Geodesic Geometry on Graphs

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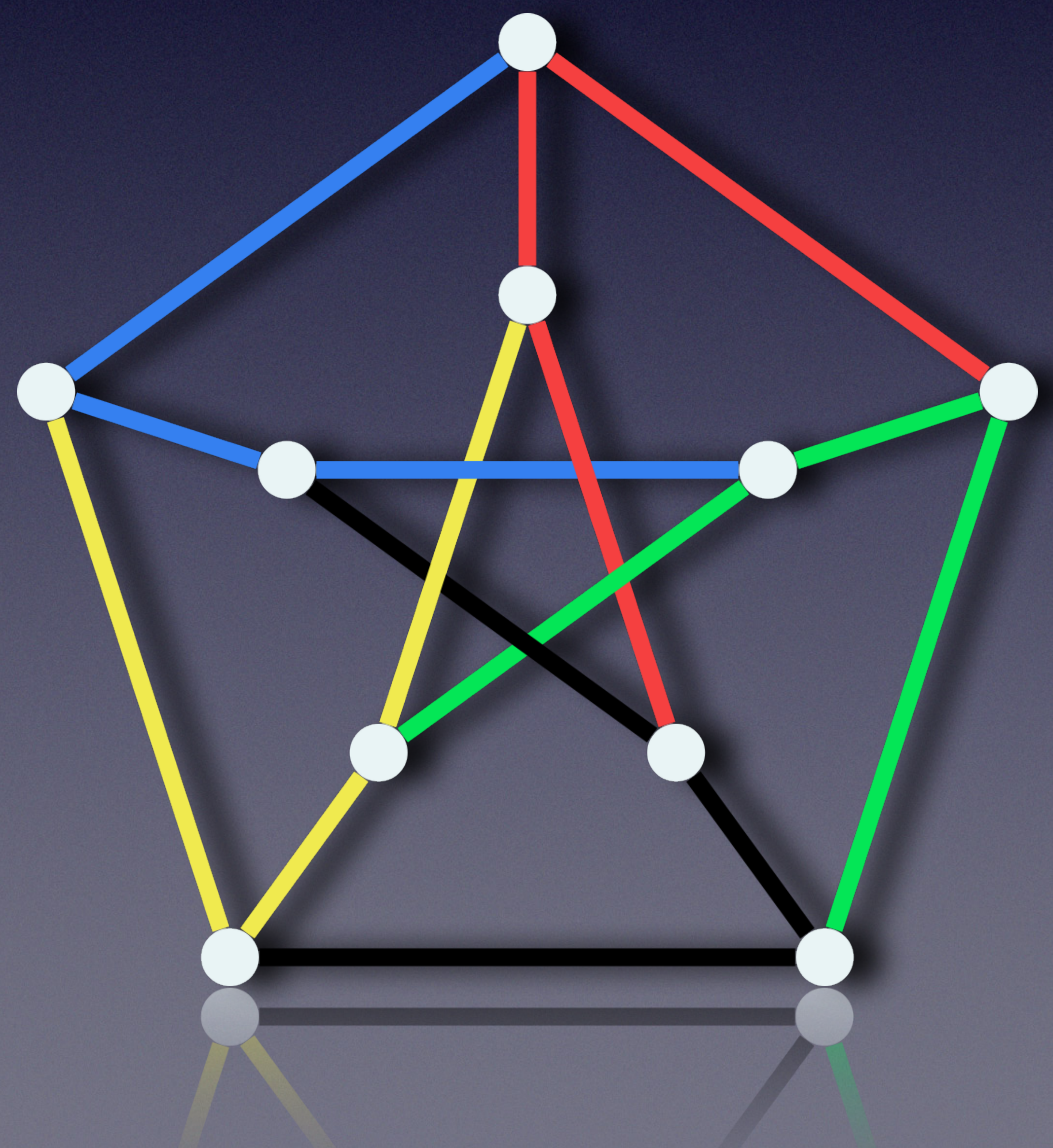
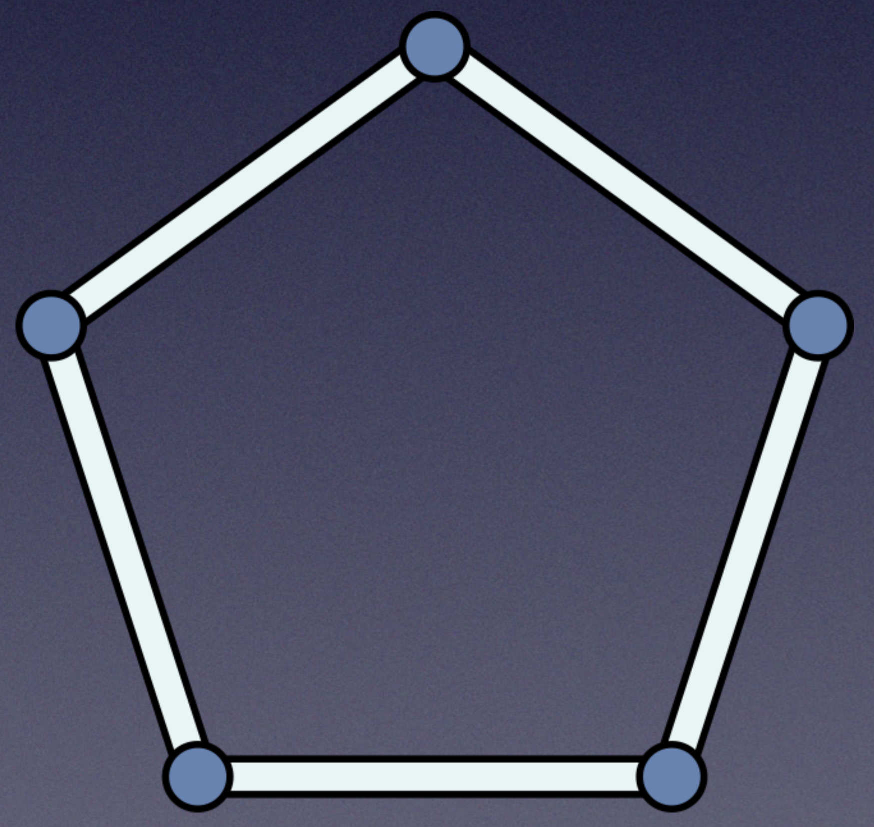
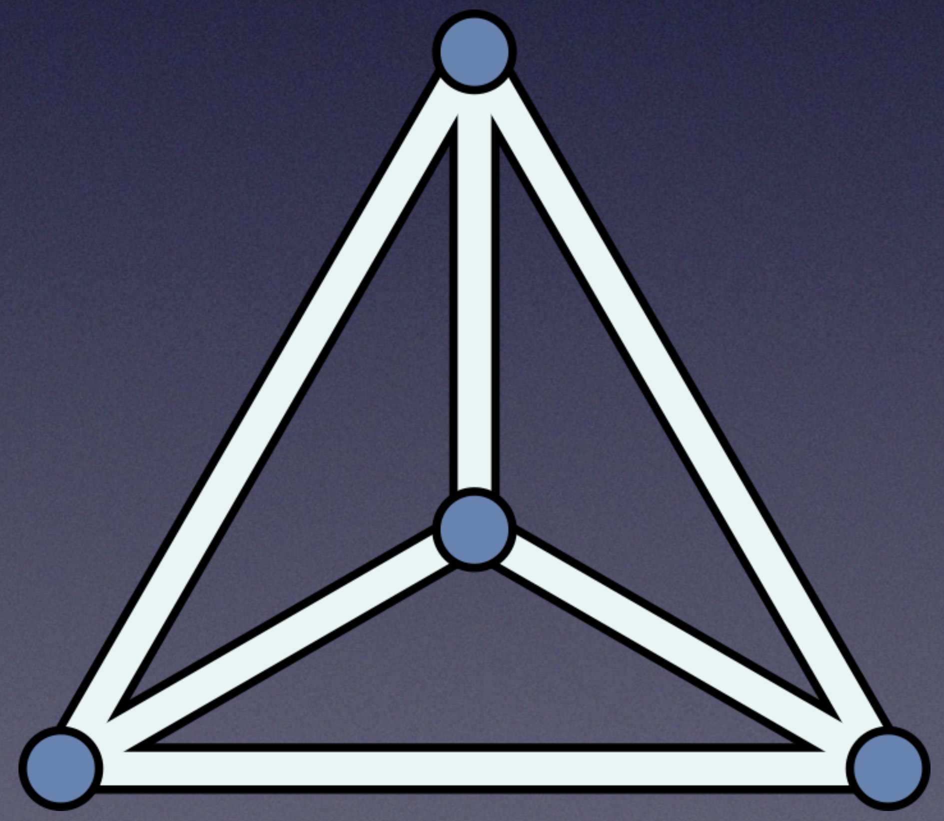


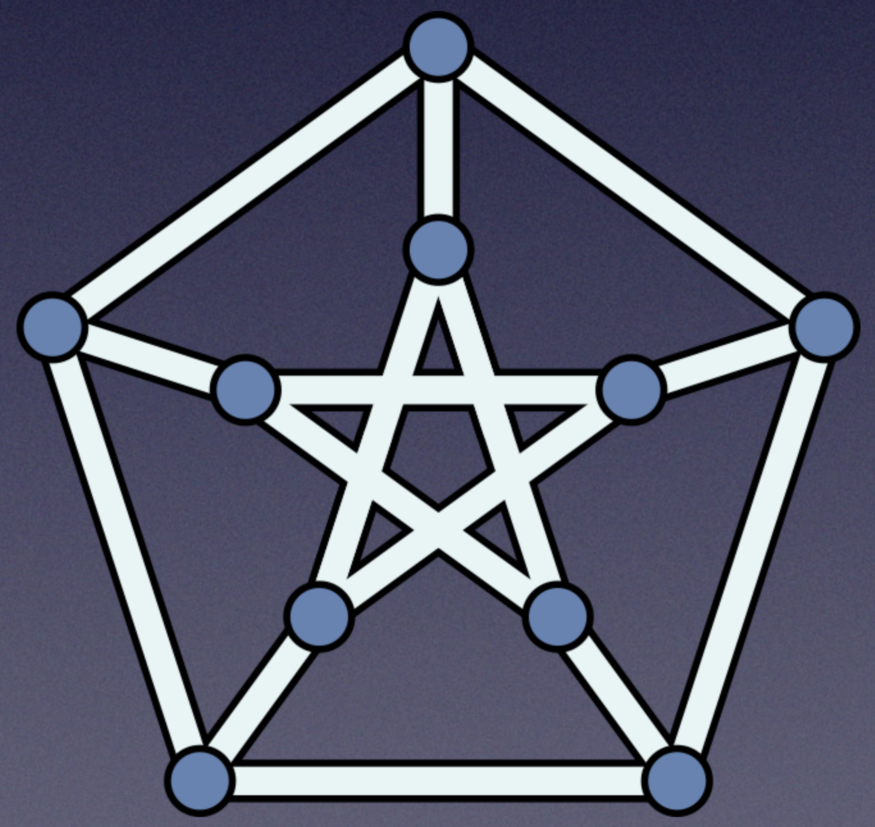
Geodetic Graphs (Ore 1962)

A graph is geodetic if there is a unique geodesic between any two vertices

E.g., trees, complete graphs, odd cycles and the Petersen graph







We still do not know to characterize such graphs (Ore's problem)

Geodetic Graphs

If $\mathscr{P}$ is the collection of all geodesics in a geodetic graph G, then

- For every two vertices u and v in G there is a unique uv path $P_{u,v}$ in $\mathscr{P}$
- If P is a path in $\mathscr{P}$ then any subpath of P is also in $\mathscr{P}$

A collection of paths $\mathscr{P}$ in a graph G that has these two properties is said to be a *consistent path system*

The Question

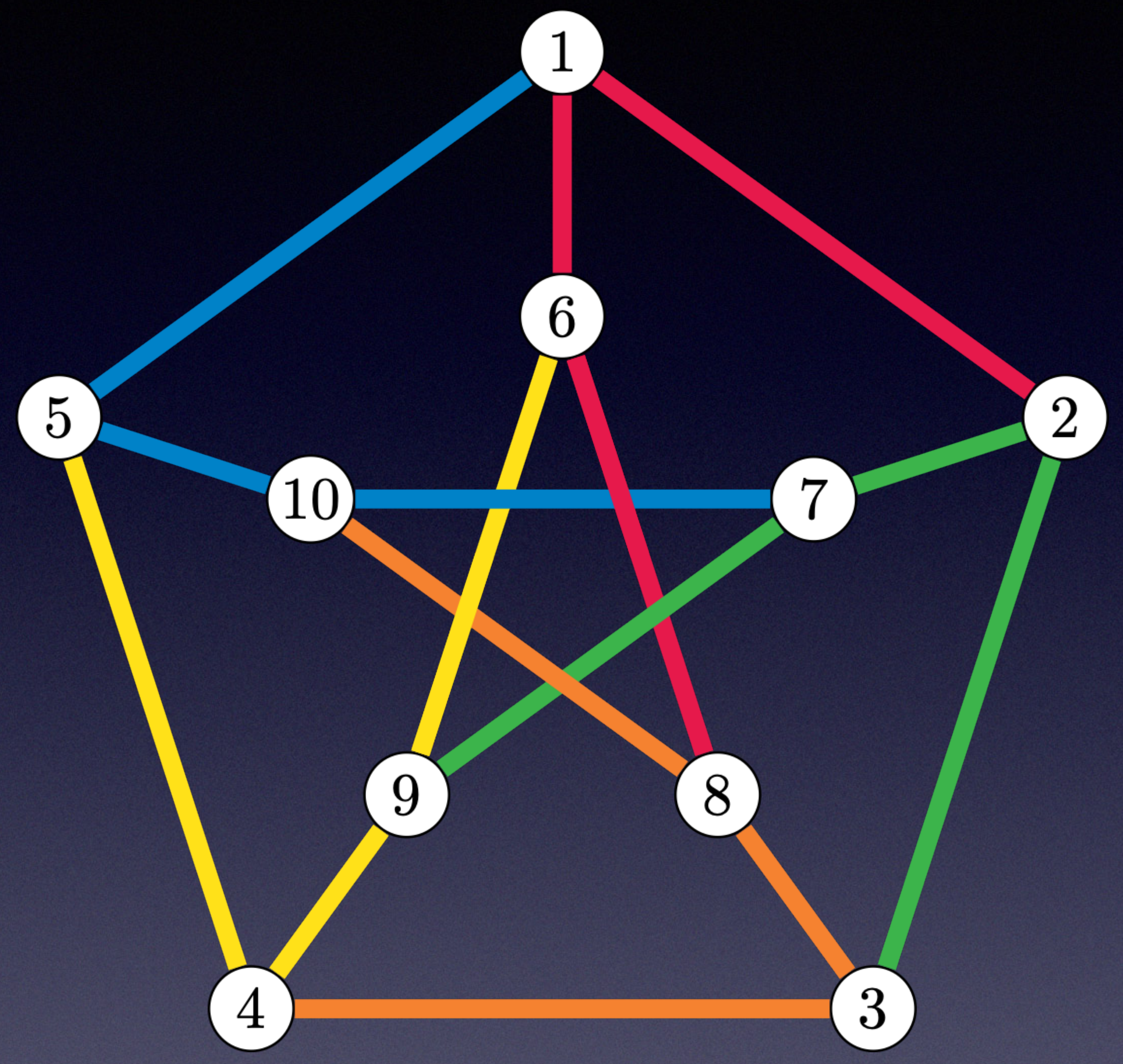
Given a consistent path system $\mathscr{P}$ in a simple undirected graph $G = (V, E)$

Does there exist a metric d on G so that each path in $\mathscr{P}$ is a d-shortest path between its end vertices?

For which graphs G is it the case that for every consistent path system there exists such a metric?

When this is the case we say that G is metrizable

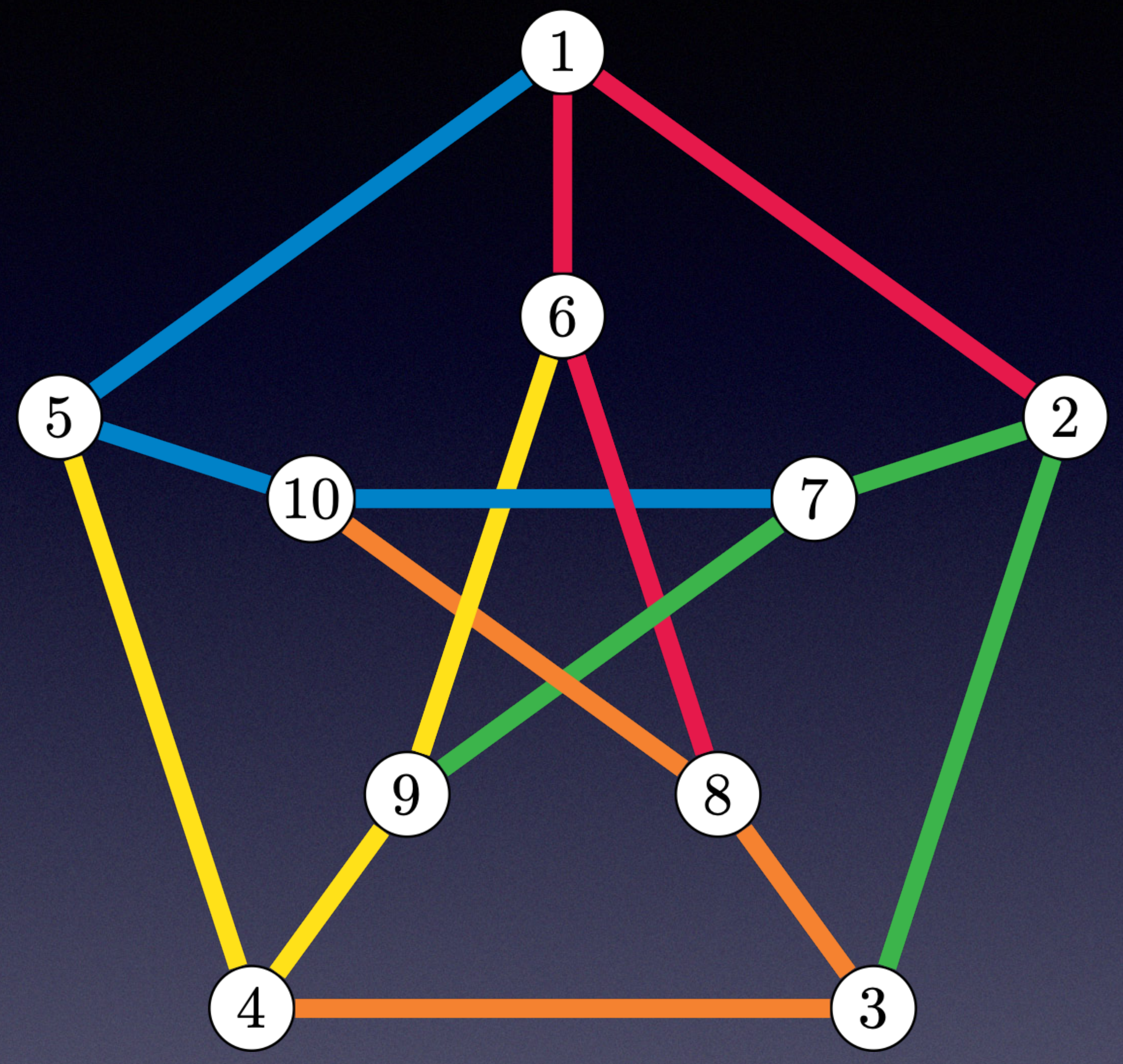
The Petersen Graph is Non-Metrizable



Define a path system $\mathscr{P}$ as follows:

- For most pairs u, v the path $P_{u,v}$ is the unique uv geodesic
- The exceptions are the five pairs of vertices connected by the colored paths

Why Petersen is Non-Metrizable



If w is a positive weight function inducing this system then

$$
\begin{aligned}
w_{1,2} + w_{1,6} + w_{6,8} &\leq w_{2,3} + w_{3,8} \\
w_{2,3} + w_{2,7} + w_{7,9} &\leq w_{3,4} + w_{4,9} \\
w_{3,4} + w_{3,8} + w_{8,10} &\leq w_{4,5} + w_{5,10} \\
w_{4,5} + w_{4,9} + w_{6,9} &\leq w_{1,5} + w_{1,6} \\
w_{1,5} + w_{5,10} + w_{7,10} &\leq w_{1,2} + w_{2,7}
\end{aligned}
$$

Adding these inequalities yields

$$w_{6,8} + w_{7,9} + w_{8,10} + w_{6,9} + w_{7,10} \leq 0,$$

contradicting w is positive!

What We Discovered

- Metrizability is very rare (details to follow)
- Yet, there are infinitely many non-trivial examples of metrizable graphs - every outerplanar graph is metrizable
- The property of being metrizable can be decided in poly-time

Path and Tree Systems

Path Systems

- A ***path system*** $\mathscr{P}$ on ***a connected graph*** $G = (V, E)$ is a collection of paths such that for every $u, v \in V$ there is a unique path $P_{u,v} \in \mathscr{P}$ which connects u and v
- A path system is ***consistent*** if for every $P \in \mathscr{P}$ and vertices u, v in P, the uv subpath of P coincides with $P_{u,v}$

Tree Systems

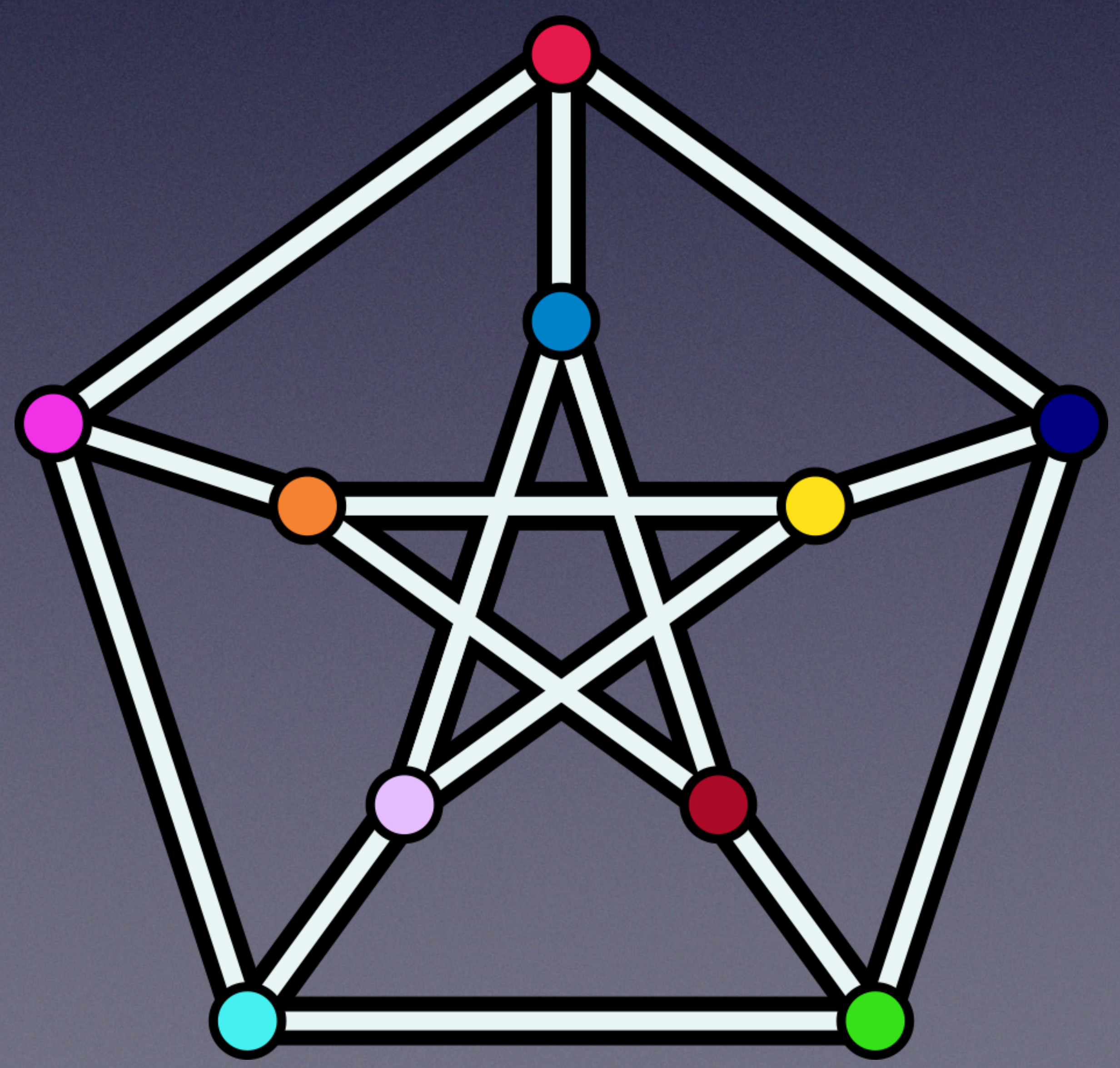
- A ***tree system*** $\mathscr{T}$ ***on a connected graph*** $G = (V, E)$ is a collection of spanning trees in G such that for every $v \in V$ there is a unique $T_v \in \mathscr{T}$
- A tree system is ***consistent*** if for every $u, v \in V$ the uv paths in T_u and T_v coincide

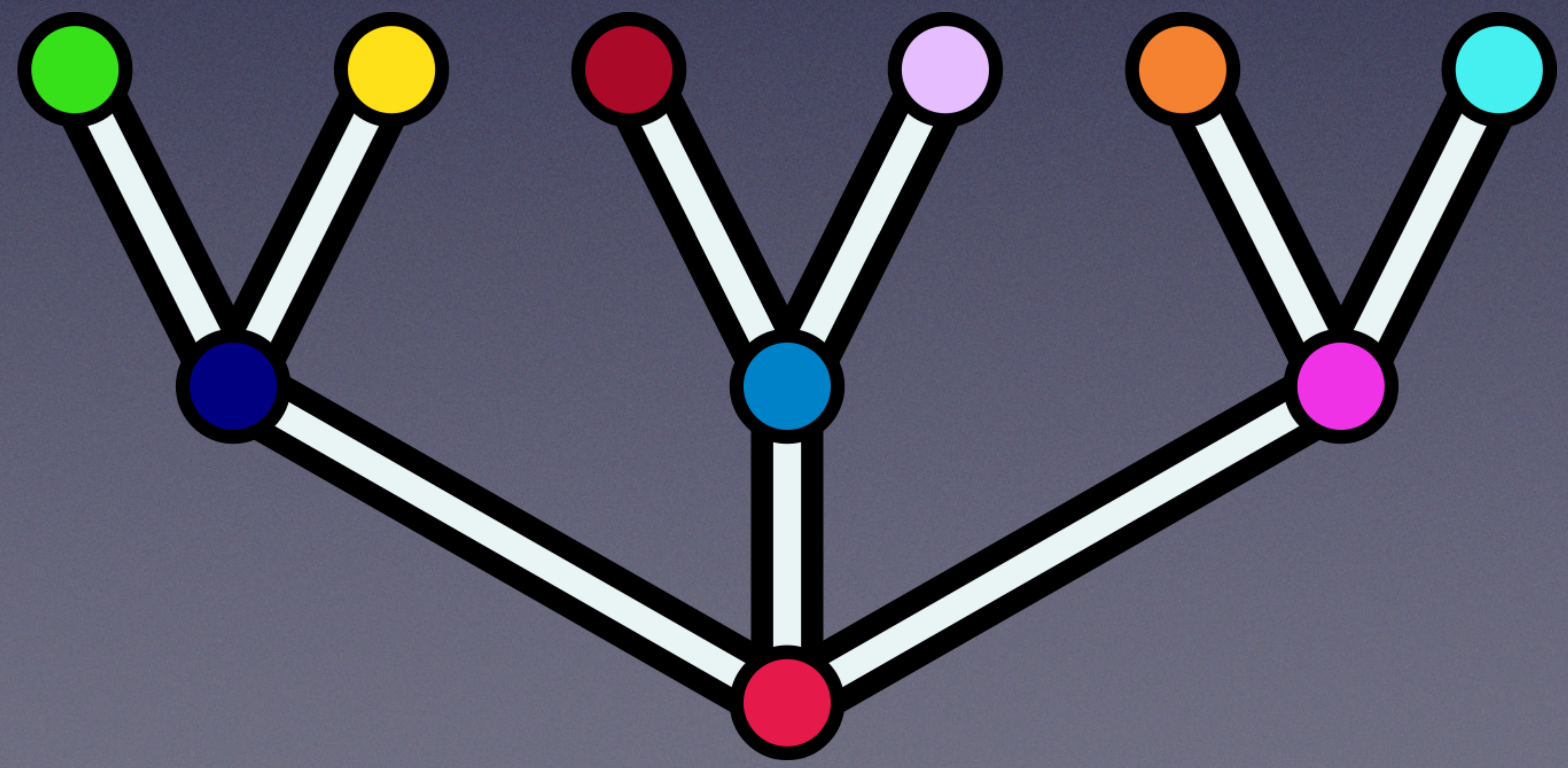
Path and Tree Correspondence

Consistent path and tree systems are in one-to-one correspondence

Given a path system $\mathscr{P}$, for $v \in V$ define T_v by $E(T_v) = \bigcup_{u \in V} E(P_{v,u})$

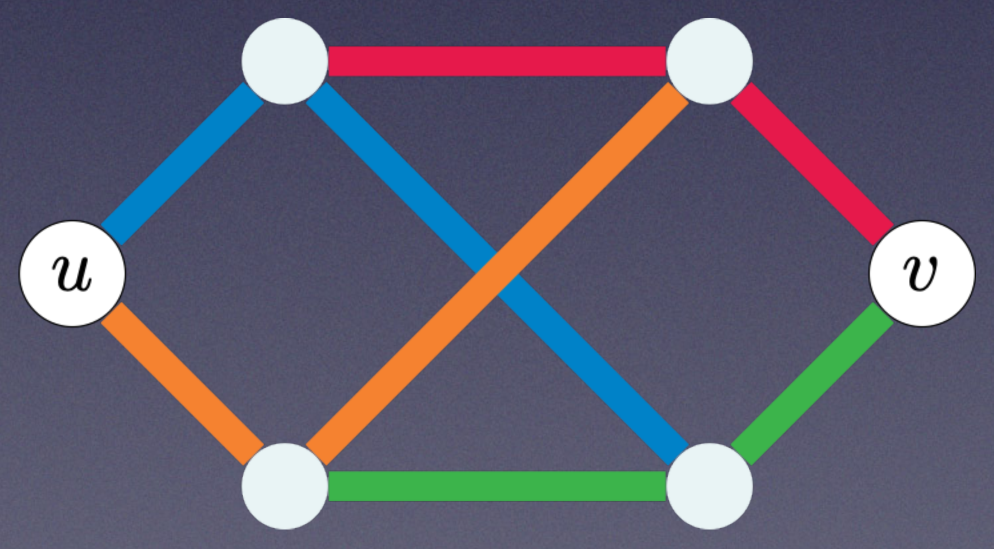
Given $\mathscr{T}$, for $u, v \in V$ let $P_{u,v}$ be the uv path in T_v (or T_u)





About Path Systems

- Every weight function $w: E \to (0,\infty)$ induces a path system in $G = (V, E)$. However,
  - The w-geodesic between vertices $u, v \in V$ need not be unique
  - We need a consistent way to break ties between geodesics
  - E.g., fix an ordering on the edges and compare paths using lexicographical ordering



- Certain partial consistent path systems do not extend to full consistent systems
- But every path systems of a subgraph does extend to the entire graph

About Path Systems

- A path system where every edge $uv \in E$ is the chosen uv path is called $neighborly$, i.e. for $uv \in E$ we have $P_{u,v} = uv$
- If H is an induced subgraph of G, then every consistent neighborly path system in H can be extended to a consistent neighborly path system in G

Metrizability - The Key Theme of this Talk

Let $G = (V, E)$ a graph and $\mathscr{P}$ a path system in G

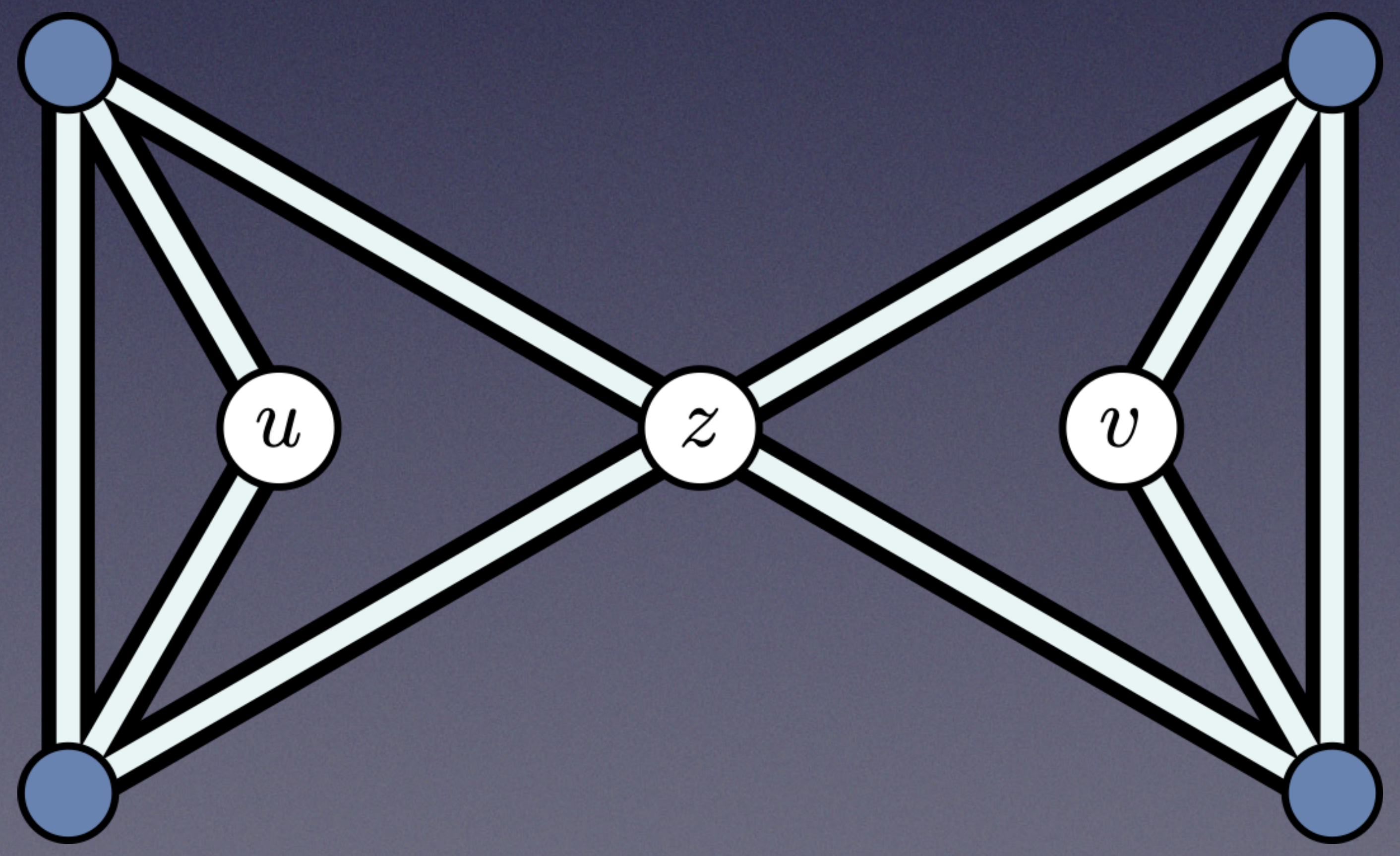
- $\mathscr{P}$ is said to be ***metrizable*** if there is a weight function $w: E \to (0,\infty)$ such that for any uv path Q in G, $w(P_{u,v}) \leq w(Q)$

- $\mathscr{P}$ is said to be ***strictly metrizable*** if there is some $w: E \to (0,\infty)$ such that for any uv path $Q \neq P_{u,v}$ in G, $w(P_{u,v}) < w(Q)$

- G is said to be ***(strictly) metrizable*** if every consistent path system in G is (strictly) metrizable

Metrizability - Remarks

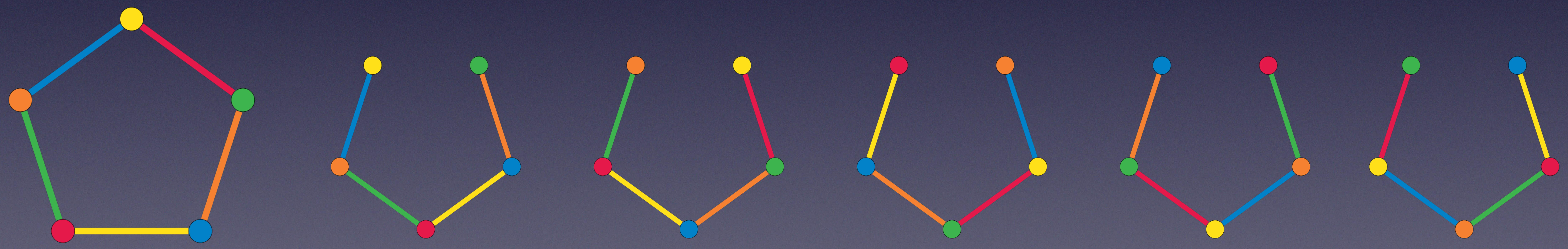
- A graph is not metrizable if it has even <u>one</u> non-metrizable path system
- We do not restrict ourselves to neighborly path systems
- A graph is metrizable iff its 2-connected components are metrizable



Cycles are Strictly Metrizable

Let us start with the (unweighted) odd cycle C_{2n+1}. It is a geodetic graph and its geodesics give rise to a path system, $\mathcal{S}_{2n+1}$

Viewing $\mathcal{S}_{2n+1}$ as a tree system, the tree T_v is the path obtained by deleting the edge antipodal to v.



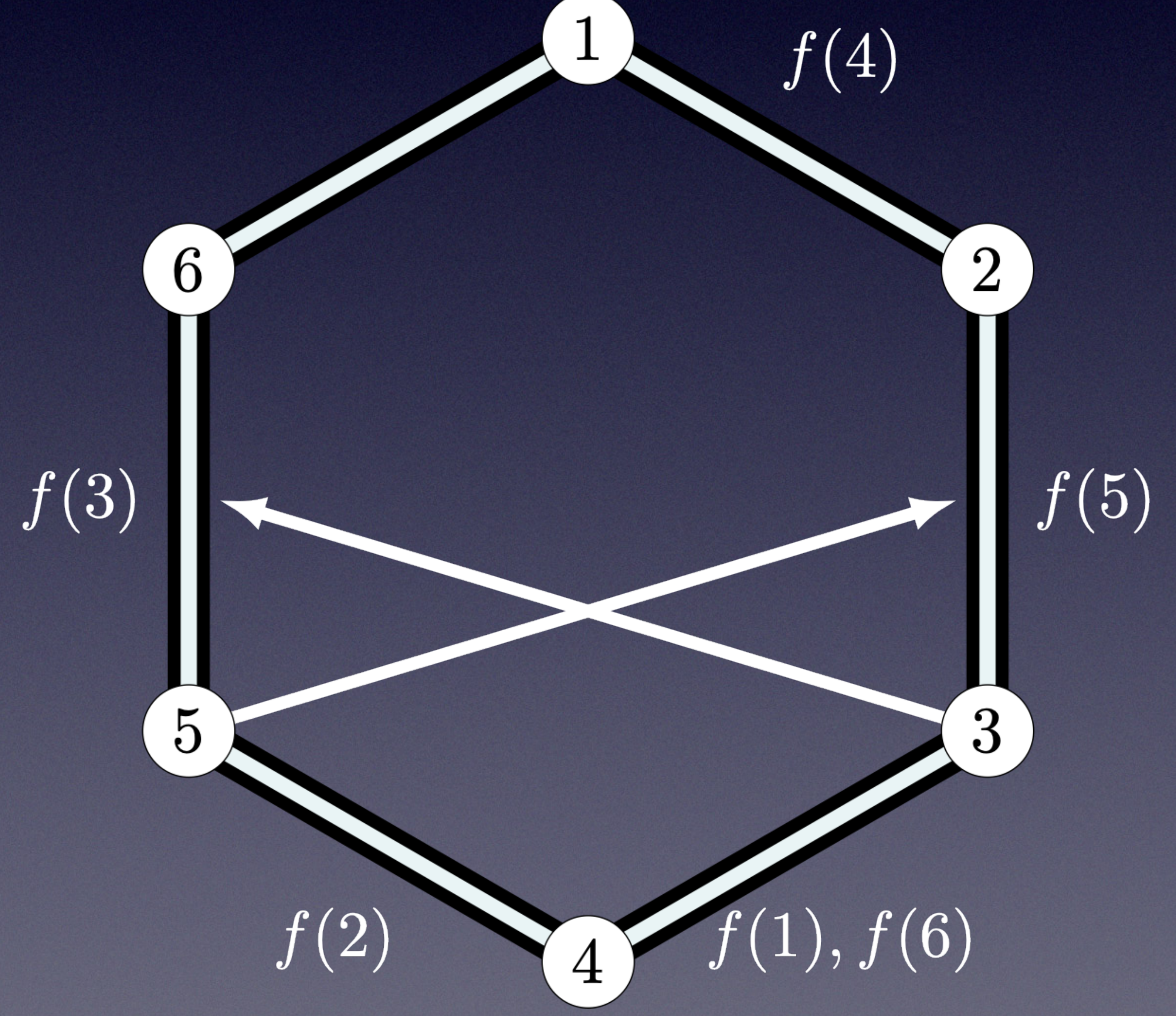
As we show, every consistent path system of the cycle results from a slight variation of this one

Cycles - Crossing Maps

Any tree system $\mathscr{T}$ of a cycle C can be viewed as a mapping $f\colon V(C) \to E(C)$, where $T_v = C \setminus f(v)$.

We say a map $f\colon V(C) \to E(C)$ is *crossing* if for all $u, v \in V(C)$ either $f(u) = f(v)$ or u and $f(u)$ separates v from $f(v)$.

Lemma. *A mapping $f\colon V(C) \to E(C)$ corresponds to a consistent path system in C if and only if it is crossing.*



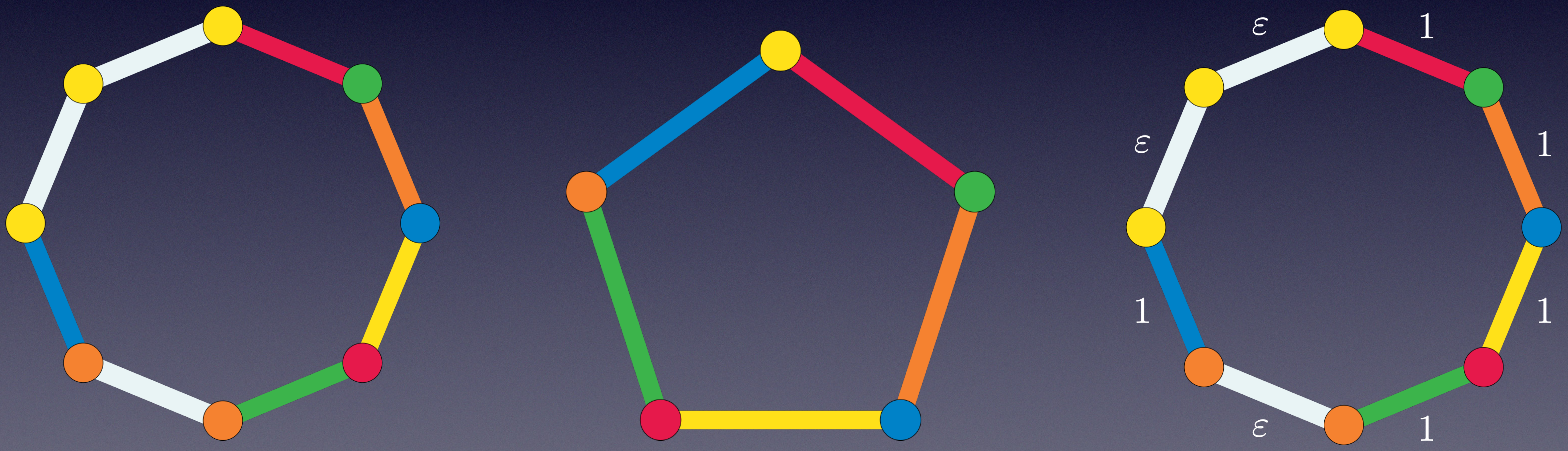
Persistent Edges

Let $\mathscr{P}$ be a path system in C and $f\colon V(C) \to E(C)$ its corresponding crossing map. An edge $uv \in E$ is called $\mathscr{P}$-*persistent* if $f(u) = f(v)$.

Claim. *Let $\mathscr{P}$ be a neighborly path system on C_n. Then, contracting a $\mathscr{P}$-persistent edge e yields a path system on C_{n-1}, denoted $\mathscr{P}/e$.*

Cycles - Path Systems

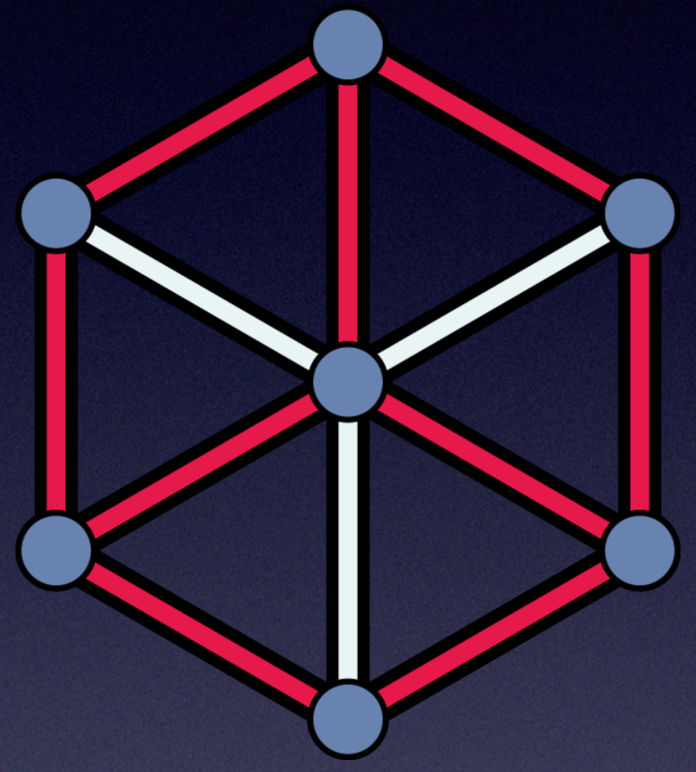
Theorem. *Let $\mathscr{P}$ be a neighborly path system in the cycle C_n , $n \geq 3$, and let $F \subseteq E(C_n)$ be the set of all $\mathscr{P}$-persistent edges. Then $\mathscr{P}/F = \mathscr{S}_m$, for some odd $3 \leq m \leq n$.*

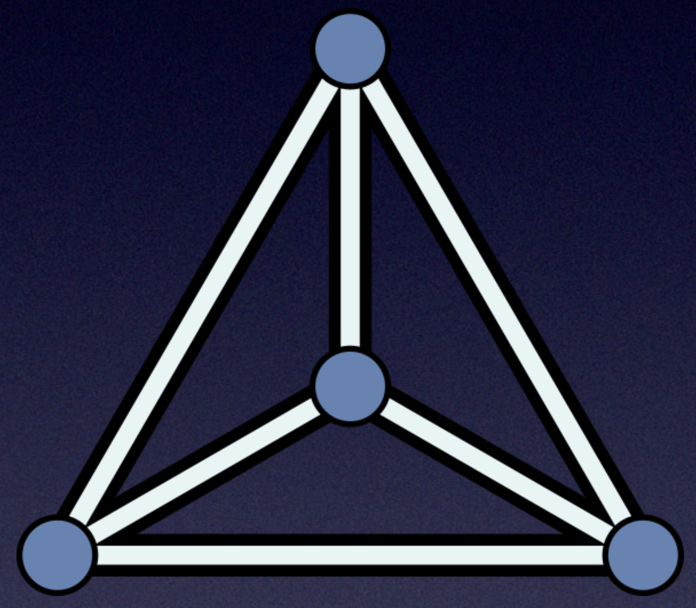


Corollary. *Cycles are strictly metrizable.*

The Class of Metrizable Graphs

If the graph G contains a subgraph which is isomorphic to a subdivision of H, we say that H is a *topological minor* of G.





Theorem. *A topological minor of a (strictly) metrizable graph is (strictly) metrizable.*

Corollary. *If a graph G contains a subdivision of a non-metrizable graph then G is not metrizable.*

The Class of Metrizable Graphs

Theorem. *A topological minor of a (strictly) metrizable graph is (strictly) metrizable.*

We need to show that G remains metrizable after either

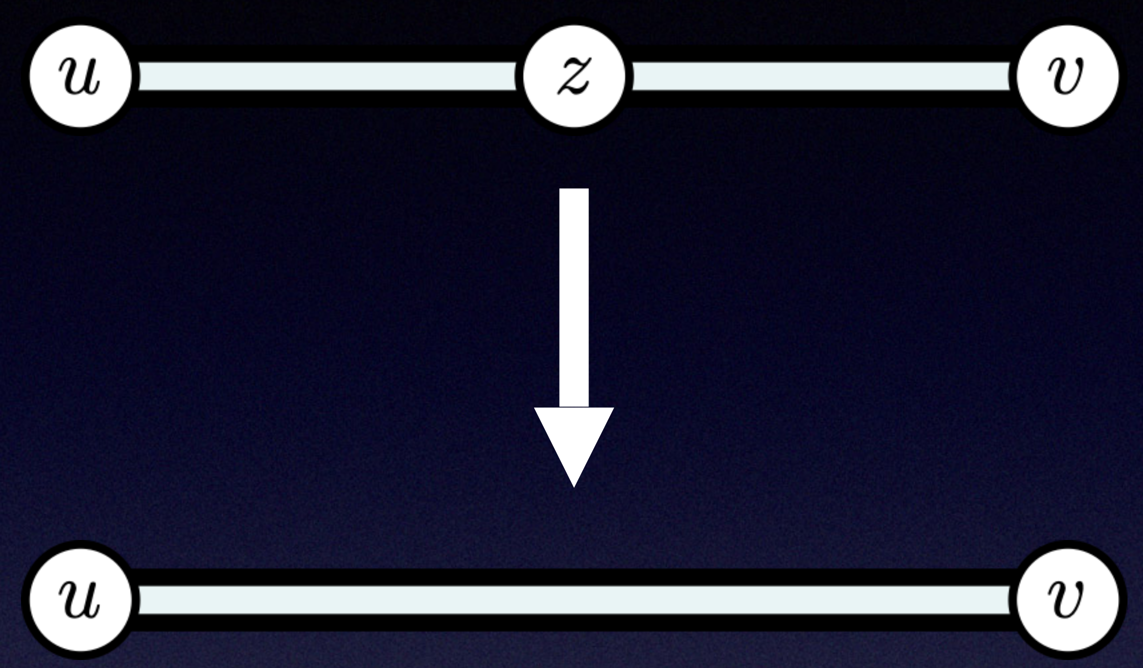
1. Edge removal
2. Suppression of a degree 2 vertex

Metrizability is Maintained under Edge Removal

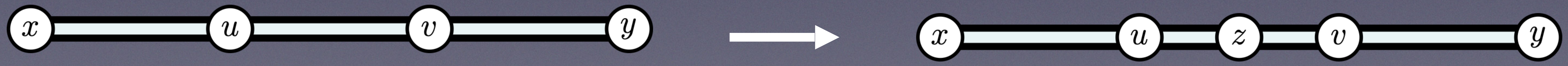
- Recall: our path systems need not be neighborly
- If $G' = G \setminus \{e\}$ for some edge e from G, then any path system of G' is also a path system in G and is metrizable by assumption

Metrizabity is Maintained under Vertex Suppression (Rough Sketch)

Let G' result from G by vertex suppression and let $\mathscr{P}'$ be a path system in G'

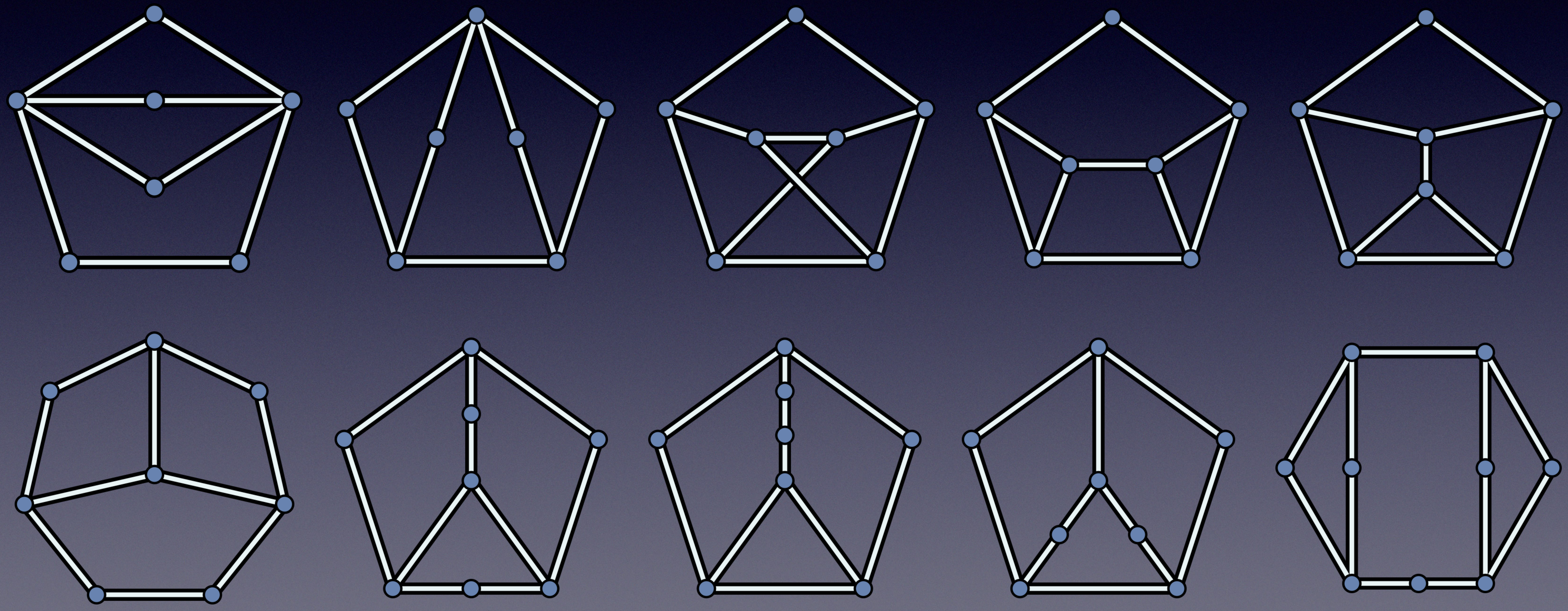


- “Extend” $\mathscr{P}'$ to a path system $\mathscr{P}$ of G
- Since G is metrizable, $\mathscr{P}$ is induced by some $w: E(G) \to (0,\infty)$
- Modify w to some $w': E(G') \to (0,\infty)$ that induces $\mathscr{P}'$



A Zoo of Non-Metrizable Graphs

We found the following non-metrizable graphs. This was done by a computer brute-force search + linear program

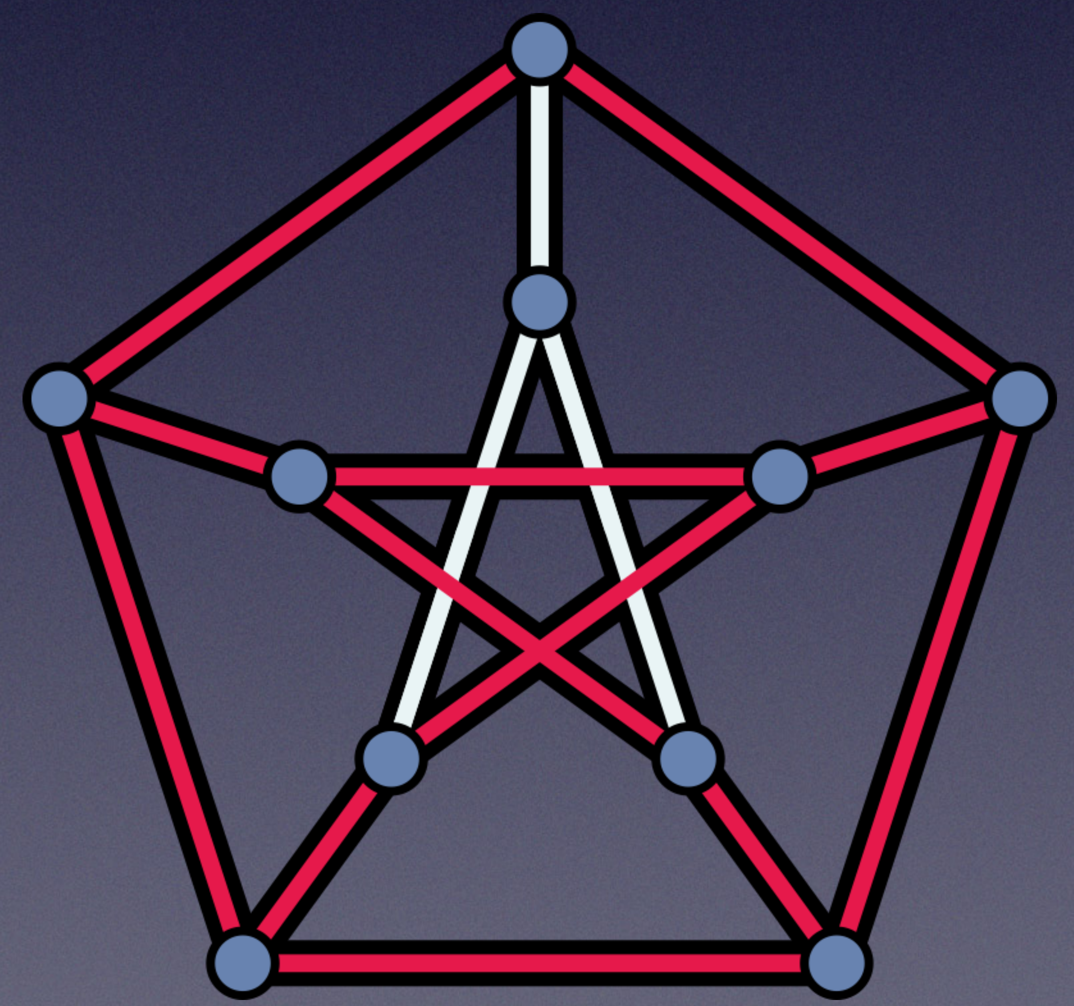


Metrizability is Rare

We now have two ways to prove a given graph G is not metrizable:

1. Construct an explicit path system in G and show that it is non-metrizable using an appropriate linear program
2. Find in G a subdivision of a non-metrizable graph from the Zoo

Method 2 can also show that Petersen's graph is not metrizable



In combination these methods show how rare metrizability is

Metrizability is Rare

Theorem. *If a 2-connected graph on at least 8 vertices contains a subdivision of a 3-connected graph other than* K_4, W_5 *and* $K_5 - e$, *then it is not metrizable.*

Specifically the following graphs are not metrizable:

- 3-connected graphs with at least 8 vertices
- 2-connected non-planar graphs with at least 8 vertices
- 2-connected graphs with minimum degree at least 3 and at least 13 vertices

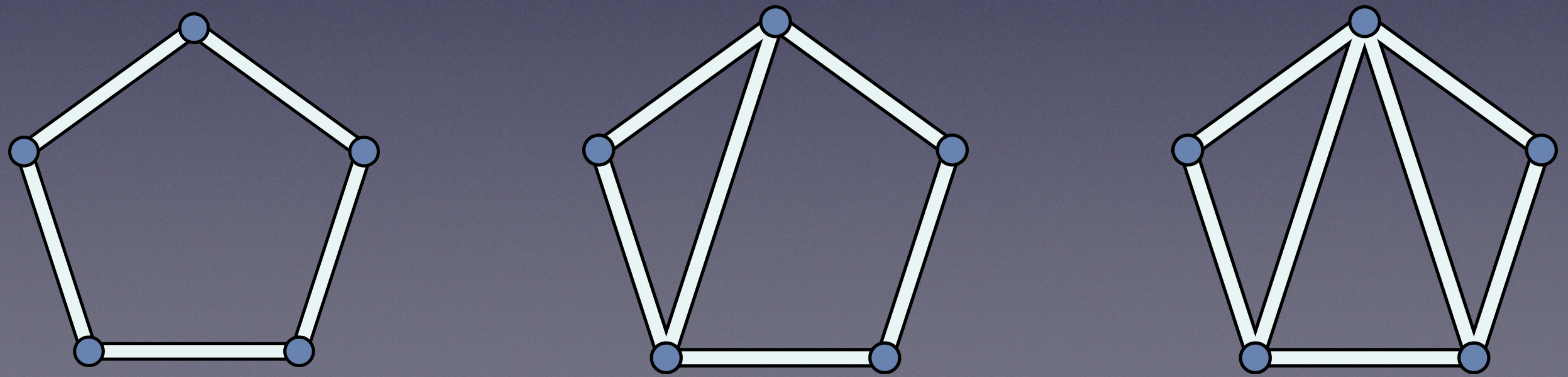
Establishing Metrizability

We saw that metrizability is rare, and that cycles are metrizable.

Is that all there is?

A path in a graph G is said to be suspended if all of its vertices, except possibly its endpoints, have degree 2 in G.

Theorem. *If G is metrizable then so is $G + uv$ provided G has a suspended path between u and v. Likewise for strict metrizability.*



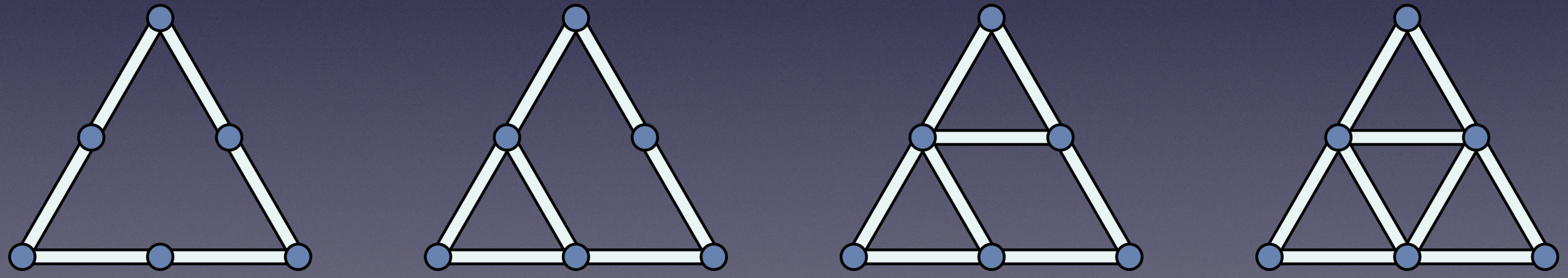
Rough Sketch of Proof

- For a path system $\mathscr{P}$ in $G + uv$ we define a "similar" path system $\mathscr{P}'$ on some topological minor H of G
- But H is metrizable, being topological minor of G, so we get a weight function w' inducing $\mathscr{P}'$
- Modify w' to a weight function on $G + uv$ that induces $\mathscr{P}$

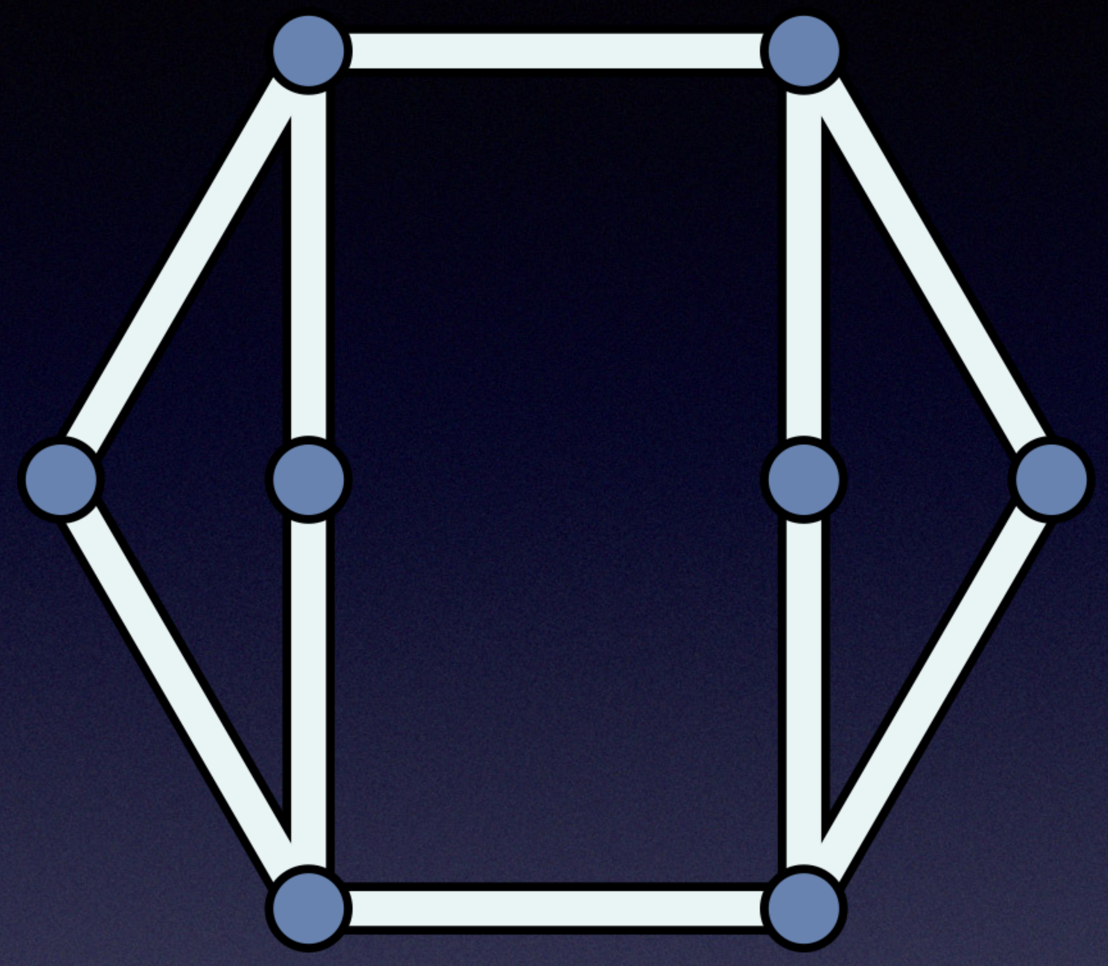
Establishing Metrizability

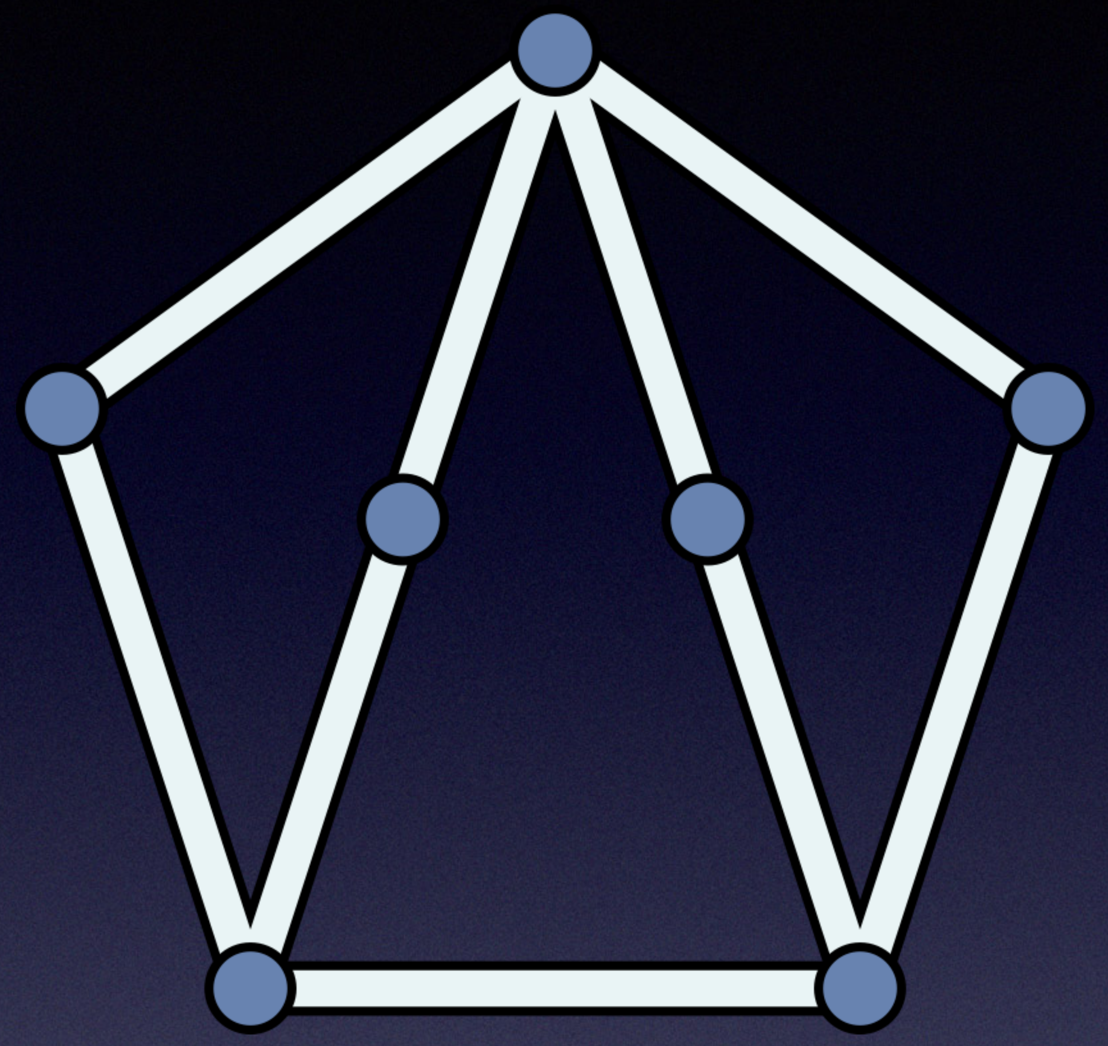
Corollary. All outerplanar graphs are strictly metrizable.

A graph is outerplanar if it can be drawn in the plane with all its vertices in the outer face



Metrizable Graphs are not Minor Closed





While the graph on the left is metrizable contracting an edge yields a non-metrizable graph.
But note: We know that the left graph is metrizable only by means of a computer run

Gate Keepers - Minors

Any graph family $\mathcal{F}$ closed under taking minors, is uniquely defined by the set of its minimal minor non-members $\mathcal{M}$,

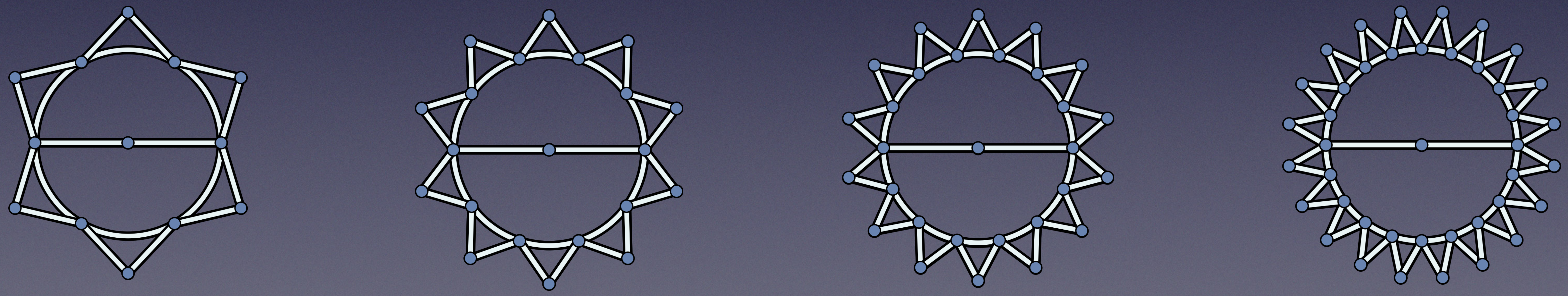
$$\mathcal{F} = \{G : \forall K \in \mathcal{M}, K \not\preceq G\}, \quad \forall H, K \in \mathcal{M}, H \not\preceq K$$

Theorem. (Robertson & Seymour) *Every minor closed family can be characterized by a <u>finite</u> set of forbidden minors.*

Gate Keepers - Topological Minors

Likewise, any graph family closed under topological minors, is uniquely defined by the set of its minimal topological minor non-members,

But unlike minors, for topological minors infinite anti-chains do exist.



Characterizing Metrizability

Let $\mathscr{M}$ denote the set of minimal forbidden topological minors characterizing the family of metrizable graphs.

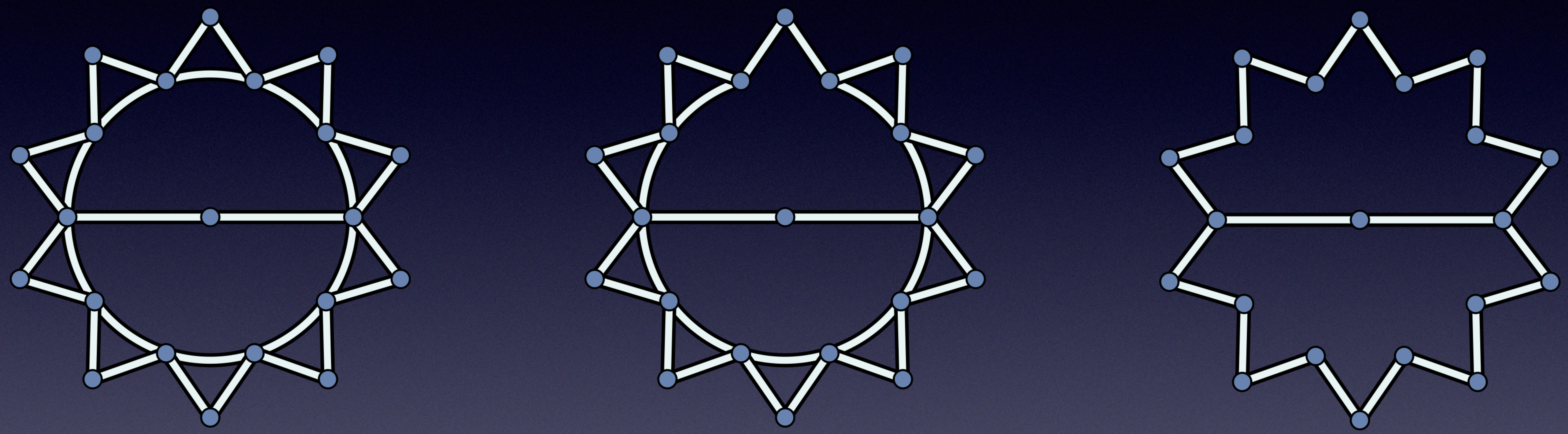
***Theorem**: The family of (strictly) metrizable graphs is characterized by a finite set of forbidden topological minors*

Claim. *If $G \in \mathscr{M}$ and G contains a suspended path of length greater than 1 with endpoints x and y then $xy \notin E(G)$.*

Proof. Suppose $xy \in G$. Since $G \in \mathscr{M}$, *it is non-metrizable. By our previous theorem $G \setminus xy$ is also not metrizable, contradicting that G is minimally non-metrizable.*

Forbidden Graphs

Can $\mathscr{M}$ contain the sort of infinite antichain we saw before?



No! By our previous claim the graph on the left can't be in $\mathscr{M}$.

While we don't know precisely what graphs are/aren't metrizable we can still say something about the sorts of graphs that can be in $\mathscr{M}$.

Forbidden Graphs

Theorem. *The class of metrizable graphs is characterized by a finite set of forbidden topological minors. The same is true for strictly metrizable graphs.*

Combined with the following result of Robertson and Seymour, this yields significant algorithmic consequences for metrizability testing

Theorem. (Roberson & Seymour)**.** *Fix a graph H. There is a polynomial time algorithm to decide whether a given graph G contains a subdivision of H.*

Corollary. *It is possible to decide in polynomial time whether a given graph is (strictly) metrizable.*

Some Open Questions & Challenges

- Let $\Theta_{a,b,c}$ denote the graph that has 2 vertices of degree 3 that are connected three openly disjoint paths of length $a, b,$ and c. We know that $\Theta_{a,b,c}$ is metrizable when $min(a, b, c) = 1$ and that $\Theta_{3,3,4}$ is non-metrizable. What about the case $min(a, b, c) = 2$?

- Find the full list of topologically minimal non-metrizable graphs.

- Do there exist "humanly verifiable" certificates for metrizability?

- What if we restrict ourselves to neighborly path systems?

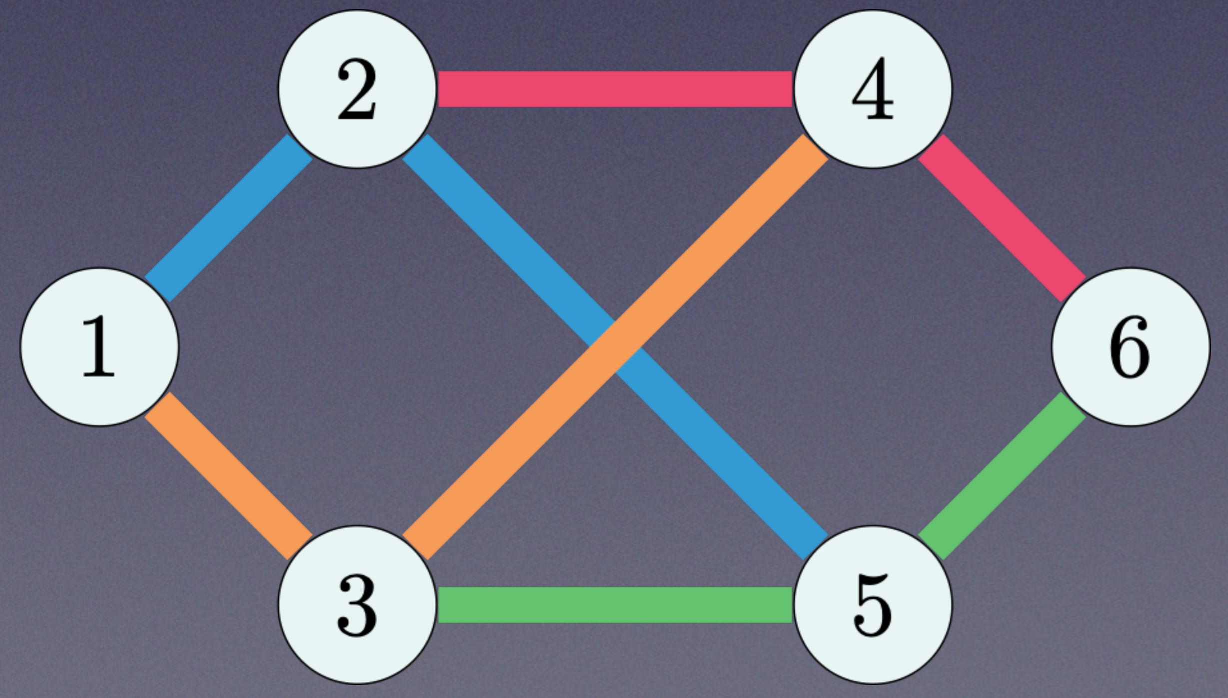
- Let Π_G be the set of all consistent path system in a graph G and $\mathscr{M}_G$ the set of all metrizable systems. As mentioned, typically $\mathscr{M}_G \subsetneq \Pi_G$. Is it even true that $|\mathscr{M}_G| \ll |\Pi_G|$ for most graphs?

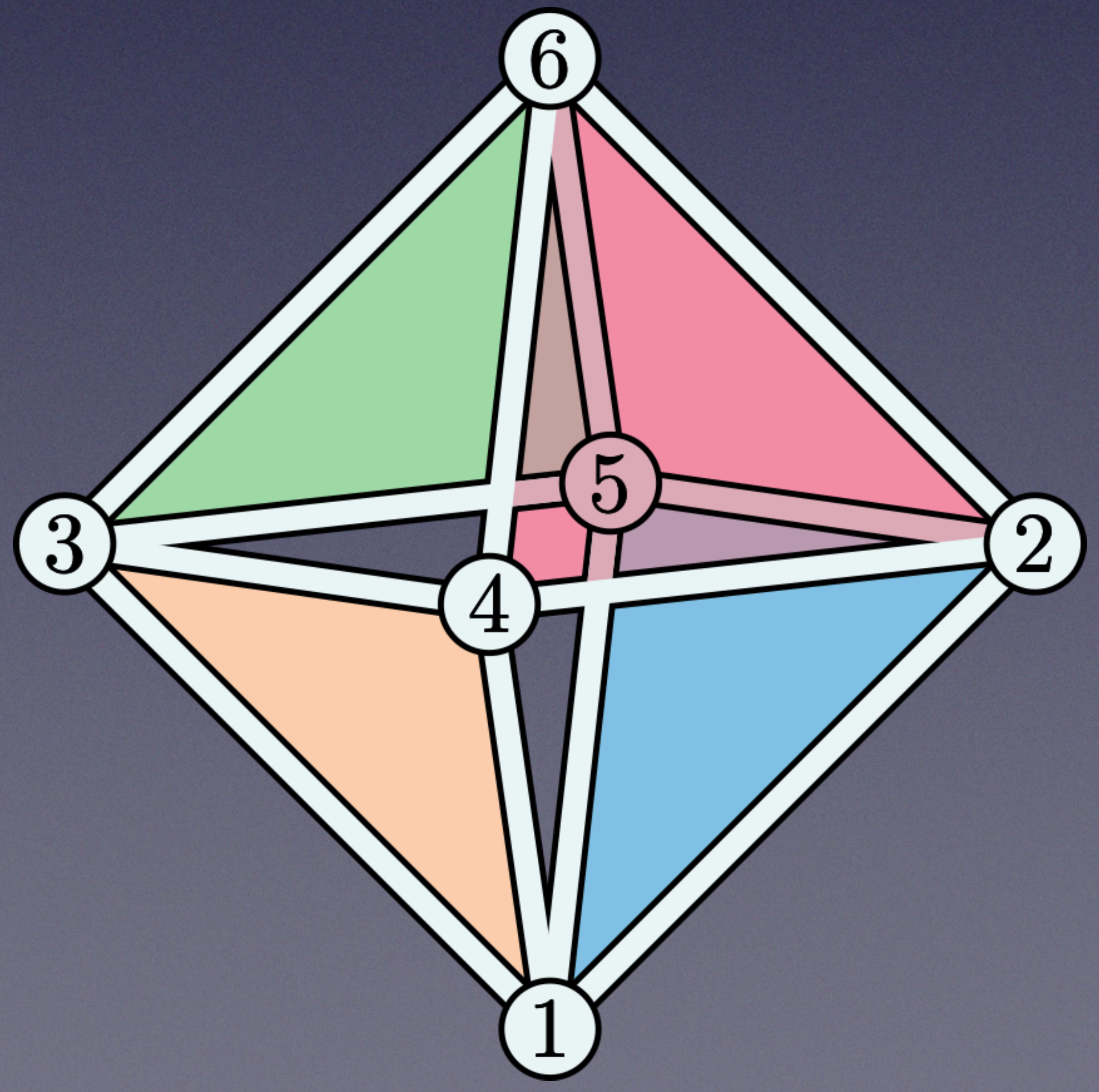
Related Work

Bodwin (2019) offers a topological characterization of partial path systems which are strictly metrizable.

He shows that a path system is strictly metrizable if and only if the system avoids an infinite family of forbidden intersection patterns.

These forbidden patterns are in a correspondence with two-colored topological 2-manifolds.





Persistent Edges

The notion of persistent edges is not limited to cycles:

Lemma. Let $\mathcal{T}$ be a tree system in a graph $G = (V, E)$, and $\mathcal{P}$ its corresponding path system. For $u, v \in V$, t.f.a.e:

1. $T_u = T_v$
2. Every tree $T_w \in \mathcal{T}$ contains the path $P_{u,v}$
3. $T_z = T_u$ for all $z \in P_{u,v}$

If $T_u = T_v$ for some edge uv then it is possible to contract it and obtain a consistent path system $\mathcal{P}/uv$ in the graph G/uv.

Deciding the Metrizability of Path Systems

Theorem. *It is possible to decide in polynomial time whether a given path system is (strictly) metrizable.*

The strict version of this theorem was also proven by Bodwin (2019) using a characterization of strict metrizability in terms of flow.

Both the strict and non-strict version of the theorem can be proven using a variant of the ellipsoid algorithm from the theory of Grötschel, Lovàsz and Schrijver (1988).

Deciding the Metrizability of Path Systems

A *strong separation oracle* for a polyhedron $K \subseteq \mathbb{R}^n$ receives an input $x \in \mathbb{Q}^n$ and either asserts that $x \in K$ or returns some $c \in \mathbb{Q}^n$ such that $c^T x < c^T y$ for all $y \in K$.

"Theorem." *(Grötschel, Lovàsz & Schrijver) Suppose the polyhedron* $K = \{x \in \mathbb{R}^n : Ax \leq b\}$ *has a strong separation oracle, where* $A \in M_{m \times n}(\mathbb{Q}), b \in \mathbb{Q}^m$*. Then it can be determined in polynomial time whether or not* K *is empty.*

To prove our theorem we need:

1. An appropriate polyhedron K
2. A strong separation oracle for K

The Polyhedron: Let $\mathscr{P}$ be a path system on a graph $G = (V, E)$ and let $\mathscr{Q}_{u,v}$ denote all the simple uv paths in G not equal to $P_{u,v}$. Define:

$$A_{u,v} := \{x \in \mathbb{R}^E : \forall Q \in \mathscr{Q}_{u,v}, \sum_{e \in P_{u,v}} x_e - \sum_{e \in Q} x_e \leq 0\}$$

$$K := B \cap \bigcap_{u,v \in V} A_{u,v}$$

$$B := \{x \in \mathbb{R}^E : x_e \geq 1 \text{for every} e \in E\}$$

K is non-empty iff there exists a positive weight function inducing $\mathscr{P}$.

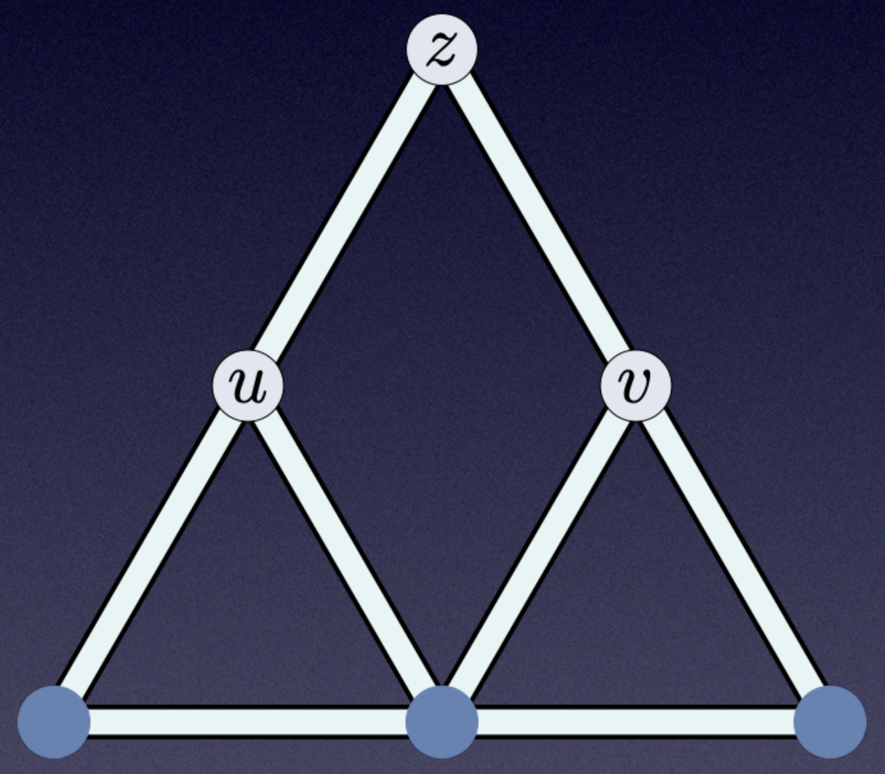
The Strong separation Oracle: Let $w \in \mathbb{R}^E$.

- If $w \notin B$ then the inequality $x_e \geq 1$ is violated for some $e \in E$
- For each $u, v \in V$ calculate the distance $d_w(u, v)$
  - If $d_w(u, v) \neq w(P_{u,v})$ then for some uv path $Q \neq P_{u,v}$, $w(Q) = d_w(u, v)$ and the inequality $\sum_{e \in P_{u,v}} x_e - \sum_{e \in Q} x_e \leq 0$ is violated
- If $d_w(u, v) = w(P_{u,v})$ for $u, v \in V$ then $w \in K$

Proof of Simple Case

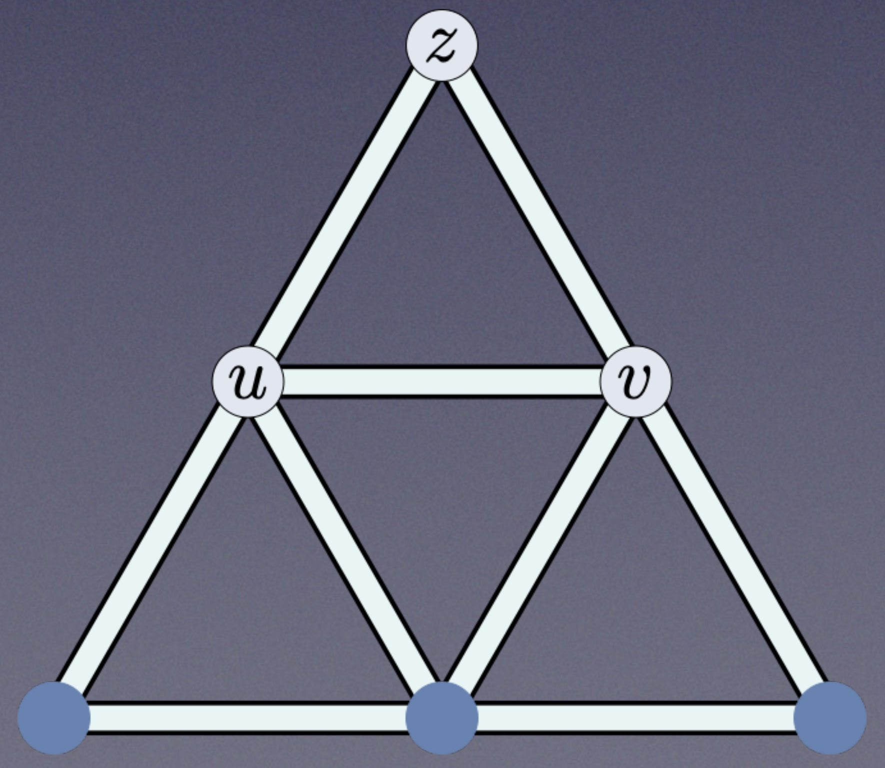
Theorem. *If G is metrizable then so is $G + uv$ provided G has a suspended path between u and v.*

We sketch a proof of the case where the suspended path P between u and v is of length 2, i.e. $P = uzv$.



Proof. Let $\mathscr{P}$ be a path system on $G + uv$. For simplicity we assume that $\mathscr{P}$ is neighborly. We define a path system $\mathscr{P}'$ on G as follows: Let $x, y \in V$,

1. If $uv \notin P_{x,y}$ then $P'_{x,y} := P_{x,y}$
2. If $uv \in P_{x,y}$ then $P'_{x,y}$ is the path obtained by replacing the edge uv by the path uzv in $P_{x,y}$



Proof of Simple Case

It can be shown that $\mathscr{P}'$ defines a consistent path system on G. As G is metrizable there is a weight function $w': E(G) \to (0,\infty)$ inducing $\mathscr{P}'$.

Define $w: E(G+uv) \to (0,\infty)$ as follows

$$w(e) := \begin{cases} w'(e) & e \neq uv \\ w'(uz) + w'(zv) & e = uv \end{cases}$$

For a path Q in $G + uv$ let Q' denote the path in G obtained by replacing any instance of the edge uv with uzv.

If Q is any xy path in $G + uv$ then

$$w(P_{x,y}) = w'(P_{x,y}) \leq w'(Q') = w(Q)$$

Therefore w induces $\mathscr{P}$.