The four dimensional uniform spanning tree

Perla Sousi [1]

Joint work with Tom Hutchcroft

[1]University of Cambridge

Spanning trees

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Definition

A tree is a connected graph with no cycles. A spanning tree of G is a subgraph of G which is a tree and has vertex set V.

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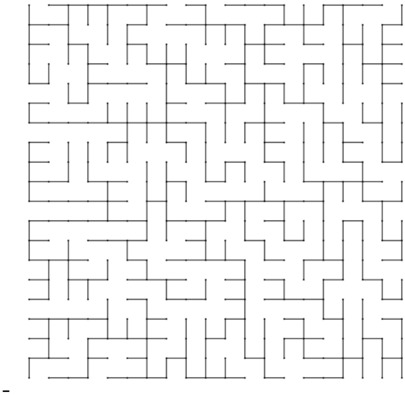


credit: Sam Watson

This is the uniform spanning tree of $\mathbb{Z}^2$

Spanning trees

The study of spanning trees goes back to the work of **Kirchhoff** in **1847**.

1847. ANNALEN *No.* 12

DER PHYSIK UND CHEMIE.

BAND LXXII.

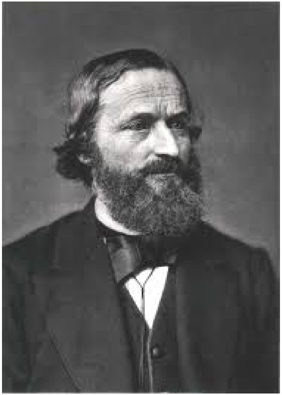
I. *Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird;*

von G. Kirchhoff.

Ist ein System von n Drähten: $1, 2 \ldots n$ gegeben, welche auf eine beliebige Weise unter einander verbunden sind, und hat in einem jeden derselben eine beliebige elektromotorische Kraft ihren Sitz, so findet man zur Bestimmung der Intensitäten der Ströme, von welchen die Drähte durchflossen werden, $I_1, I_2 \ldots I_n$, die nöthige Anzahl linearer Gleichungen durch Benutzung der beiden folgenden Sätze [1]):

I. Wenn die Drähte $k_1, k_2, \ldots$ eine geschlossene Figur bilden, und w_k bezeichnet den Widerstand des Drahtes k, E_k die elektromotorische Kraft, die in demselben ihren Sitz hat, nach derselben Richtung positiv gerechnet als I_k, so ist, falls $I_{k1}, I_{k2}, \ldots$ alle nach *einer* Richtung als positiv gerechnet werden:

Gustav Kirchhoff



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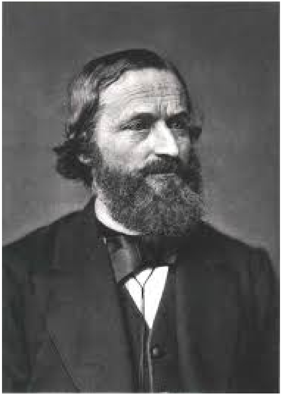

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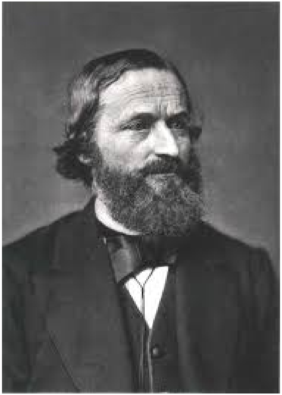

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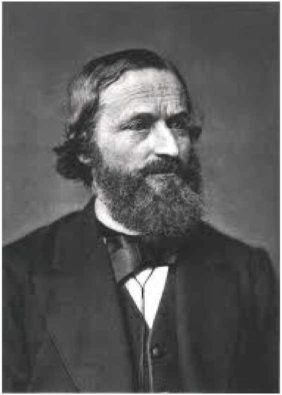
- In his **8** page long paper, he also proved the **Matrix Tree Theorem** – counting the number of spanning trees of a graph.



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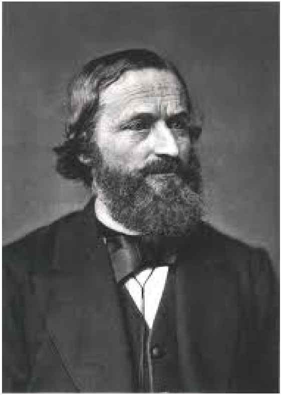

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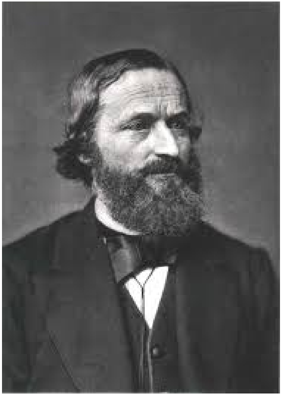

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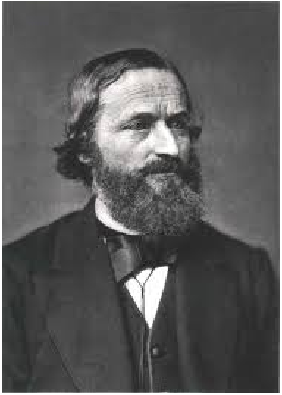

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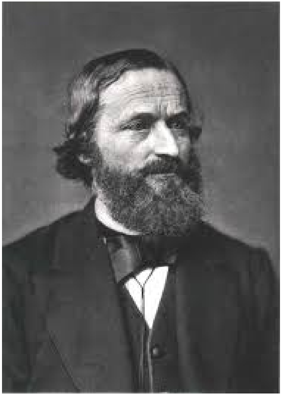

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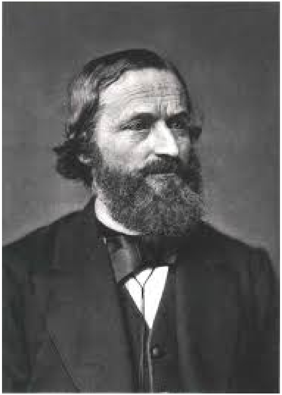
- Electrical networks are an important tool to understand the geometry of **large UST's**.



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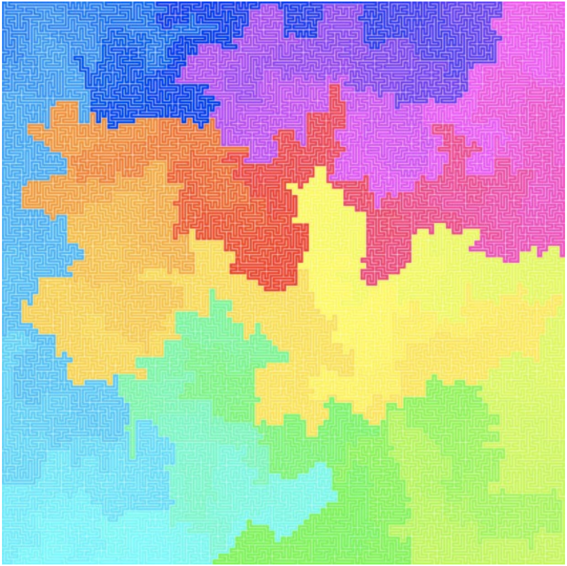
- It is connected to models in statistical physics, such as the random cluster model.
- Connections between UST, electrical networks and random walks.
- Study of scaling limit of UST led **Oded Schramm** to develop the beautiful theory of **SLE** that describes the scaling limits of conformally invariant processes on the plane.

SLE(8)



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- First sampling algorithm (1847) using **Matrix Tree Theorem** – Kirchhoff
- Wilson's algorithm using **loop erased walks**
- Aldous – Broder (and Diaconis) algorithm

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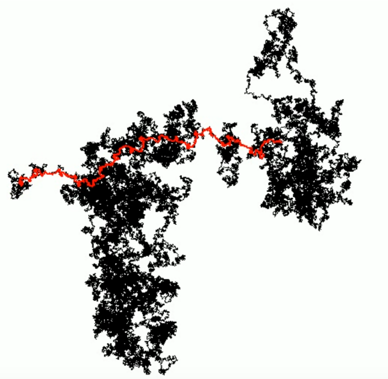
Take a loop erased random walk on $\mathbb{Z}^2$ and rescale space $\rightsquigarrow$ SLE(**2**) curve.



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Theorem (Wilson)

The tree we obtained has the same distribution as the **UST**.

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- Designate 0 as the root vertex and order the rest.
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- Start a SRW from the next vertex in the ordering till it hits the previous path. Erase loops.

UST on $\mathbb{Z}^2$

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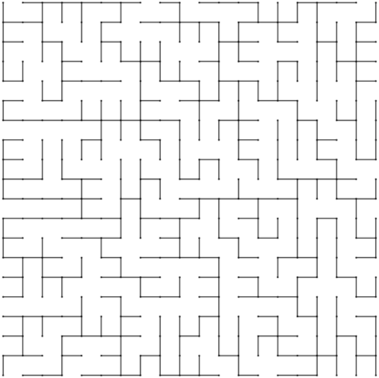

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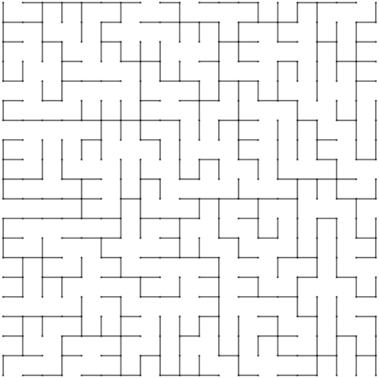

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Theorem (Pemantle (1991))

The USF on $\mathbb{Z}^d$ *has one tree with probability* 1 *for* $d \leq 4$ *and infinitely many trees for* $d \geq 5$*.*

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Theorem (Pemantle (1991); Benjamini, Lyons, Peres and Schramm (2001))

All trees in the USF in $\mathbb{Z}^d$ are one-ended for all $d \geq 2$.

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What happens at $\mathbf{d = 4}$?

$d = 4$

Theorem (Hutchcroft and S. (2020) d=4)

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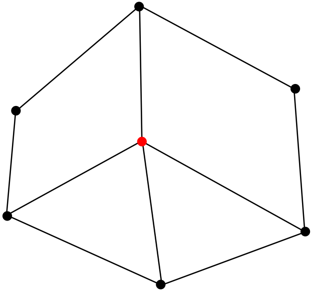
$$\mathbb{P}(\textit{past of } 0 \;\cap \partial B(0,n) \neq \emptyset) \asymp \frac{(\log n)^{2/3+o(1)}}{n^2}$$

$$\mathbb{P}(|\textit{past of } 0| \geq n) \asymp \frac{(\log n)^{1/6}}{\sqrt{n}}$$

Proof ideas

Small detour

Aldous - **Broder** algorithm for generating a **UST** of a finite graph G



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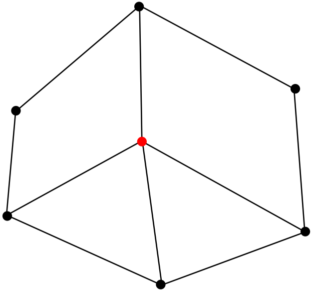

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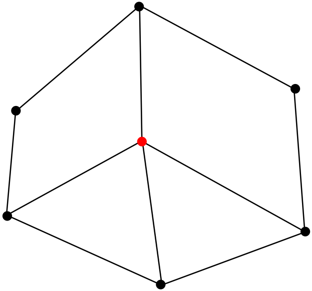

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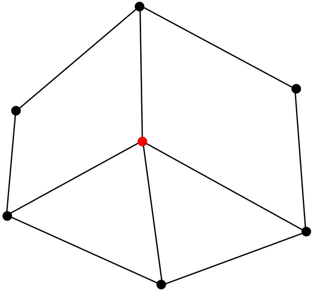

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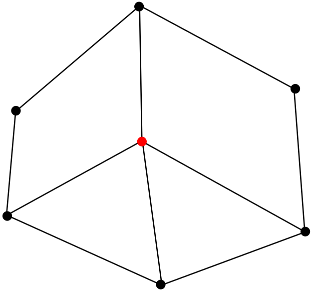

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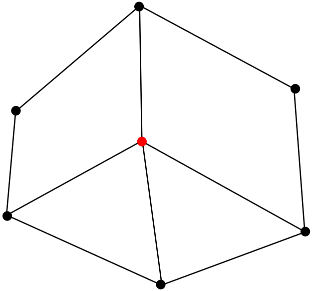

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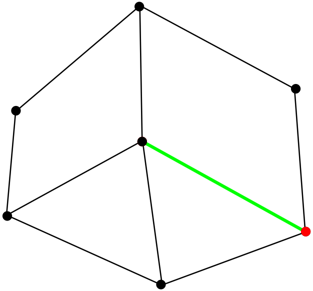

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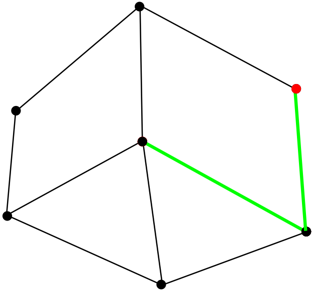

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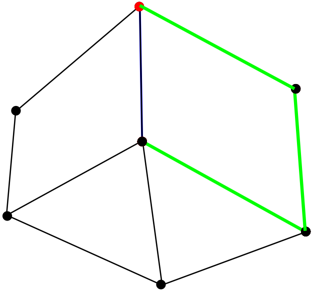

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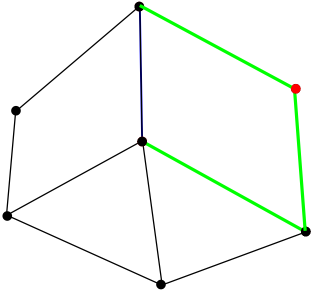

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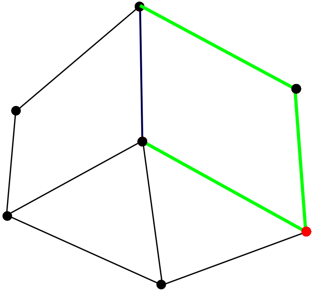

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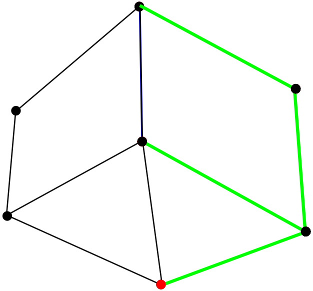

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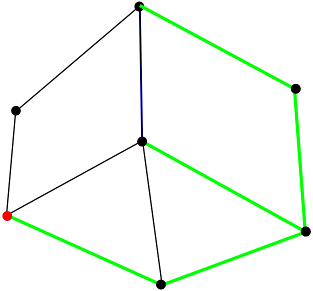

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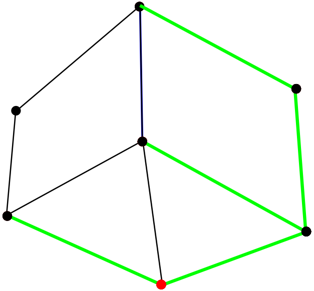

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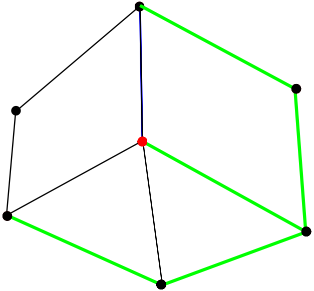

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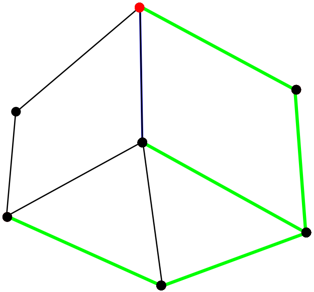

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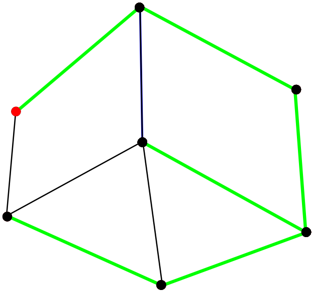

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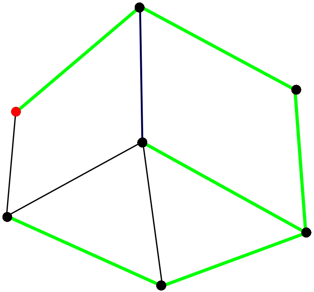

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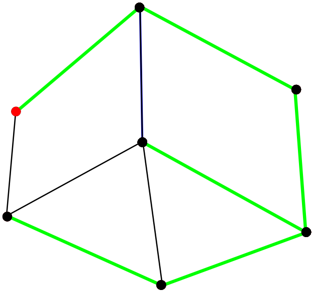

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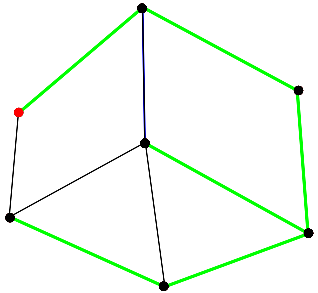

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Clear how to generalise the algorithm for an ∞ *recurrent graph* (walk visits every vertex with probability 1).

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Discrete analogue of the Newtonian capacity: for $A \subseteq \mathbb{R}^d$ compact

$$\frac{1}{\mathrm{Cap}(A)} = \inf\left\{\int\int G(x,y)d\mu(x)d\mu(y) : \mu \text{ prob. measure on } A\right\}$$

(G is the Green kernel)

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Need to estimate $\mathbb{E}\left[e^{-\varepsilon \mathrm{Cap}(\eta[0,n])}\right]$, where η LERW in $\mathbb{Z}^4$.

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This easily then yields

$$\mathbb{E}[\mathrm{Cap}(X[0, n])] \asymp \frac{n}{\log n}.$$

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Theorem (Lyons, Peres and Schramm)

Let X and Y be transient chains. Then

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Useful tools developed in work with **Asselah and Schapira** on the capacity of the range of a SRW.

$d = 4$

Theorem (Hutchcroft and S. (2020) d=4)

$$\mathbb{P}(\textit{past of } 0 \text{ contains a path of length } n) \asymp \frac{(\log n)^{1/3}}{n}$$

$$\mathbb{P}(\textit{past of } 0 \cap \partial B(0,n) \neq \emptyset) \asymp \frac{(\log n)^{2/3+o(1)}}{n^2}$$

$$\mathbb{P}(|\textit{past of } 0| \geq n) \asymp \frac{(\log n)^{1/6}}{\sqrt{n}}$$