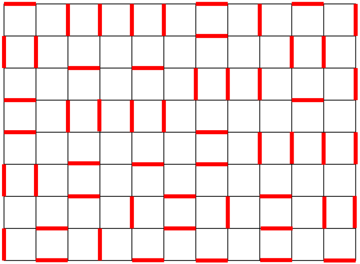
Height function

Introduced by Thurston. Hence view as a random surface.

Note: depends on the choice of a reference frame.

Height function

Take two dimer covers...

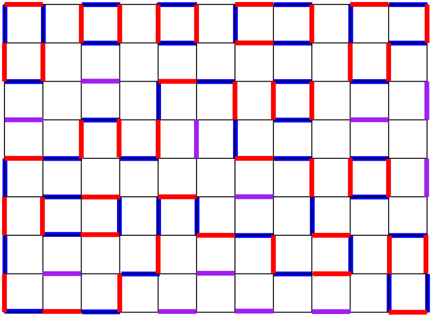
 

one fixed and one random $\Longrightarrow$ *single dimer model*

both random and independent $\Longrightarrow$ *double dimer model*

Height function

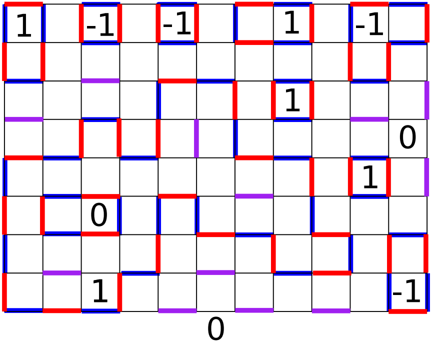
... and superimpose them...



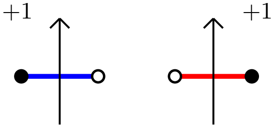
loops and *doubled edges*

Height function

... and interpret the loops as level lines of the height function.

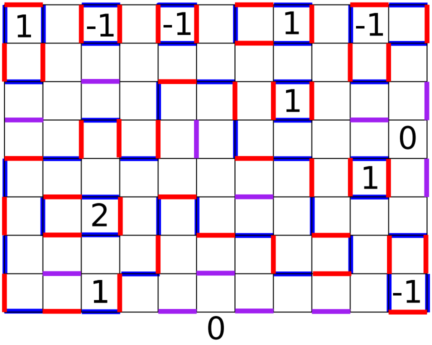


rule:

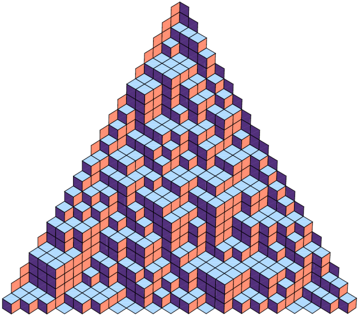


Height function

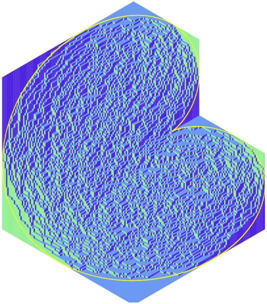
... and interpret the loops as level lines of the height function.



Large scale behaviour?



Kenyon



Kenyon–Okounkov–Sheffield 2006

The effect of boundary conditions is, however, not entirely trivial and will be discussed in more detail in a subsequent paper.

P. W. Kasteleyn, 1961

The Dirichlet Gaussian free field

Let $\mathcal{D}$ be a domain in the plane with boundary and let

$$G_{\mathcal{D}}(x,y) = \int_0^\infty p_t(x,y)dt$$

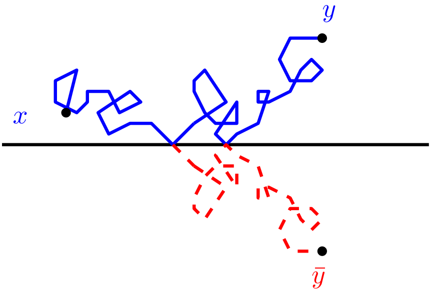
be the *Green function* of Brownian motion killed upon hitting $\partial\mathcal{D}$.

The *Gaussian free field with Dirichlet b.c.* $\Phi_{\mathcal{D}}^{\mathrm{Dir}}$ is a random *distribution* satisfying

- $\Phi_{\mathcal{D}}^{\mathrm{Dir}}(f)$ is a mean-zero Gaussian for all test functions f,
- $\mathbf{E}[\Phi_{\mathcal{D}}^{\mathrm{Dir}}(f)\Phi_{\mathcal{D}}^{\mathrm{Dir}}(g)] = \iint_{\mathcal{D}^2} f(x)g(y)G_{\mathcal{D}}^{\mathrm{Dir}}(x,y)dxdy$ for all f, g.

The value at a point does not make sense!

The Dirichlet GFF in the half-plane $\mathbb{H}$



We have

$$p_t(x,y) = p_t^{\mathbb{C}}(x,y) - p_t^{\mathbb{C}}(x,\bar{y})$$

and hence

$$G_{\mathbb{H}}^{\mathrm{Dir}}(x,y) = -\frac{1}{2\pi}\log|x-y| + \frac{1}{2\pi}\log|x-\bar{y}|$$
$$= \frac{1}{2\pi}\log\left|\frac{x-\bar{y}}{x-y}\right|.$$

The dimer model h.f. and the Dirichlet GFF

Theorem (Kenyon '99)

Let $\mathcal{D} \subset \mathbb{C}$ bounded domain, $\mathcal{D}^\delta = \mathcal{D} \cap \delta\mathbb{Z}^2$ with *Temperleyan* boundary conditions. Let h^δ be the associated height function. Then,

$$h^\delta - \mathbb{E}(h^\delta) \to \frac{1}{\sqrt{\pi}} \Phi_{\mathcal{D}}^{\mathrm{Dir}} \qquad \text{as} \qquad \delta \to 0,$$

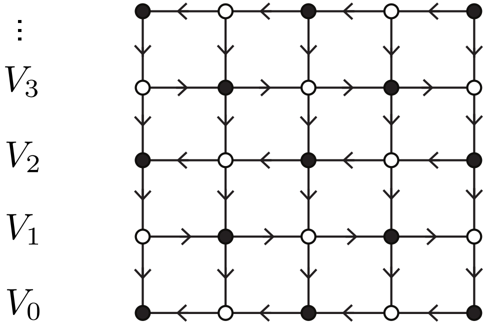
in distribution.

Main ingredients of the proof:

- Kasteleyn theory (exact solvability)
- Analysis of boundary conditions
- Computation of moments

Kasteleyn theory

A *Kasteleyn orientation*: orientation of edges, such that around every face, *odd number* of clockwise arrows.

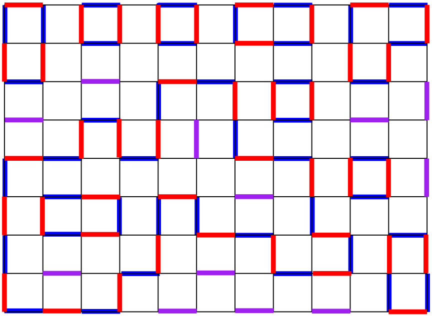


Kasteleyn theory

Kasteleyn matrix K: adjacency matrix, with Kasteleyn signs (antisymmetric).

Theorem (Temperley–Fisher, ’61& Kasteleyn, ’61)

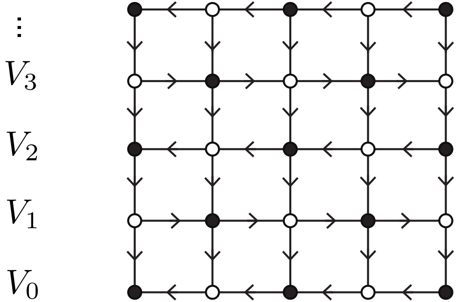
We have $Z = |\mathrm{Pf}(K)|$ and therefore the probabilities $\mathbf{P}(e_1, \ldots, e_n \in \mathbf{m})$ are given by $\mathrm{Pf}(K^{-1})$.



Consequence: we need to understand K^{-1}!

Random walk representation (Kenyon)

We restrict our attention to $\mathcal{D} = \mathbb{H}$.



Let $\mathcal{L} = K^*K$. Then $\mathcal{L}$ nonzero only from $W \to W$, $B \to B$.

Diagonal contributions vanish so really $W_0 \to W_0, \ldots, B_1 \to B_1$.

Then $\mathcal{L} =$ discrete Laplacian on each four sublattices, with appropriate boundary conditions.

Random walk representation (Kenyon)

From the definition $\mathcal{L} = K^* K$ we get

$$K^{-1} = \mathcal{L}^{-1} K^*$$

Moreover

- $\mathcal{L}^{-1}$ is the *Green function*
- K^* is the *discrete derivative*

We can understand K^{-1}, and hence Kasteleyn theory leads to the scaling limit for n-point correlation function!

2) Free boundary dimer model

aka: dimer model with boundary monomers,
or boundary monomer-dimer model

Free boundary dimer model

Let $\partial_m \subseteq V$ be a fixed part of the boundary of G.

A *monomer-dimer cover* $\mathbf{m}$: a set of vertices from ∂_m (called *monomers*) and edges, such that each vertex covered by exactly one monomer or dimer.

The *(boundary) monomer-dimer model* for vertex weights z_v and edge weights w_e:

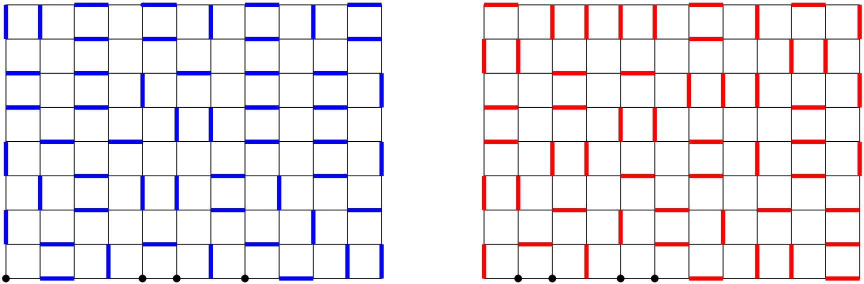
$$\mathbb{P}(\mathbf{m}) = \frac{1}{Z} \prod_{v \in \mathbf{m}} z_v \prod_{e \in \mathbf{m}} w_e.$$

We will consider the uniform model $w_e \equiv 1$, $z_v \equiv z > 0$.

(The monomers can be considered as impurities in the medium)

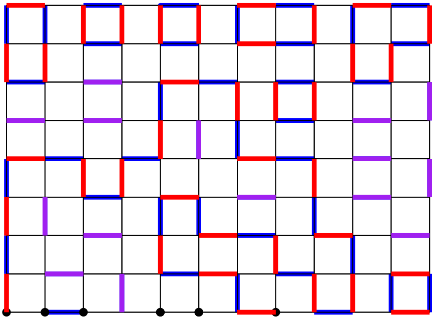
Double free boundary dimers

Take two free boundary dimer covers...



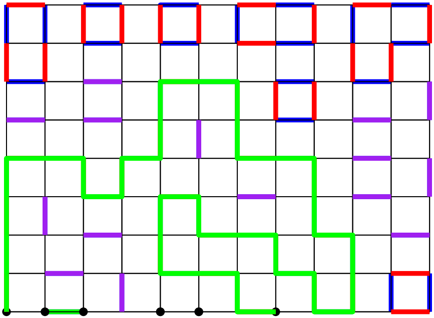
Double free boundary dimers

... and superimpose them...



Double free boundary dimers

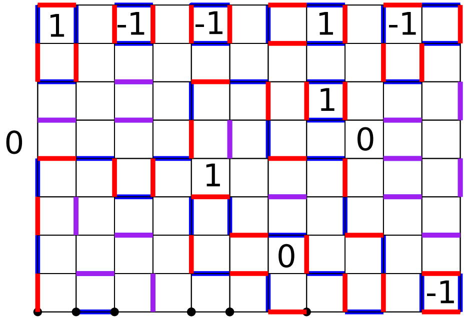
... and superimpose them...



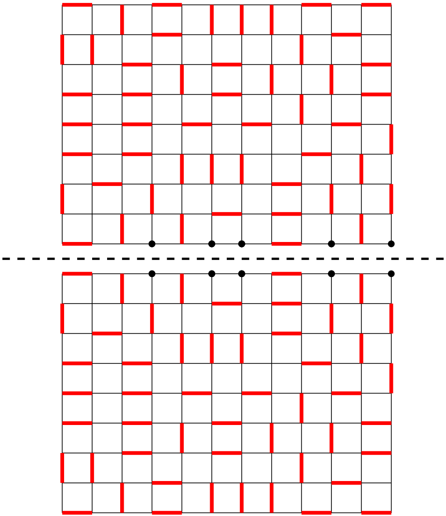
Not only loops but also *arcs* connecting ∂_m to ∂_m.
(We have a conjecture for scaling limit of arcs).

Double free boundary dimers

... and interpret the arcs and loops as level lines of the height function.



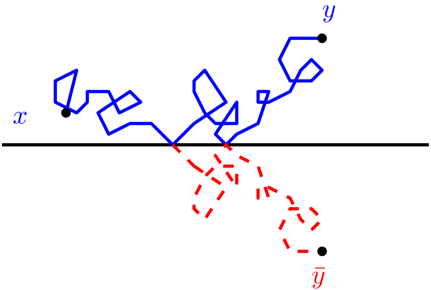
Guessing the scaling limit



The monomer-dimer h.f. is the dimer h.f. conditioned to be *even*.

The Neumann GFF in $\mathbb{H}$

The Neumann GFF is the *even part* of the Dirichlet GFF (up to a constant).



$$p_t(x,y) = p_t^{\mathbb{C}}(x,y) + p_t^{\mathbb{C}}(x,\bar{y})$$

So

$$\begin{aligned} G_{\mathbb{H}}^{\mathrm{Neu}}(x,y) &= -\frac{1}{2\pi}\log|x-y| - \frac{1}{2\pi}\log|x-\bar{y}| \\ &= -\frac{1}{2\pi}\log|(x-\bar{y})(x-y)|\,. \end{aligned}$$

Main result

Theorem (B.–Lis–Qian, ’21)

- The infinite volume limit of the monomer-dimer model $\mathcal{D}_n \uparrow \mathbb{H}$ exists.
- In the scaling limit, for *any* $z > 0$, we have

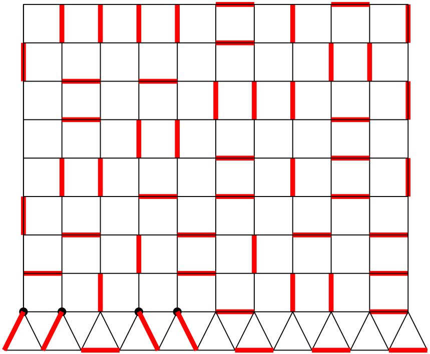
$$h^\delta - \mathbb{E}(h^\delta) \to \frac{1}{\sqrt{\pi}} \Phi_{\mathbb{H}}^{\text{Neu}} \qquad \text{as} \qquad \delta \to 0,$$

Note: first result where the limit doesn’t have Dirichlet b.c.

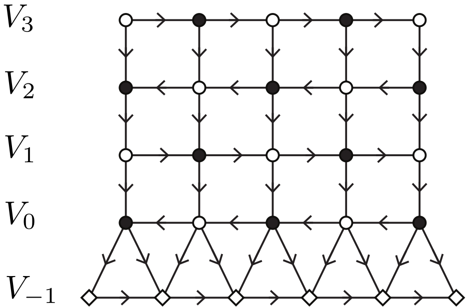
Sketch of proof of main result

Lemma (Giuliani–Jauslin–Lieb, ’15)

There is a bijection between the monomer-dimer model and a non-bipartite dimer model.

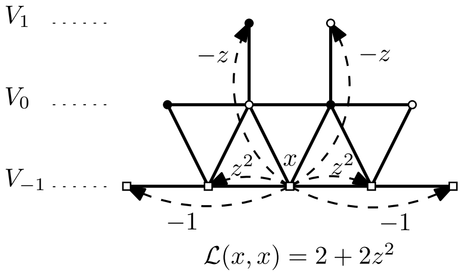


Kasteleyn orientation



Random walk representation??

Analysis breaks down near the boundary:



and

$$-\sum_{y\sim x} \mathcal{L}(x,y) = 2 - 2z^2 + 2z.$$

Let $P(x,y) = -\mathcal{L}(x,y)/\mathcal{L}(x,x)$: can be signed, do not sum to 1...

Question

Make sense of Green function?

Random walk representation!!

Proposition (Berestycki–L.–Qian, '21)

Grouping walks by their excursions to the boundary: an effective random walk with no killing.
Jump probabilities on the boundary have an exponential tail.

Proof: It is a miracle! From a computation we obtain:

- weights are positive
- weights sum up to one.

Open question

Find a conceptual proof...

Towards scaling limit

Notice that $\mathcal{L}^{-1}$ not restricted to $B \to B, W \to W$:
However paths must go through boundary!

E.g.: $e = (w,b); e' = (w',b')$

$$\mathbb{P}(e,e' \in \mathbf{m}) = \mathrm{Pf}\begin{pmatrix} 0 & K^{-1}(w,b) & K^{-1}(w,w') & K^{-1}(w,b') \\ & 0 & K^{-1}(b,w') & K^{-1}(b,b') \\ & & 0 & K^{-1}(w',b') \\ & & & 0 \end{pmatrix}$$

$$= \overbrace{K^{-1}(w,b)}^{\mathbb{P}(e\in\mathbf{m})}\overbrace{K^{-1}(w',b')}^{\mathbb{P}(e'\in\mathbf{m})} + K^{-1}(b,w')K^{-1}(w,b') - K^{-1}(w,w')K^{-1}(b,b')$$

so

$$\mathrm{Cov}(1_{e\in\mathbf{m}}; 1_{e'\in\mathbf{m}}) = K^{-1}(b,w')K^{-1}(w,b') - K^{-1}(w,w')K^{-1}(b,b')$$

Leads to scaling limit eventually...! □

Thank you!

Double dimer model

Conjecture (Kenyon '10)

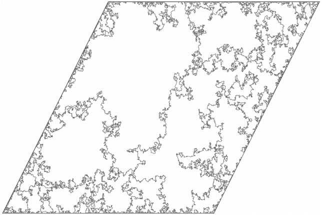
The scaling limit of the loops in the double dimer model should be the CLE_4 process introduced by Sheffield and Werner.

- the scaling limit can only be CLE_4 (Kenyon '10, Dubédat '14, Basok–Chelkak '18)
- tightness of loops is missing!
- analogous picture in the continuum: CLE_4 are the level lines of GFF (Miller–Sheffield).

Double monomer-boundary dimer model

Cojecture (Berestycki-L.-Qian)

The scaling limit of the arcs in the double free boundary dimer should be the ALE process introduced by Aru, Sepulveda and Werner.



- analogous picture in the continuum: ALE is boundary touching level lines of Neumann GFF (Qian–Werner, '18)

Thank you!