# Liouville quantum gravity with matter central charge in $(1, 25)$: a probabilistic approach

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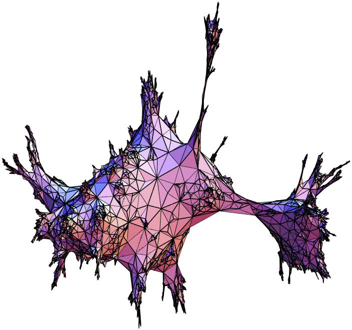
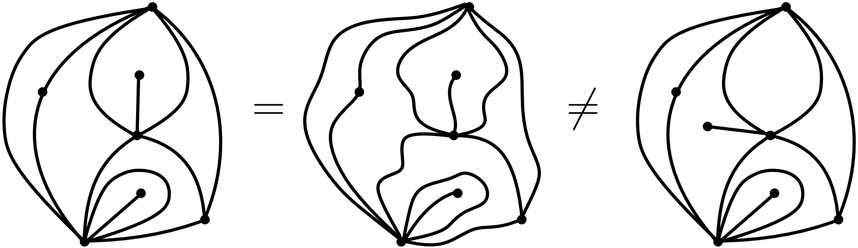


A uniform planar map

# Random planar maps

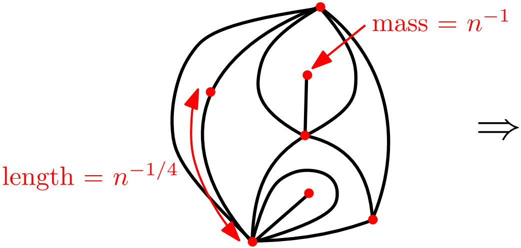
- A **planar map** is a graph drawn on the sphere, viewed modulo continuous deformations.
- For $n \in \mathbb{N}$ sample $M_n$ **uniformly** at random from the collection of planar maps with $n$ edges.



# Scaling limits of uniform planar maps

$M_n$ uniform planar map with $n$ edges. Does $M_n$ converge as $n \to \infty$?

- Gromov-Hausdorff-Prokhorov topology for metric measure spaces (Le Gall'13, Miermont'13, ...)




$\Rightarrow$ The Brownian map

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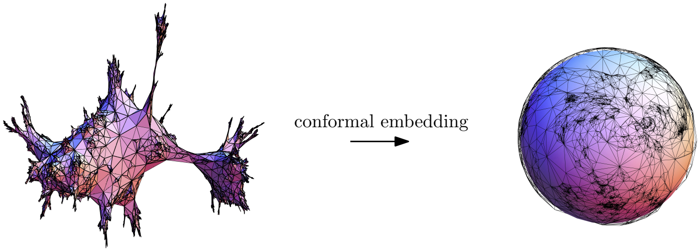


Figure by Nicolas Curien.

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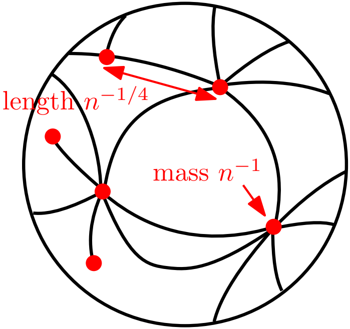



conformally embedded planar map

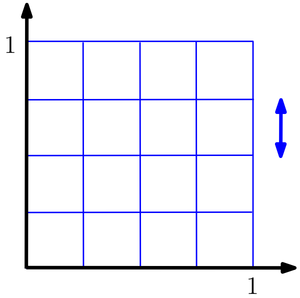
$\Rightarrow$

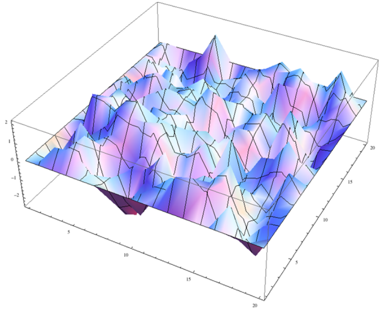
Liouville quantum gravity (LQG) surface

# The Gaussian free field (GFF)

- Hamiltonian $H(f)$ quantifies how much $f$ deviates from being harmonic

$$H(f) = \frac{1}{2}\sum_{x\sim y}(f(x)-f(y))^2, \qquad f : \frac{1}{n}\mathbb{Z}^2 \cap [0,1]^2 \to \mathbb{R}.$$



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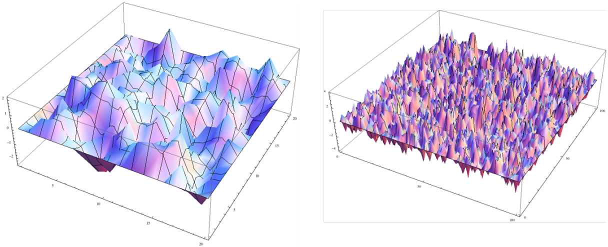

$n = 20,\ \ n = 100$

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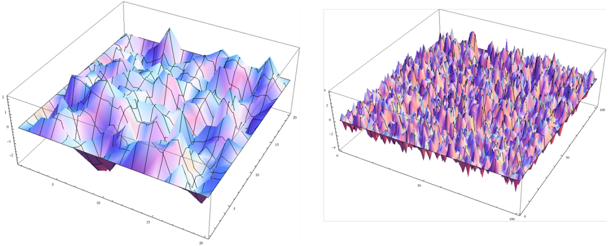

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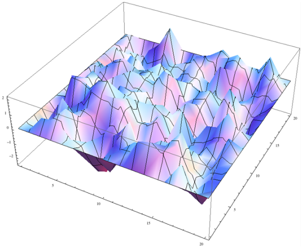

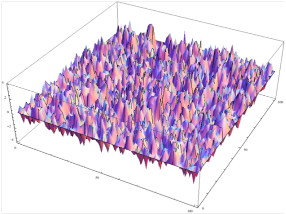

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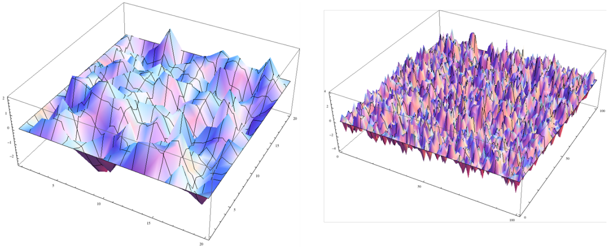
- The GFF is a **random distribution (i.e., random generalized function)**.



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# Liouville quantum gravity (LQG)

- Let $\gamma \in (0,2)$ and let $h$ be the Gaussian free field (GFF).
- LQG surface: $e^{\gamma h}(dx^2 + dy^2)$.
- The definition of an LQG surface does not make literal sense since $h$ is a distribution and not a function.
- Measure $\mu$ and distance function (metric) $D$ defined by considering regularization $h_\epsilon$ of $h$.[1]

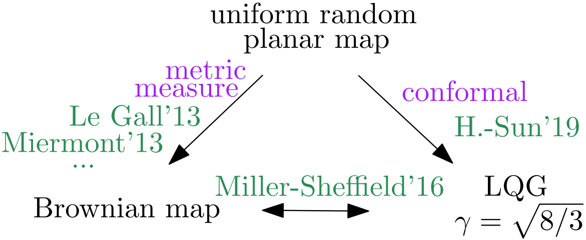
$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \quad U \subset \mathbb{C},$$

$$D(z_1, z_2) = \lim_{\epsilon \to 0} a_\epsilon \inf_{P: z_1 \to z_2} \int_P e^{\gamma h_\epsilon(z)/d}\, dz, \quad z_1, z_2 \in \mathbb{C}.$$

- LQG for $\gamma = \sqrt{8/3}$ describes the scaling limit of uniform planar maps.

[1] Metric construction: Gwynne-Miller'19, Ding-Dubedat-Dunlap-Falconet'19, Dubedat-Falconet-Gwynne-Pfeffer-Sun'19. Hausdorff dim. $(\mathbb{C}, D)$ denoted by $d = d(\gamma)$.

# Equivalence of Brownian map & Liouville quantum gravity



# Planar maps reweighted by Laplacian determinant

- Let $\mathbf{c} \in \mathbb{R}$ be a matter central charge.[2]
- Let $M$ be a random planar map of size $n$ such that

$$\mathbb{P}[M = \mathfrak{m}] \propto (\det \Delta_{\mathfrak{m}})^{-\mathbf{c}/2}$$

  where $\Delta_{\mathfrak{m}}$ is a linear operator derived from the adjacency matrix of $\mathfrak{m}$.
- Physics heuristic: The law of $M$ has been "reweighted by the number of ways to embedded $M$ in $\mathbf{c}$-dimensional space".
- Kirchhoff's matrix-tree theorem: $\det \Delta_{\mathfrak{m}} = \#$ spanning trees on $\mathfrak{m}$.
- David-Distler-Kawai (DDK) ansatz: $M \Rightarrow e^{\gamma h} d^2 z$ as $n \to \infty$ for $\mathbf{c} \leq 1$, where

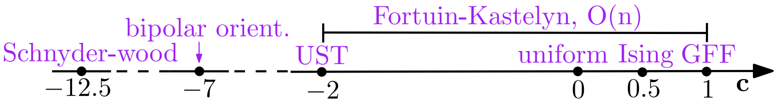
$$\mathbf{c} = 25 - 6(\gamma/2 + 2/\gamma)^2, \qquad \gamma \in (0,2].$$

- DDK ansatz best understood mathematically for $\mathbf{c} = 0$ ($\gamma = \sqrt{8/3}$).

[2]Note that this is different from the Liouville central charge $\mathbf{c}_L = 26 - \mathbf{c}$.

# Mathematical progress on DDK ansatz for $\mathbf{c} \leq 1$

- Convergence to LQG in the mating-of-trees topology
  - Duplantier-Miller-Sheffield'14, Sheffield'16 ($\mathbf{c} \in (-2, 1)$), Gwynne-Mao-Sun'19 ($\mathbf{c} \in (-2, 1)$), Gwynne-Kassel-Miller-Wilson'18 ($\mathbf{c} < -2$), Kenyon-Miller-Sheffield-Wilson'19 ($\mathbf{c} = -7$), Gwynne-H.-Sun'16 ($\mathbf{c} = -7$), Li-Sun-Watson'17 ($\mathbf{c} = -12.5$), Bernardi-H.-Sun'18 ($\mathbf{c} = 0$)




Belief: $(\det \Delta)^{-\mathbf{c}/2} \approx$ statistical physics model partition function

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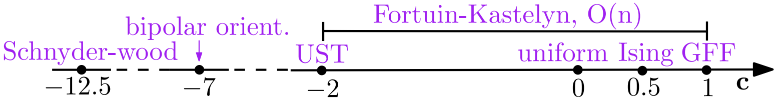
- Random planar map exponents consistent with LQG predictions, e.g.
  - various results on stable maps (Borot, Bouttier, Budd, Chen, Curien, Guitter, Kortchemski, Le Gall, Maillard, Miermont, Richier, etc.) and CLE on LQG (Duplantier, Miller, Sheffield, Werner)
  - nesting statistics in $O(n)$ loop model (Borot-Bouttier-Duplantier'16)
  - volume growth exponent $\mathbf{c} \in (-\infty, -2] \cup \{0\}$ (Gwynne-H.-Sun'20)
  - Ising perimeter and interface exponent (Chen-Turunen'18, Turunen'20)

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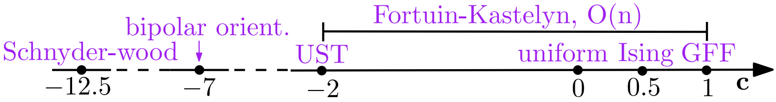
- Reweighting discretized LQG surface by $(\det \Delta)^{-\mathbf{c}/2}$
  - Ang-Park-Pfeffer-Sheffield'20

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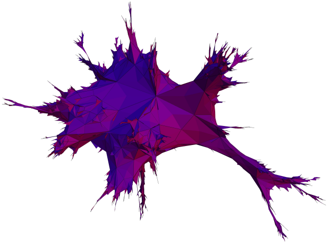
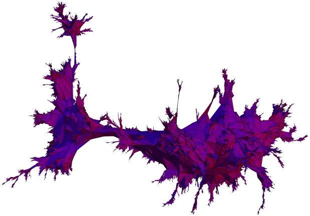
# Reweighted planar maps

$\mathbf{c} = -7$
$\gamma = \sqrt{4/3} \approx 1.15$



$\mathbf{c} = -5$
$\gamma \approx 1.24$



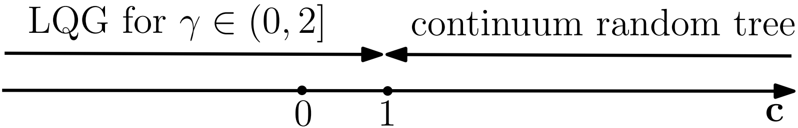
$\mathbf{c} = 0$
$\gamma = \sqrt{8/3} \approx 1.63$

As $\mathbf{c} \to -\infty$, $\gamma \to 0$ and $e^{\gamma h}(dx^2 + dy^2)$ approaches Euclidean geometry.

Physics conj.: For $\mathbf{c} > 1$, random planar map $\Rightarrow$ continuum random tree.

Simulations by Bettinelli

# Planar maps reweighted by $(\det \Delta)^{-\mathbf{c}/2}$




Can we define some non-trivial geometry for LQG with $\mathbf{c} > 1$?

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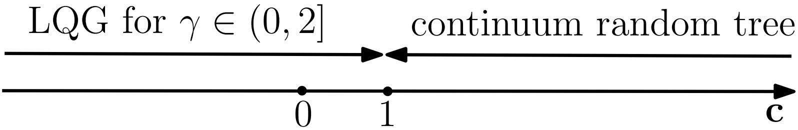
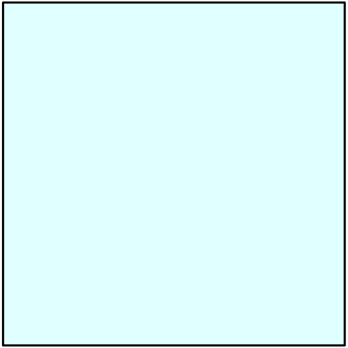



Can we define some non-trivial geometry for LQG with $\mathbf{c} > 1$?

Yes, when $\mathbf{c} \in (1, 25)$.

# Square subdivision: discretization of LQG surface

Let $\mu = e^{\gamma h} d^2 z$ be the $\mathbf{c}$-LQG area measure in $[0,1]^2$ for $\mathbf{c} < 1$.



Fix $\epsilon > 0$. Divide a square $S$ iff $\mu(S) > \epsilon$.

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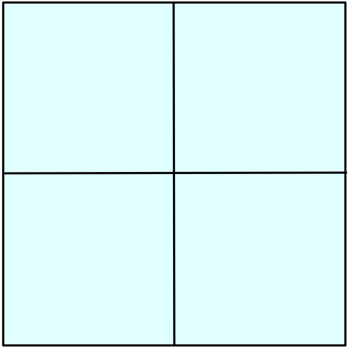

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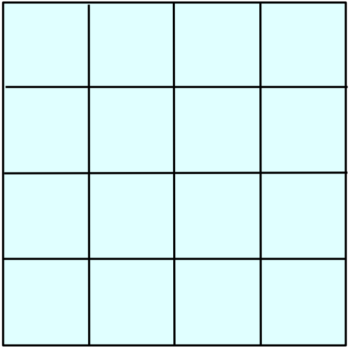

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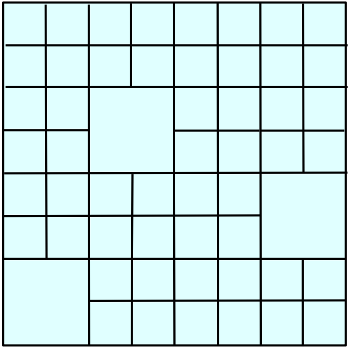

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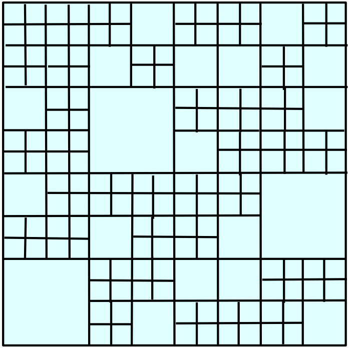

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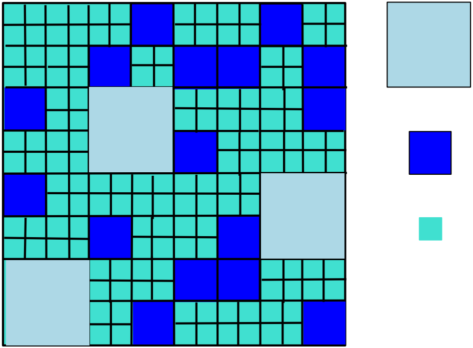

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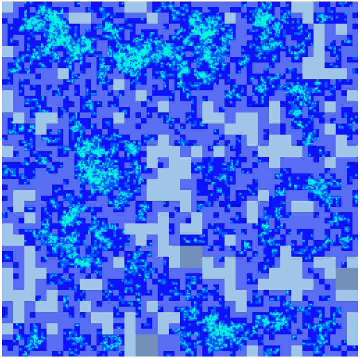
# Illustration of LQG area measure



Area measure $\mu = e^{\gamma h} d^2 z$, $\gamma = 1.5$, $\mathbf{c} = -1.04$

(simulation by Miller and Sheffield)

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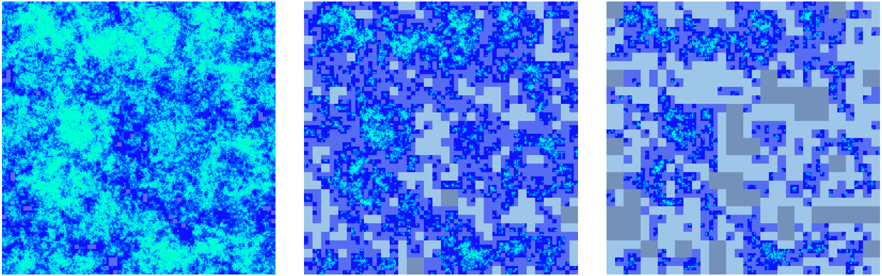

$\gamma = 1$, $\mathbf{c} = -12.5$ $\quad$ $\gamma = 1.5$, $\mathbf{c} = -1.04$ $\quad$ $\gamma = 1.75$, $\mathbf{c} = -0.57$

Area measure $\mu = e^{\gamma h} d^2 z$

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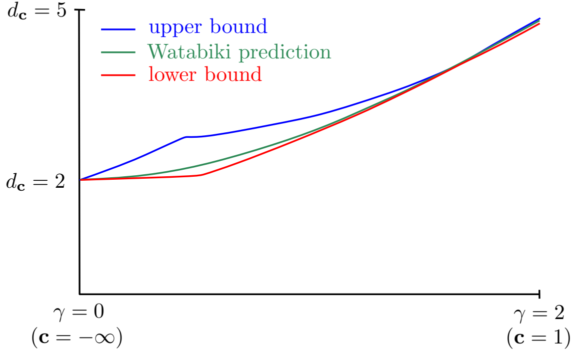
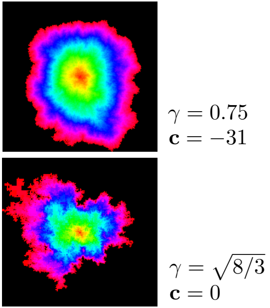
# The Hausdorff dimension of an LQG surface

- Recall: A $\mathbf{c}$-LQG surf. has measure $\mu$ and distance func. (metric) $D$.
- $d_{\mathbf{c}}$ = Hausdorff dimension of metric space $(\mathbb{C}, D)$.
- Volume of metric ball: $\mu(\mathcal{B}(z,r)) = r^{d_{\mathbf{c}}+o(1)}$ (Ang-Falconet-Sun'20).

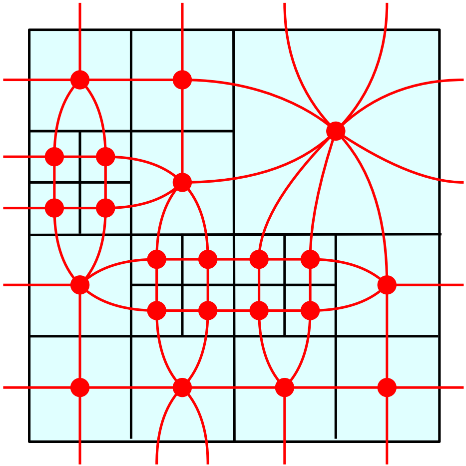



LQG metric balls (by Miller)

Bounds for $d_{\mathbf{c}}$: Ang'19, Ding-Goswami'19, Ding-Gwynne'18, Gwynne-Pfeffer'19

# The square subdivision as a planar map



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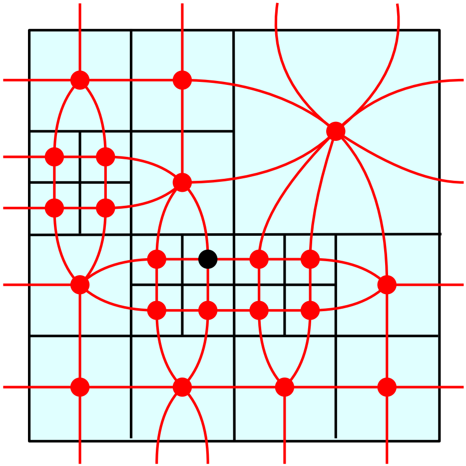

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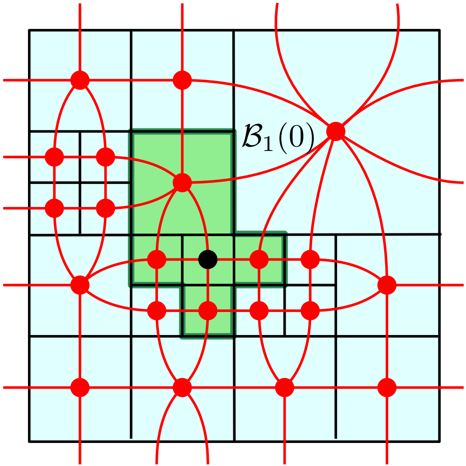

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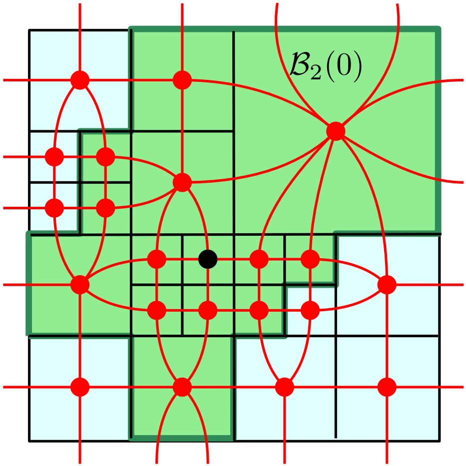

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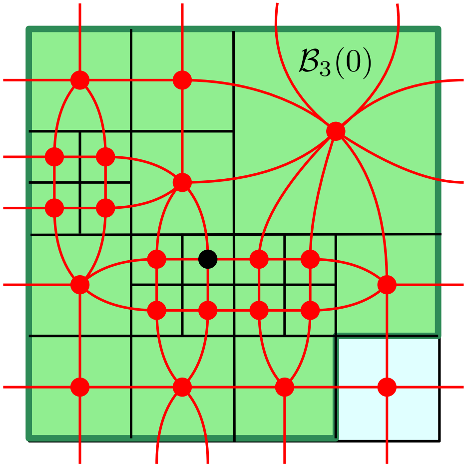


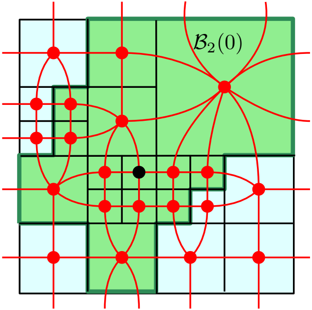
# The square subdivision as a discrete approximation to LQG

Let $\mathbf{c} < 1$.

By Ding-Zeitouni-Zhang'18, Ding-Gwynne'18,

$$\#\mathcal{B}_r(0) = r^{d_{\mathbf{c}}+o(1)},$$

where $\mathcal{B}_r(0)$ is the graph metric ball of radius $r$ and $d_{\mathbf{c}} > 2$ is the Hausdorff dimension of $\mathbf{c}$-LQG.



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By Ding-Zeitouni-Zhang'18, Ding-Gwynne'18,

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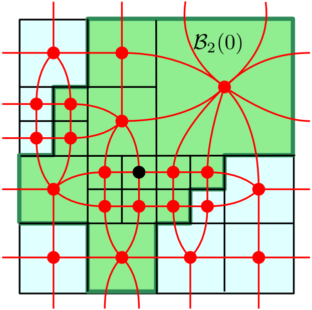

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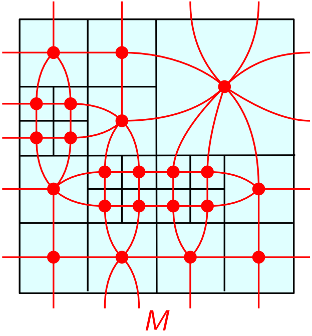

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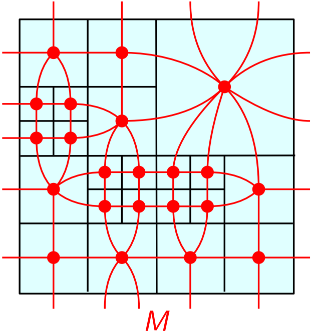
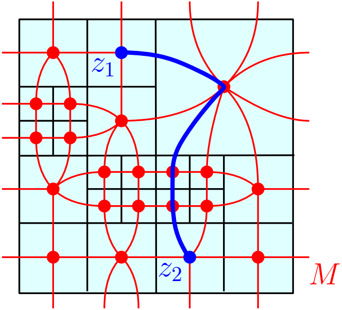



These results suggest that the square subdivision planar map $M$ is in the **c**-universality class of planar maps.

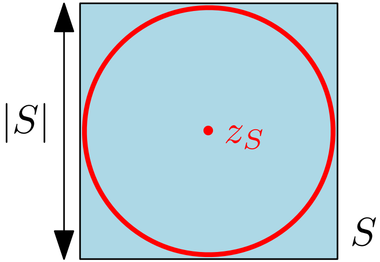
# The square subdivision as a discrete approximation to LQG




Conjecture: The graph metric of $M$ appropriately rescaled converges to the **c**-LQG metric $D$ associated with the GFF $h$ as $\epsilon \to 0$.

Ding-Dunlap'20 proves tightness for a related metric.

# Approximate LQG area measure via GFF circle average

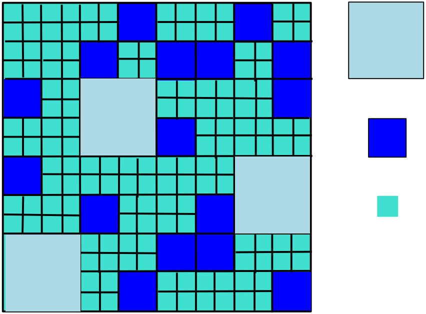



LQG area measure: $\mu = e^{\gamma h} d^2 z$; GFF circle average: $h_{|S|}(z_S)$.

$$\mu(S)^{1/\gamma} \approx |S|^Q e^{h_{|S|}(z_S)}, \qquad Q = 2/\gamma + \gamma/2$$

| | | | |
|---|---|---|---|
| Coupling constant | $\gamma$ | $\gamma \in (0,2]$ | $\lvert\gamma\rvert = 2$ |
| Background charge | $Q = 2/\gamma + \gamma/2$ | $Q \geq 2$ | $Q \in (0,2)$ |
| Matter central charge | $\mathbf{c} = 25 - 6Q^2$ | $\mathbf{c} \leq 1$ | $\mathbf{c} \in (1,25)$ |

# The square subdivision model with GFF circle averages



Fix $\epsilon > 0$. Divide a square $S$ iff $|S|^Q e^{h_{|S|}(z_S)} > \epsilon$.

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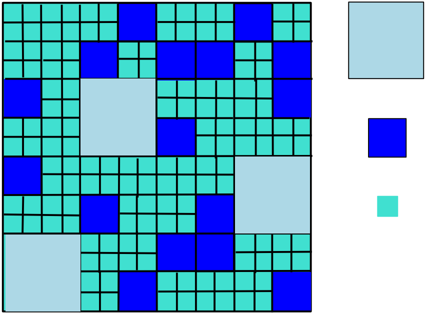

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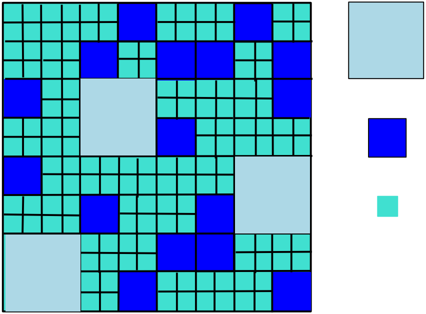

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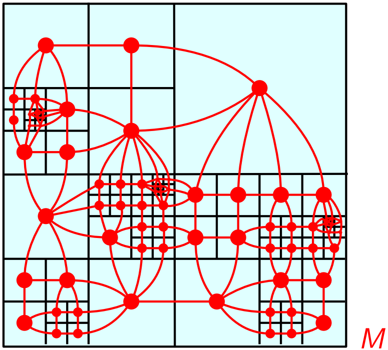
$\gamma \in \mathbb{C}, \ Q \in \mathbb{R}$

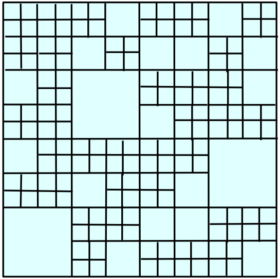
# Our model for LQG with $\mathbf{c} \in (1, 25)$



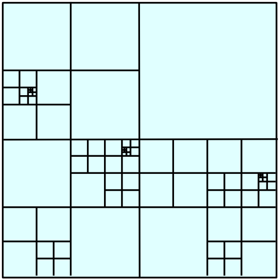

Let $M$ be planar map of squares defined using GFF circle averages.

$M$ is our model for LQG with $\mathbf{c} \in (1, 25)$.

# Phase transition at $\mathbf{c} = 1$: Infinite-volume surface



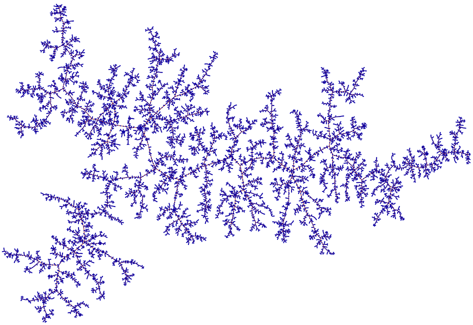
For $\mathbf{c} < 1$ the square subdivision terminates with probability 1.



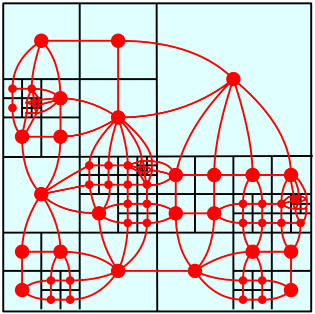
For $\mathbf{c} \in (1, 25)$, as $\epsilon \to 0$ the probability that the square subdivision terminates goes to 0.

Dense set of "infinite mass" points ($d = 2 - Q^2/2$, Hu-Miller-Peres'10).

# Finite and infinite volume models for $\mathbf{c} \in (1, 25)$



Finite volume: continuum random tree



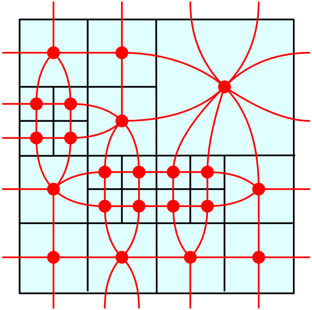
Infinite volume

Left figure due to Kortchemski.

# Is our model the "correct" model for $\mathbf{c} \in (1, 25)$?

Ang-Park-Pfeffer-Sheffield'20 argue as follows:

- Let $\mathbf{c} \in \mathbb{R}$, $\epsilon > 0$, and $n \in \mathbb{N}$.
- Let $M$ be obtained from square subdivision for central charge 0 and square size $\epsilon$, conditioned on $\#V(M) = n$.
- Reweight the prob. meas. by Laplacian determinant (defined via smooth approx. to $h$ and Polyakov-Alvarez) to power $-\mathbf{c}/2$.
- For the resulting probability measure, $M$ has the law of square subdivision for central charge $\mathbf{c}$, conditioned on $\#V(M) = n$.

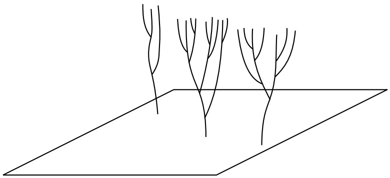


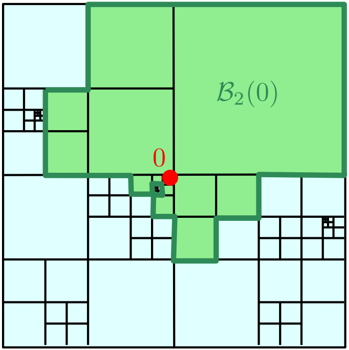
$M$

# Superpolynomial ball volume growth

**Theorem (Gwynne-H.-Pfeffer-Remy'19, Infinite dimension)**

*Let* $\mathbf{c} \in (1, 25)$*. Almost surely,* $\lim_{r\to\infty} \frac{\log \#\mathcal{B}_r(0)}{\log r} = \infty.$





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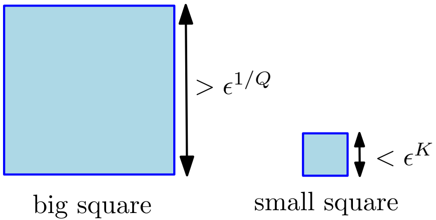

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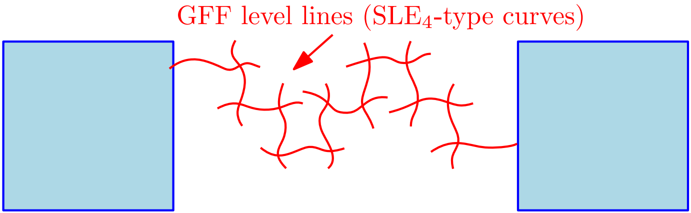

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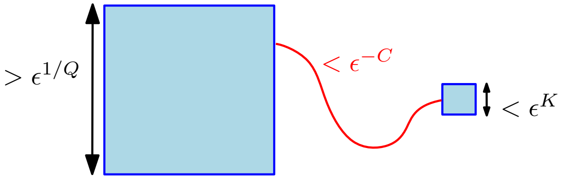

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By the triangle inequality and the above, $\#\mathcal{B}_r(0) > \epsilon^{-cK}$ for $r = 3\epsilon^{-C}$.

# Point-to-point distances grow polynomially

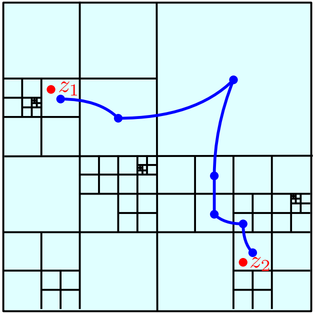
Let $D_h^\epsilon(\cdot,\cdot)$ denote the graph metric in the planar map of squares.

## Proposition 1 (Gwynne-H.-Pfeffer-Remy'19)

*For* $\mathbf{c} < 25$, *there exists* $\underline{\alpha}, \overline{\alpha} > 0$ *s.t. for fixed* $z_1, z_2 \in \mathbb{C}$, *a.s.*

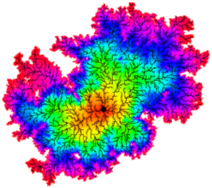
$$\epsilon^{-\underline{\alpha}+o(1)} \le D_h^\epsilon(z_1, z_2) \le \epsilon^{-\overline{\alpha}-o(1)} \quad \text{as } \epsilon \to 0.$$

Ding-Gwynne'20 gets tightness for point-to-point distances in Liouville first passage percolation for $\mathbf{c} \in (1, 25)$.

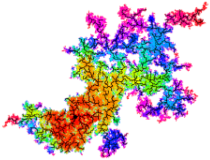


# Liouville first passage percolation

- $D_h^{\mathrm{LFPP},\epsilon}(z_1, z_2) := a_\epsilon \inf_{P: z_1 \to z_2} \int_P e^{\xi h_\epsilon(z)}\, dz$, $\xi > 0$.
- If $\xi = \gamma_{\mathbf{c}}/d_{\mathbf{c}}$ and $\mathbf{c} < 1$ then $D_h^{\mathrm{LFPP},\epsilon}$ converges to the LQG metric in probability as $\epsilon \to 0$.
- Ding-Gwynne'20: If $\xi > \xi_{\mathrm{crit}} := \gamma_1/d_1$ then $D_h^{\mathrm{LFPP},\epsilon}$ is tight for the topology on lower semi-continuous functions.
- If $D_h^{\mathrm{LFPP}}$ is a subsequential limit then
  - $D_h^{\mathrm{LFPP}}(z_1, z_2) < \infty$ a.s. for fixed $z_1, z_2$ and
  - the "infinite mass" points have infinite $D_h^{\mathrm{LFPP}}$-distance from all other points.
- Conjecture: $D_h^{\mathrm{LFPP}}$ is unique.
- $\xi > \xi_{\mathrm{crit}}$ corresponds to $\mathbf{c} > 1$ via analytic continuation.



$\xi = 0.5$

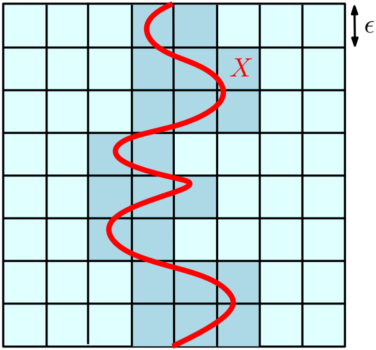


$\xi = 1.7$

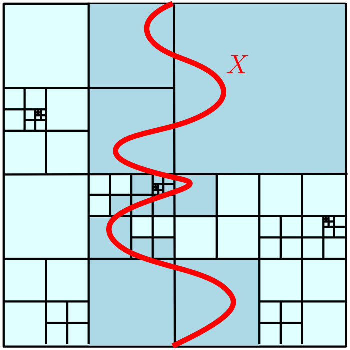
Metric balls
(Miller, Ding-Gwynne)

# KPZ (Knizhnik-Polyakov-Zamolodchikov) formula

- Let $X$ be a fractal independent of the Gaussian free field $h$.
- Let $N_0^\epsilon(X)$ and $N_h^\epsilon(X)$ denote the number of squares intersecting $X$.
- Let $d_X$ (resp. $d_X^{\mathbf{c}}$) denote Euclidean (resp. **c**-LQG) dimension of $X$.
- KPZ formula: $d_X = Q d_X^{\mathbf{c}} - 0.5(d_X^{\mathbf{c}})^2$
- KPZ formula used in physics to predict exponents and dimensions.




$N_0^\epsilon(X) = \epsilon^{-d_X + o(1)}$




$N_h^\epsilon(X) = \epsilon^{-d_X^{\mathbf{c}} + o(1)}$

# KPZ (Knizhnik-Polyakov-Zamolodchikov) formula

Recall that $N_h^\epsilon(X)$ is the number of squares intersecting $X$.

**Theorem (Gwynne-H.-Pfeffer-Remy'19; KPZ formula for $\mathbf{c} < 25$)**

*If* $\dim_{\mathrm{Haus}}(X) = \dim_{\mathrm{Mink}}(X) = d_X$ *then a.s. for sufficiently small* $\epsilon > 0$,

$$N_h^\epsilon(X) = \begin{cases} \epsilon^{-(Q-\sqrt{Q^2-2d_X})+o_\epsilon(1)} & \text{if } d_X < Q^2/2, \\ \infty & \text{if } d_X > Q^2/2. \end{cases}$$

*Furthermore,* $\mathbb{E}[N_h^\epsilon(X)] = \epsilon^{-(Q-\sqrt{Q^2-2d_X})+o_\epsilon(1)}$ *for* $d_X < Q^2/2$.

- $X \cap$ "infinite mass points" $\neq \emptyset \Leftrightarrow d_X > Q^2/2 \Leftrightarrow$ exponent complex
- Variants of KPZ formula for $\mathbf{c} \leq 1$: Benjamini-Schramm'09, Duplantier-Sheffield'11, Rhodes-Vargas'11, Barral-Jin-Rhodes-Vargas'13, Aru'15, Gwynne-H.-Miller'15, Berestycki-Garban-Rhodes-Vargas'16, Gwynne-Pfeffer'19, etc.

# Open problems and further directions

- Scaling limit results for planar maps reweighted by $(\det\Delta)^{-\mathbf{c}/2}$.

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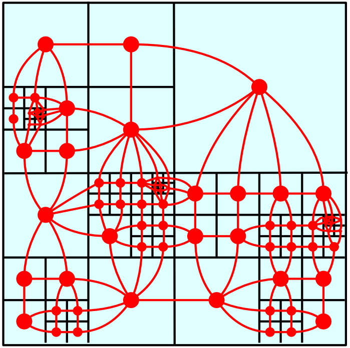

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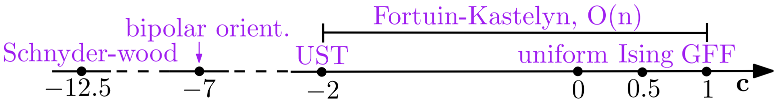


Belief: $(\det \Delta)^{-\mathbf{c}/2} \approx$ statistical physics model partition function

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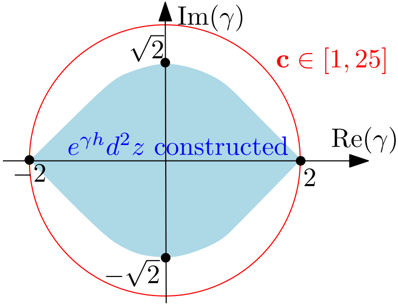

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- Path integral approach $e^{-S_{\mathrm{L}}(\varphi)} D\varphi$ to $\mathbf{c} > 1$, where $D\varphi$ is "Lebesgue measure on the space of functions" and

$$S_{\mathrm{L}}(\varphi) := \frac{1}{\pi} \int |\partial_z \varphi(z)|^2 + \pi \widetilde{\mu} e^{\gamma \varphi(z)} \, d^2 z, \qquad \widetilde{\mu} > 0.$$

  - see David-Kupiainen-Rhodes-Vargas'16 and related works for $\mathbf{c} < 1$

# Open problems and further directions (cont.)

- Schramm-Loewner evolution (SLE) for $\mathbf{c} > 1$.
  - SLE is a one parameter family of random fractal curves describing the scaling limit of statistical physics models. Parameter $\kappa > 0$ or $\mathbf{c} \leq 1$.
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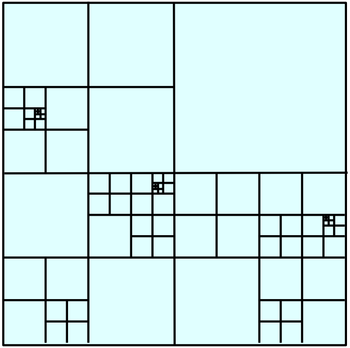
- Physical meaning of complex dimensions, for example in the KPZ formula $d_X^{\mathbf{c}} = Q - \sqrt{Q^2 - 2d_X}$ for $d_X > Q^2/2$.



# Thanks!