Integer distance sets

Rachel Greenfeld
Northwestern University

Joint work with
Marina Iliopoulou and Sarah Peluse

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Why? $E(\Lambda)$ is orthogonal in $L^2(D_r)$.

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$$\forall \lambda' \neq \lambda \in \Lambda: \ \widehat{\mathbb{1}_{D_r}}(\lambda' - \lambda) = \int_D e^{2\pi i x \cdot (\lambda' - \lambda)} dx = 0$$

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$\|\Lambda - \Lambda\|$: distance set

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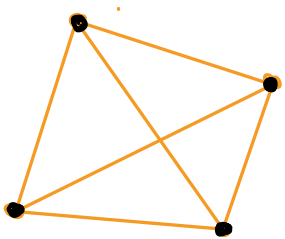

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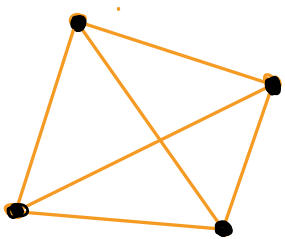

distances are algebraically related

Can zeros of a Bessel function be algebraically related???

$E(\Lambda)$ is orthogonal in $L^2(D_r) \Leftrightarrow \|\Lambda - \Lambda\| \subseteq \|\{\hat{\mathbb{1}}_{D_r} = 0\}\| = \frac{1}{2\pi r}\{J_1 = 0\}$

Question: How large can $|\Lambda|$ be?

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1st attempt:

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Determinant method (Bombieri - Pila)

↳ There are very few lattice points on a manifold unless there is an algebraic reason.

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- Encode Λ as lattice points on an analytic manifold.
- Analyse transcendentality of the manifold
- Apply the determinant method.

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2nd attempt:

$$\|\{\hat{\mathbb{1}}_{D_r} = 0\}\| = \left\{\tfrac{1}{2\pi}\left(n + \tfrac{1}{4}\right) + O\left(\tfrac{1}{n}\right) \mid n \in \mathbb{Z}\right\}.$$

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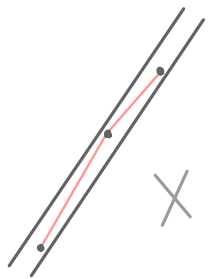

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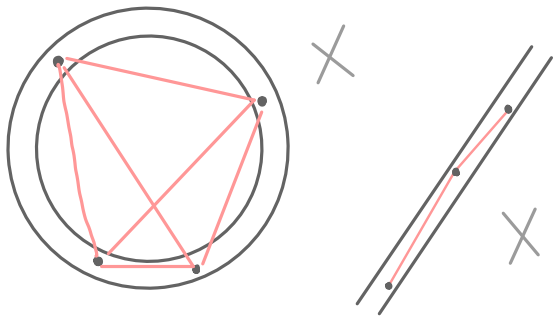

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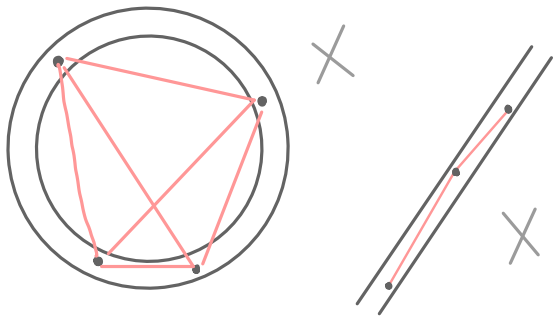

$\Rightarrow$ no 3 points in a thin tube
no 4 points in a thin annulus.

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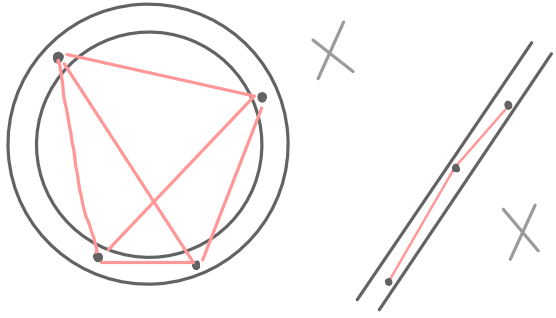

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This links to another famous problem:

The size and structure of integer distance sets.

Integer distance sets

$S \subseteq \mathbb{R}^2$ is an integer distance set if the distance between any pair of points $s', s \in S$, $\|s' - s\| \in \mathbb{Z}$, is an integer.

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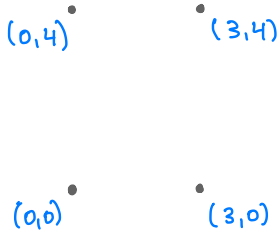

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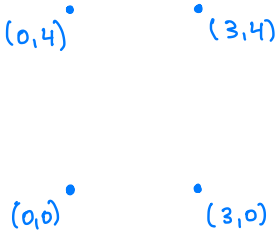

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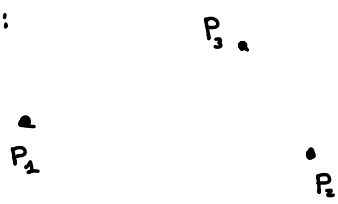

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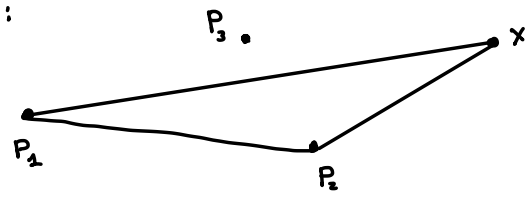

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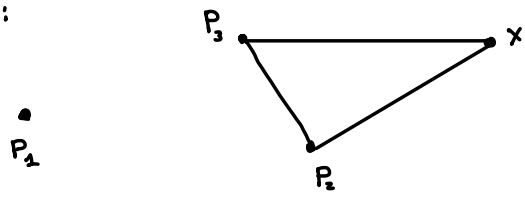

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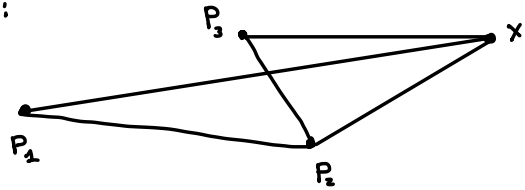

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$\|P_1 - P_2\| + 1$ hyperbolas

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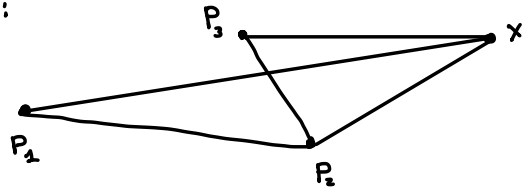

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$$x \in H_1 \cap H_2$$

$$|H_1 \cap H_2| \leq 4(\|P_1 - P_2\| + 1)(\|P_2 - P_3\| + 1) < \infty .$$

Bézout's theorem

∎

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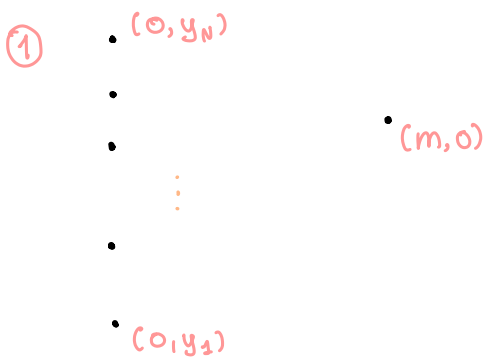

$m^2 = (x_i - y_i)(x_i + y_i)$ $\quad x_i, y_i \in \mathbb{Z}$

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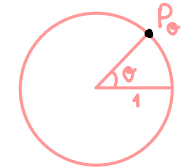
② concyclic:

θ such that $\tan \frac{\theta}{4} \in \mathbb{Q}$

$P_\theta = (\cos\theta, \sin\theta)$



$\{P_\theta\}$ is dense in the unit circle;

$$\|P_\theta - P_{\theta'}\| = 2\left|\sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2} - \sin\tfrac{\theta'}{2}\cos\tfrac{\theta}{2}\right| \in \mathbb{Q}$$

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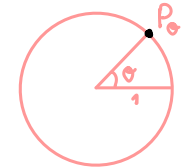

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S is concentrated on one line/circle.

All so-far-known integer distance sets have all but up to 4 of their points on a single line / circle.

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Conditional on Bombieri-Lang's conjecture, under these assumptions, $|S|$ is bounded by a constant. [Ascher - Braune - Turchet, 2020]

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Theorem (G-Iliopoulou-Peluse, 2024):

Let $S \subset [-N, N]^2$ be an integer distance set. Then there exists a line or circle $C \subseteq \mathbb{R}^2$ such that: $|S \setminus C| = O((\log N)^{O(1)})$.

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In particular, we obtain:

Corollary: Let $S \subseteq [-N,N]^2$ be an integer distance set with no 3 of its points on a line and no 4 points on a circle. Then

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The height of $\frac{m}{n} \in \mathbb{Q}$ with $(m,n)=1$ is $\max\{|m|,|n|\}$.

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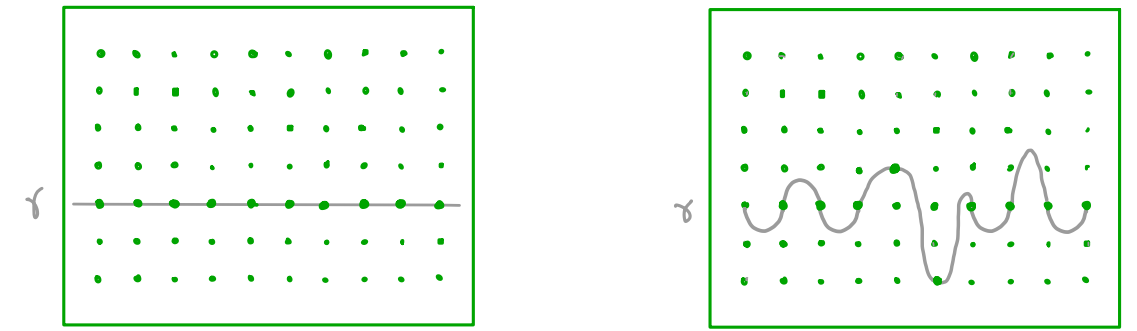
Rational points of height $\leq H$



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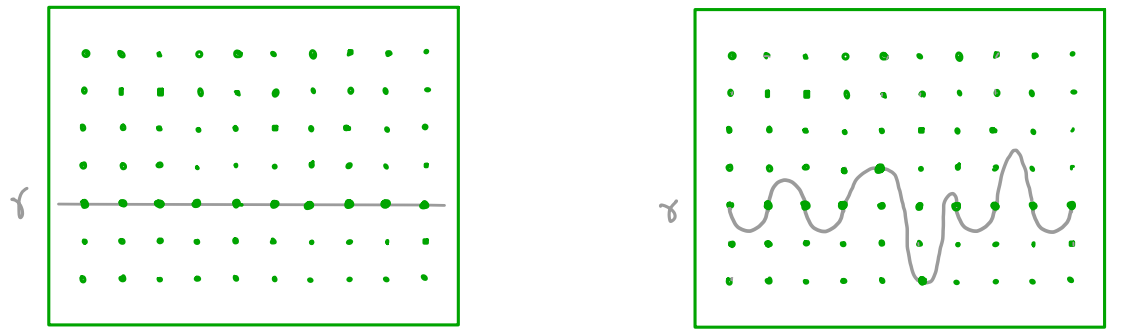

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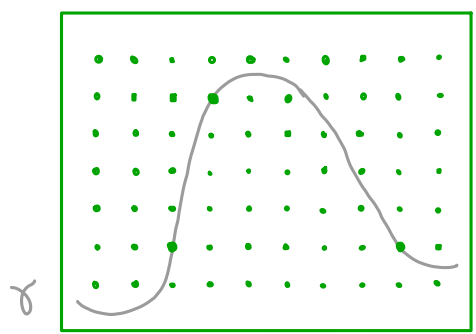

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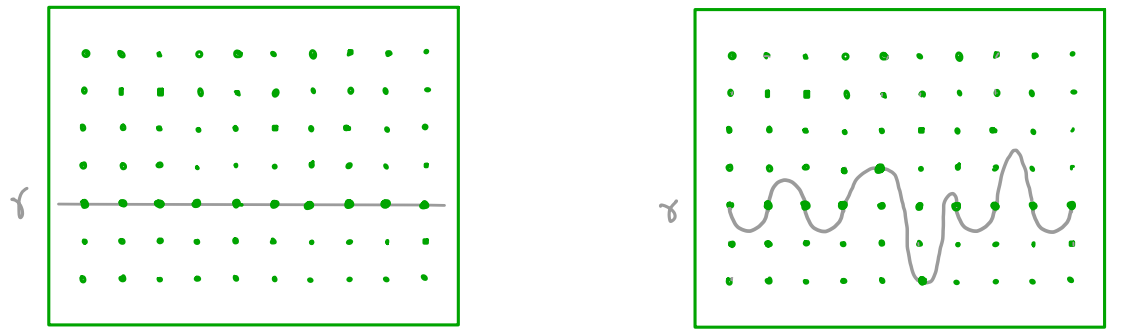

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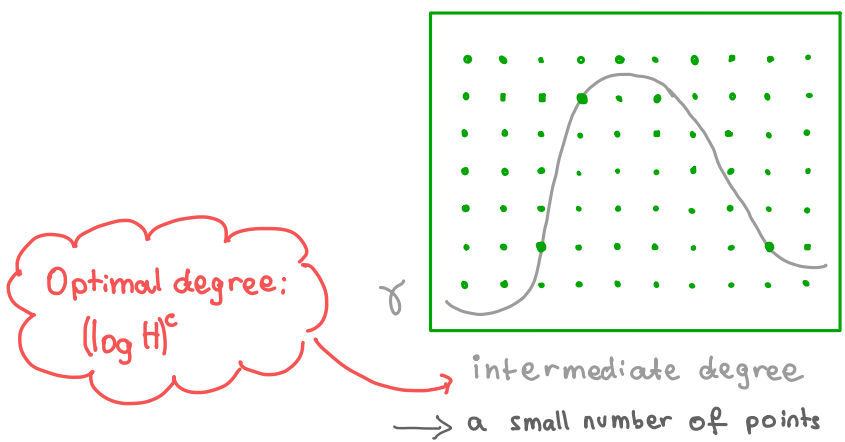

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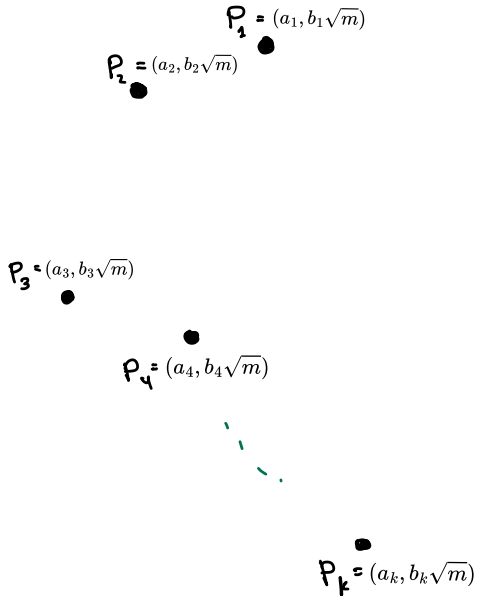

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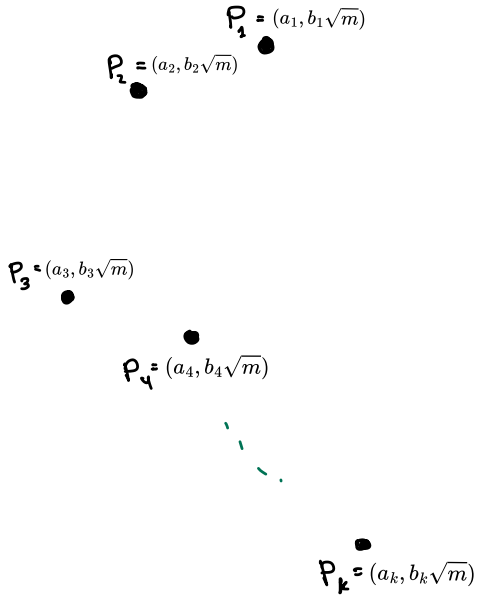

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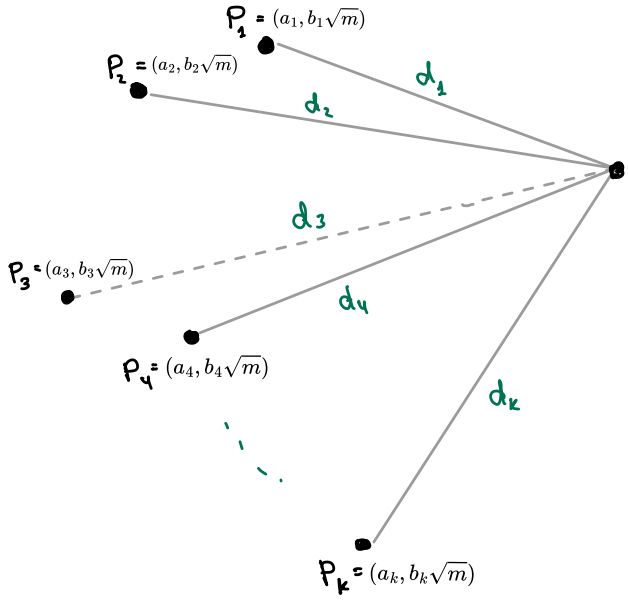

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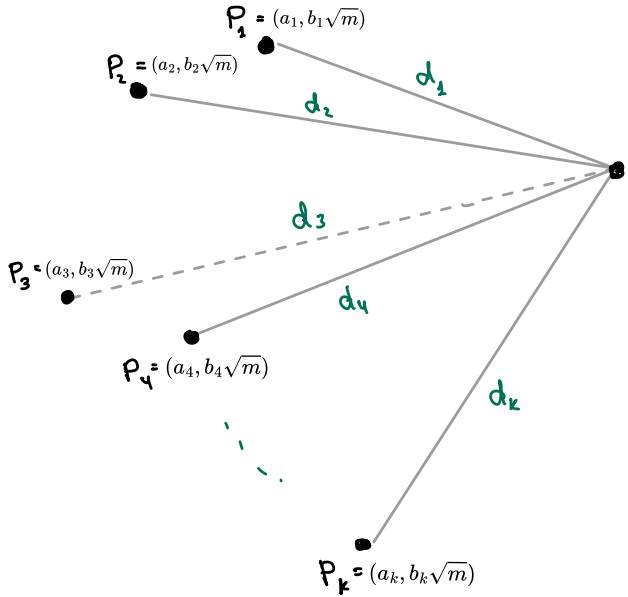

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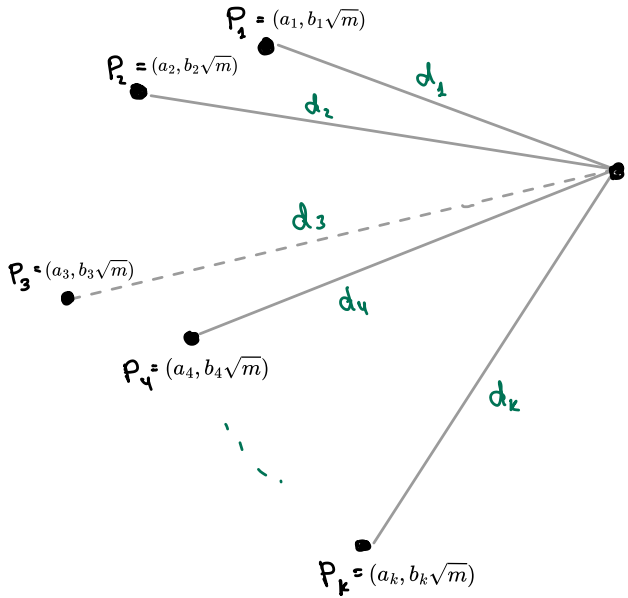

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Points of $S \xrightarrow[\text{injective}]{\pi^{-1}} \tilde{S}$: rational points of height $O(N^2)$ on $\overline{X}_k \subseteq \mathbb{P}^{k+2}$.

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$\dim \overline{X}_k = 2$



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[Heath-Brown, 2002]

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might be too low ☹

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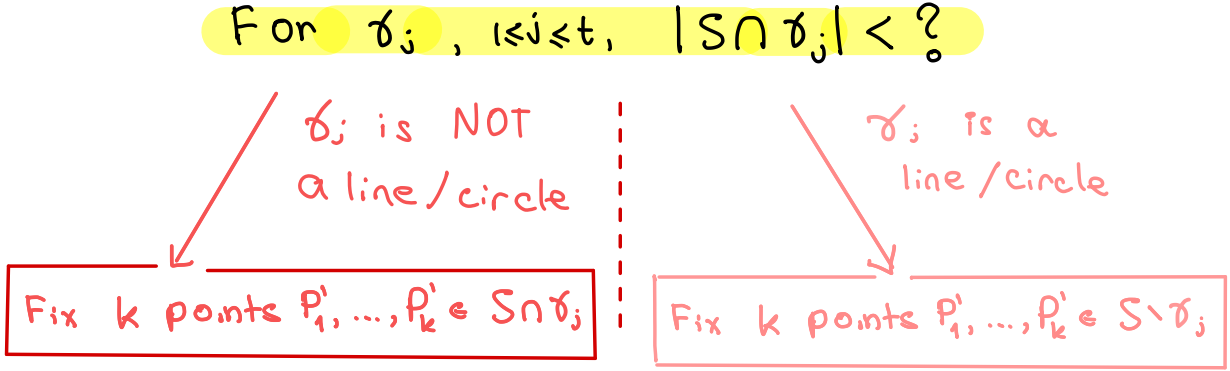
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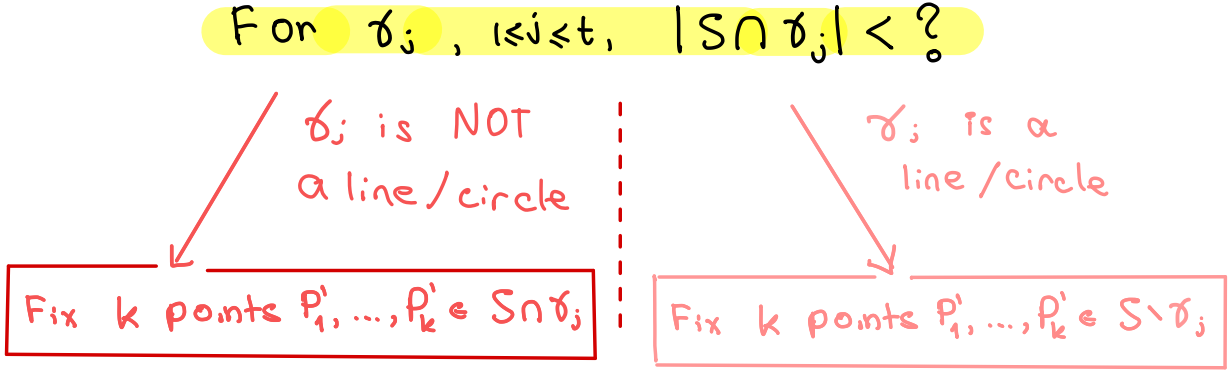

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$$S \cap \gamma_j \overset{\pi^{-1}}{\rightsquigarrow} \tilde{S}_j := \{\text{rational points of height } O(N^2) \text{ on } \overline{C_k} \subseteq \mathbb{P}^{k+2}\}.$$

For γ_j, $1 \le j \le t$, $|S \cap \gamma_j| < ?$

γ_j is NOT a line/circle

Fix k points $P_1', \dots, P_k' \in S \cap \gamma_j$

γ_j is a line/circle

Fix k points $P_1', \dots, P_k' \in S \setminus \gamma_j$

$$P_i' = (a_i', \sqrt{m}\, b_i'), \quad 1 \le i \le k$$

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$$|S \cap \gamma_j| \le |\widetilde{S}_j| = ?$$

If $(P_i')_{i=1}^{k}$ are in "general position", $\overline{C}_k$ is an irreducible curve of degree $\approx 2^k$ defined over $\mathbb{Q}$.

γ_j is line / circle and $|S \setminus \gamma_j| > ck^2$

γ_j isn't line / circle and $|S \cap \gamma_j| > \log N^c k^2$

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Use Bombieri–Pila's-type result

$|\widetilde{S}_j| =$?

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[Castryk, Cluckers, Dittmann, Nguyen, 2020]

$$\Downarrow$$

$$|\widetilde{S}_j| = O\left(e^{O(k)} N^{O(2^{-k})}\right)$$

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$$\Updownarrow$$

$$|\widetilde{S}_j| = O(e^{o(k)} N^{O(2^{-k})}) = O((\log N)^{O(1)})$$

choose $k \approx \log\log N$

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$\Leftrightarrow$

$$|\gamma_j \cap S| \leq |\widetilde{S}_j| = O(e^{O(k)} N^{O(2^{-k})}) = O((\log N)^{O(1)})$$

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choose $k \approx \log\log N$

as long as:

γ_j isn't line/circle and $|S \cap \gamma_j| > \log N^c k^2$

or

γ_j is line/circle and $|S \setminus \gamma_j| > ck^2$

as $k^2 \asymp (\log\log N)^2$, we have:

If γ_j is <u>not</u> a line/circle, then $|\gamma_j \cap S| = O((\log N)^{o(1)})$.

Otherwise, either $|S \setminus \gamma_j| = O((\log\log N)^2)$ or $|\gamma_j \cap S| = O((\log N)^{o(1)})$.

As there are $O((\log N)^{o(1)})$ curves, this concludes the proof.



Theorem (G-Iliopoulou-Peluse, 2024):

Let $S \subset [-N, N]^2$ be an integer distance set. Then there exists a line or circle $C \subseteq \mathbb{R}^2$ such that: $|S \setminus C| = O((\log N)^{O(1)})$.

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In fact, we have that either $|S| = O((\log N)^{O(1)})$, or there is a line/circle C s.t. $|S \setminus C| = O((\log \log N)^2)$.

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Let $S \subset [-N, N]^2$ be an integer distance set. Then there exists a line or circle $C \subseteq \mathbb{R}^2$ such that: $|S \setminus C| = O((\log N)^{O(1)})$.

Corollary: Let $S \subseteq [-N, N]^2$ be an integer distance set with no 3 of its points on a line and no 4 points on a circle. Then

$$|S| = O((\log N)^{O(1)}).$$

Corollary: Let S be a noncollinear integer distance set. If $|S| = N$ then:

$$\operatorname{diam} S \geq N^{c(\log\log N)}.$$

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Does there exist a dense rational distance set in $\mathbb{R}^2$?

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What is the size & structure of **higher-dimensional** integer distance sets?

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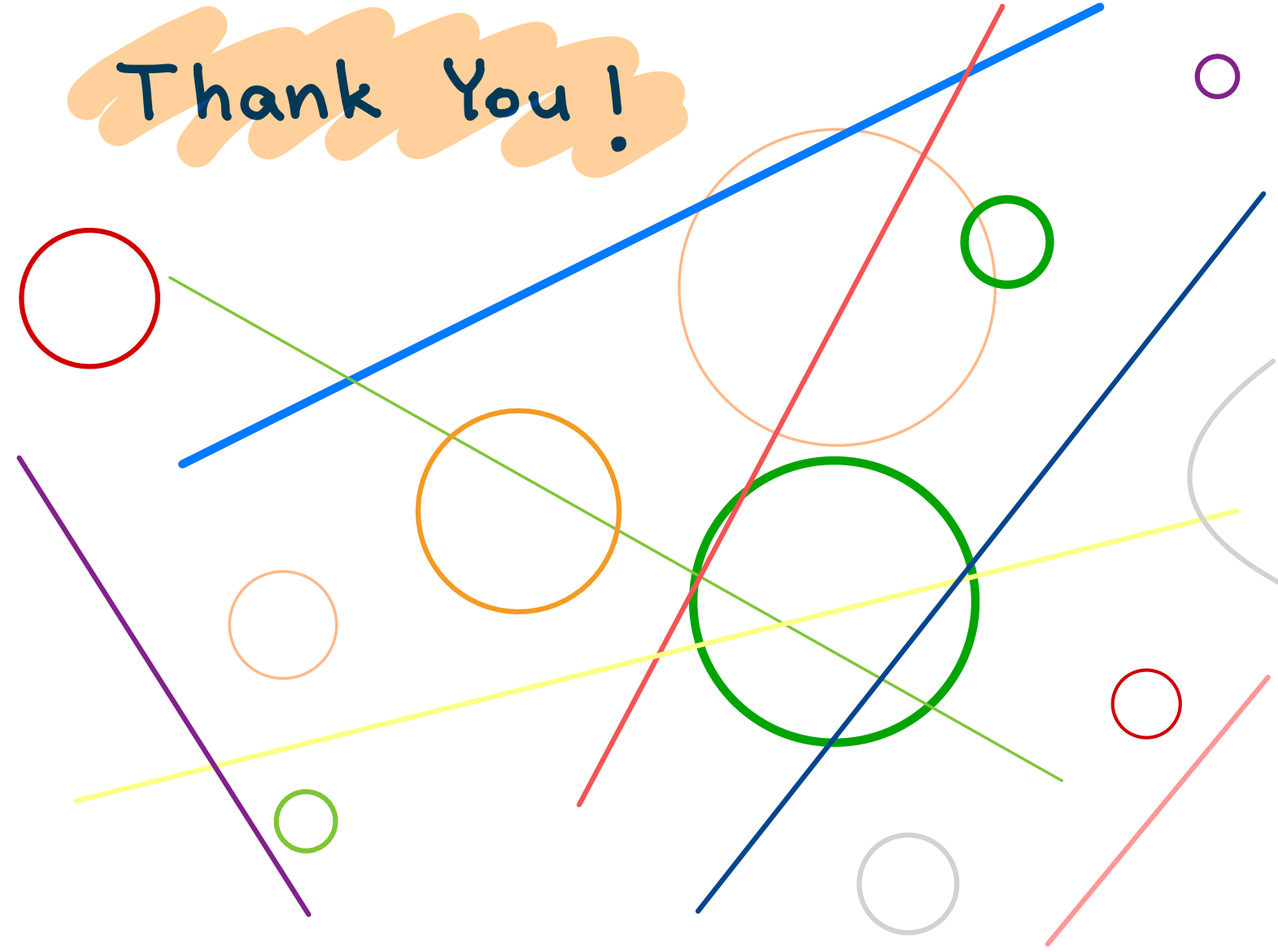
What is the size & structure of **higher-dimensional** integer distance sets?

Can our method be adapted to other long-standing problems?


Thank You !