Integer distance sets

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Joint work with
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<u>Why?</u> $E(\Lambda)$ is orthogonal in $L^2(D_r)$.

$$\Updownarrow$$

$$\forall \lambda' \neq \lambda \in \Lambda: \ \widehat{\mathbb{1}_{D_r}}(\lambda' - \lambda) = \int_D e^{2\pi i x \cdot (\lambda' - \lambda)} dx = 0$$

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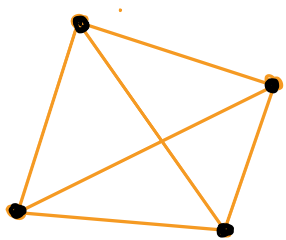

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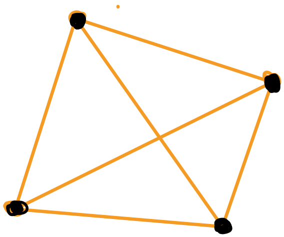

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$\boxed{\text{Can zeros of a Bessel function be algebraically related???}}$

$E(\Lambda)$ is orthogonal in $L^2(D_r) \Longleftrightarrow \| \Lambda - \Lambda \| \subseteq \| \{ \hat{\mathbb{1}}_{D_r} = 0 \} \| = \frac{1}{2\pi r} \{ J_1 = 0 \}$

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There are very few lattice points on a manifold unless there is an algebraic reason.

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- Encode Λ as lattice points on an analytic manifold.
- Analyse transcendentality of the manifold
- Apply the determinant method.

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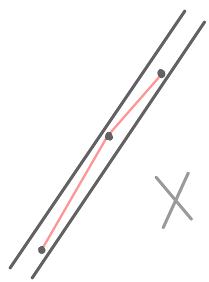

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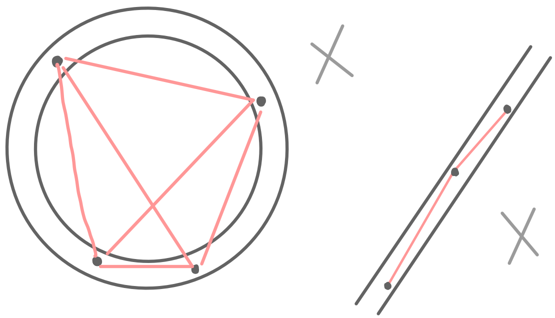

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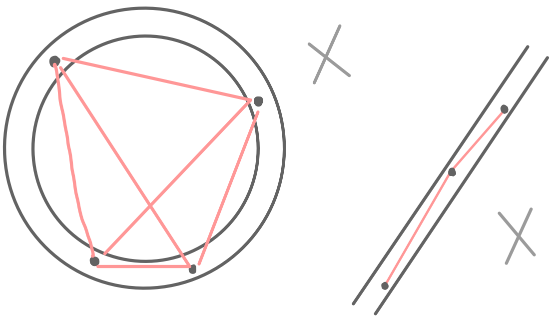

$\implies$ no 3 points in a thin tube
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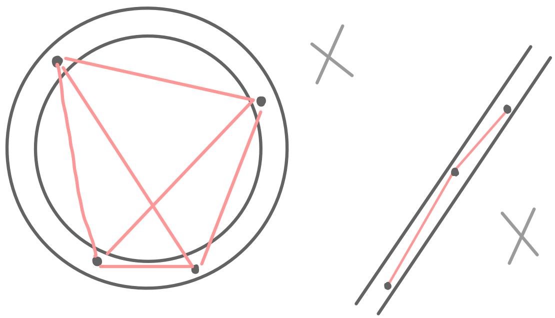

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This links to another famous problem:

The size and structure of integer distance sets.

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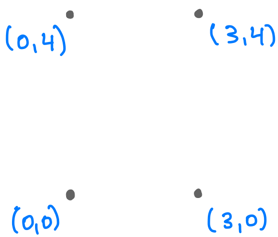
2) 

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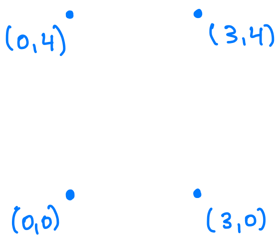
2) 

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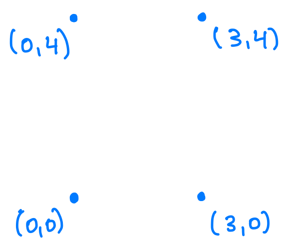
2) 

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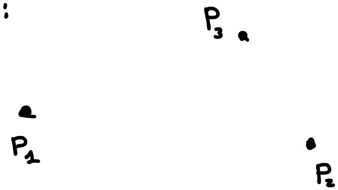

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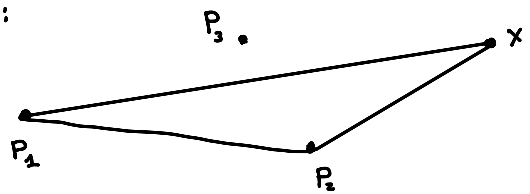

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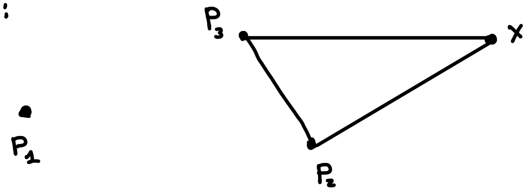

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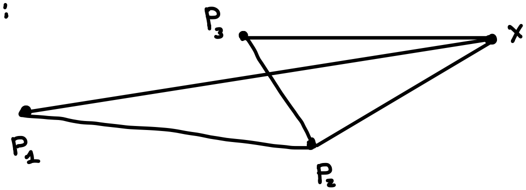

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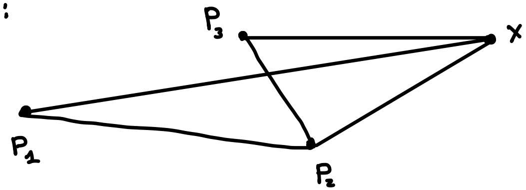

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$$|H_1 \cap H_2| \leq 4(\|P_1 - P_2\| + 1)(\|P_2 - P_3\| + 1) < \infty .$$

Bézout's theorem

$\blacksquare$

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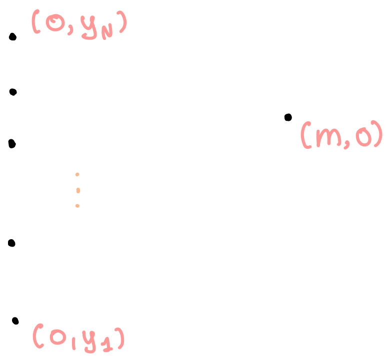

$m^2 = (x_j - y_j)(x_j + y_j) \qquad x_j, y_j \in \mathbb{Z}$

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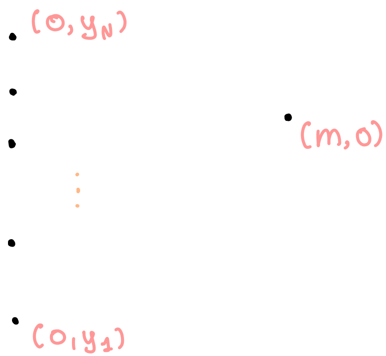

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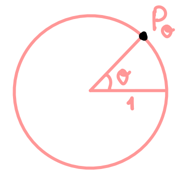
$P_\theta = (\cos\theta, \sin\theta)$



$\{P_\theta\}$ is dense in the unit circle;

$$\|P_\theta - P_{\theta'}\| = 2\left|\sin\frac{\theta}{2}\cos\frac{\theta'}{2} - \sin\frac{\theta'}{2}\cos\frac{\theta}{2}\right| \in \mathbb{Q}$$

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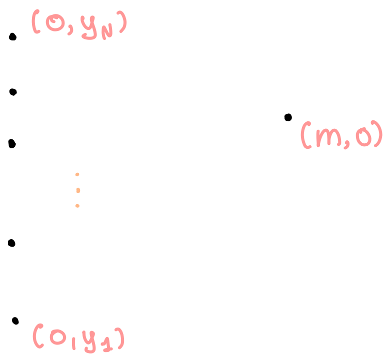
① 

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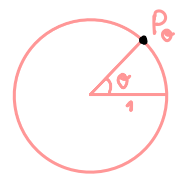

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S is concentrated on one line/circle.

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<u>Conjecture:</u> Yes.

<u>Question</u> (Erdös): How large can an integer distance set S be if it has no 3 points on a line and no 4 points on a circle?

All so-far-known integer distance sets have all but up to 4 of their points on a single line / circle.

Question: Is this ALWAYS the case?

Conjecture: Yes.

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Conditional on Bombieri-Lang's conjecture, under these assumptions, $|S|$ is bounded by a constant. [Ascher - Braune - Turchet, 2020]

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	$\lvert S\rvert = O((\log N)^c)$ [G-Iliopoulou-Peluse, 2024]	$\lvert\Lambda\rvert = O_\varepsilon(N^{\frac{3}{5}+\varepsilon})$ [Zakharov, 2024]

Theorem (G-Iliopoulou-Peluse, 2024):

Let $S \subset [-N,N]^2$ be an integer distance set. Then there exists a line or circle $C \subseteq \mathbb{R}^2$ such that: $|S \setminus C| = O((\log N)^{O(1)})$.

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In particular, we obtain:

Corollary: Let $S \subseteq [-N,N]^2$ be an integer distance set with no 3 of its points on a line and no 4 points on a circle. Then

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In particular, we obtain:

Corollary: Let S be a noncollinear integer distance set. If $|S| = N$ then:

$$\operatorname{diam} S \gtrsim N^{c(\log\log N)}.$$

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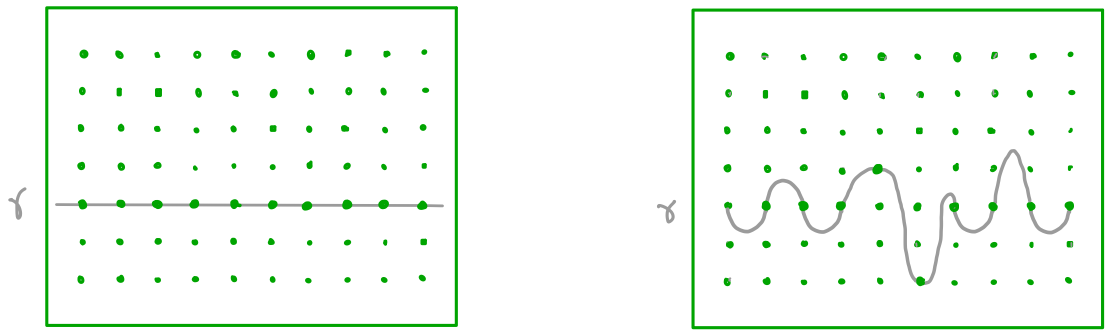
A well-developed theory - originally due to Bombieri - Pila (1989) - provides sharp bounds on the number of lattice points on any irreducible curve V of degree d defined over $\mathbb{Q}$.

The height of $\frac{m}{n} \in \mathbb{Q}$ with $(m,n)=1$ is $\max\{|m|,|n|\}$.

The height of $(q_1,\dots,q_k) \in \mathbb{Q}^k$ is the maximal height of $q_1,\dots,q_k$.

Rational points of height $\le H$



γ has too low degree $\longrightarrow$ many points $\longleftarrow$ γ has too high degree

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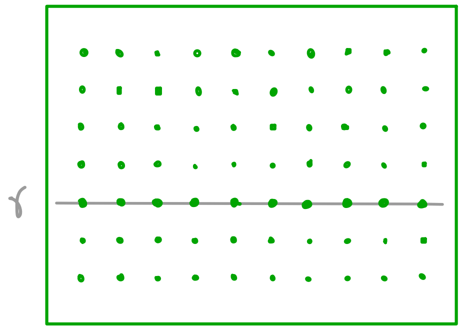

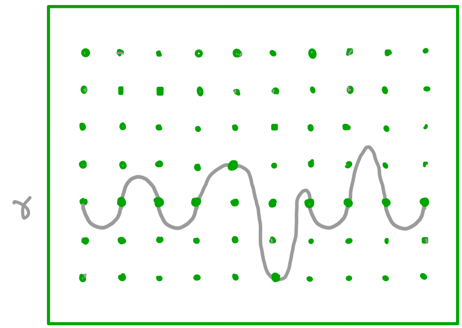

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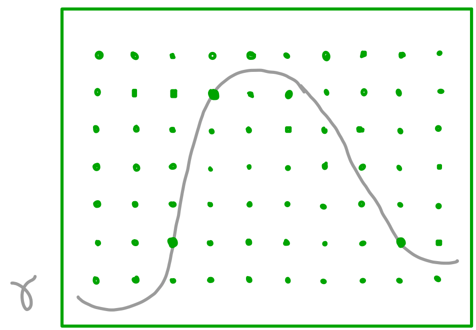

intermediate degree

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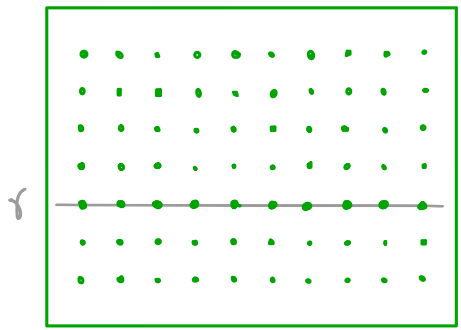

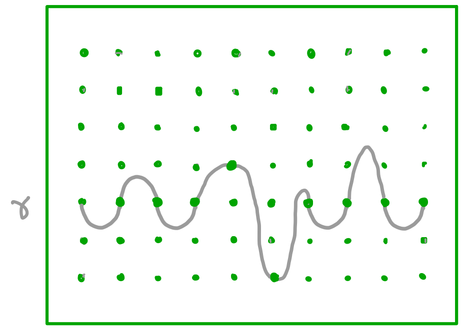

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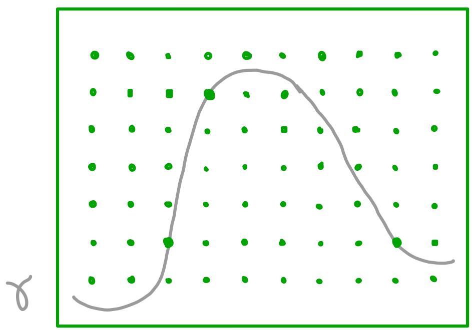

Optimal degree: $(\log H)^c$

intermediate degree

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By translation and rotation, we can assume without loss of generality that:

$$0 \in S \subseteq \left\{ (x, \sqrt{m}\, y) \mid x, y \in \tfrac{1}{M}\mathbb{Z} \right\}$$

for squarefree $m = m(S)$, and integer $M = O(N)$. [Kemnitz, 88]

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We define the affine variety:

$$X_k := \left\{ (x, y, d_1, \dots, d_k) \mid (x-a_j)^2 + (y-b_j)^2 m = d_j^2 ;\ j = 1, \dots, k \right\} \subseteq \mathbb{C}^{k+2}.$$

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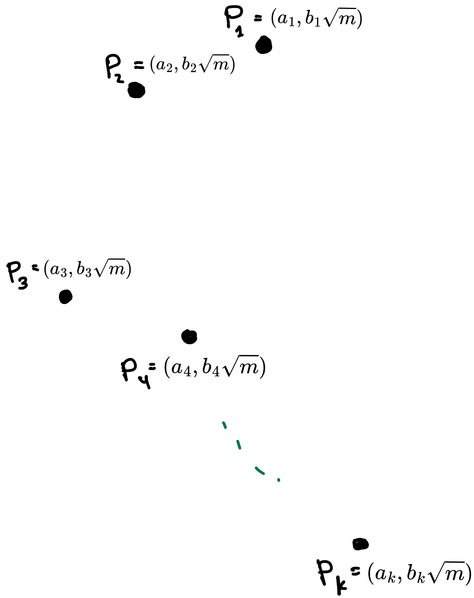

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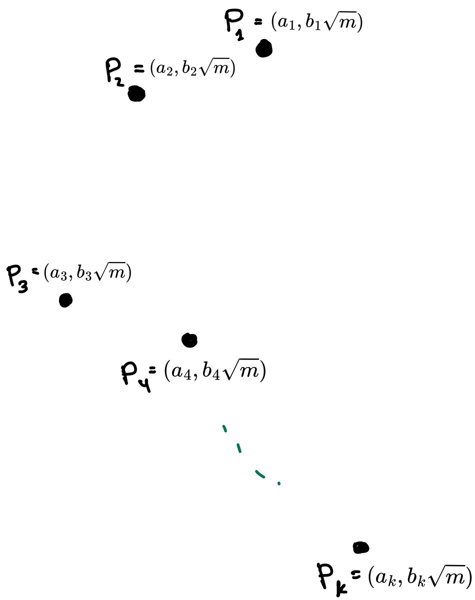

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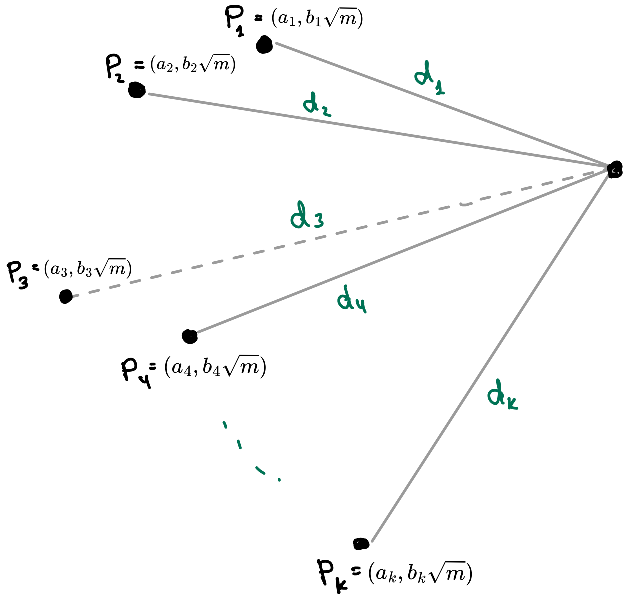

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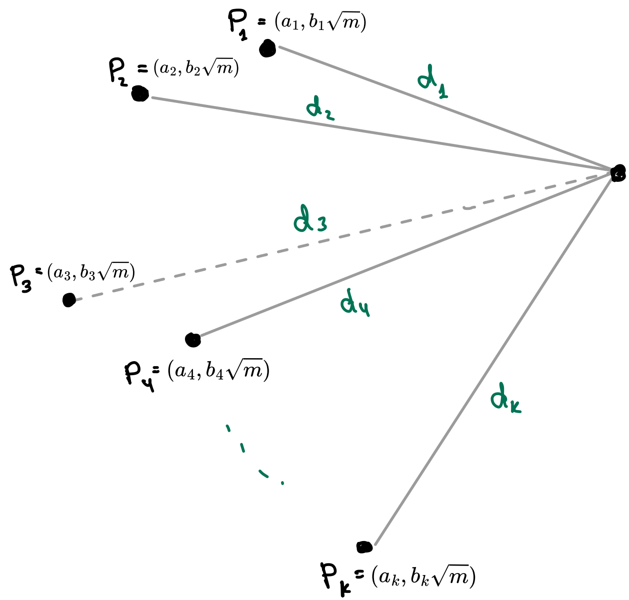

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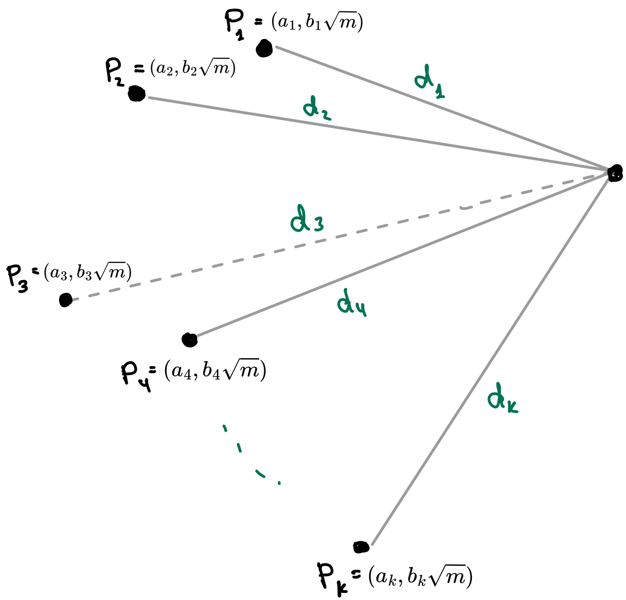

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$$\boxed{\text{Points of } S \xrightarrow[\text{injective}]{\pi^{-1}} \tilde{S} : \text{rational points of height } O(N^2) \text{ on } \overline{X_k} \subseteq \mathbb{P}^{k+2}.}$$

$\overline{X}_k \subseteq \mathbb{P}^{k+2}$ is an <u>irreducible</u> <u>surface</u> of <u>degree 2^k</u> defined over $\mathbb{Q}$.

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$\dim \overline{X}_k = 2$

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[Heath-Brown, 2002]

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$$\Big\downarrow \pi(\tilde{S}) : (x, y, d_1, \dots, d_k) \mapsto (x, \sqrt{m}\, y)$$

$S \subseteq \mathbb{R}^2$ can be covered by $t = O((\log N)^{O(1)})$ irreducible curves $\gamma_1, \dots, \gamma_t \subset \mathbb{C}^2$ of degree $O((\log N)^{O(1)})$.

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might be too low ☹

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$$C_k := \left\{ (x, y, d_1, \dots, d_k) \mid (x - a_i')^2 + (y - b_i')^2 m = d_i^2 ; \ \overset{i=1,\dots,k}{(x, \sqrt{m}\, y) \in \gamma_j} \right\}$$

$$S \cap \gamma_j \overset{\pi^{-1}}{\rightsquigarrow} \tilde{S}_j := \left\{ \begin{array}{c} \text{rational points of height } O(N^2) \\ \text{on } \overline{C_k} \subseteq \mathbb{P}^{k+2} \end{array} \right\}$$

$$|S \cap \gamma_j| \le |\tilde{S}_j|$$

For γ_j, $1 \leq j \leq t$, $|S \cap \gamma_j| < ?$

γ_j is NOT a line/circle

$\boxed{\text{Fix } k \text{ points } P'_1, \ldots, P'_k \in S \cap \gamma_j}$

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$$|S \cap \gamma_j| \leq |\tilde{S}_j| = ?$$

If $(P_i^1)_{i=1}^k$ are in "general position", $\overline{C_k}$ is an irreducible curve of degree $\approx 2^k$ defined over $\mathbb{Q}$.

γ_j is line / circle and $|S \setminus \gamma_j| > ck^2$

γ_j isn't line / circle and $|S \cap \gamma_j| > \log N^c k^2$

If $(P_i')_{i=1}^k$ are in "general position", $\overline{C}_k$ is an irreducible curve of degree $\approx 2^k$ defined over $\mathbb{Q}$.

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Use Bombieri–Pila's-type result

$$|\tilde{S}_j| = ?$$

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[Castryk, Cluckers, Dittmann, Nguyen, 2020]

$$|\tilde{S}_j| = O\left(e^{O(k)} N^{O(2^{-k})}\right)$$

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$\Downarrow$

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choose $k \approx \log\log N$

as long as:

γ_j isn't line/circle and $|S \cap \gamma_j| > \log N^c k^2$

or

γ_j is line/circle and $|S \setminus \gamma_j| > ck^2$

as $k^2 \asymp (\log\log N)^2$, we have:

If γ_j is <u>not</u> a line/circle, then $|\gamma_j \cap S| = O((\log N)^{O(1)})$.

Otherwise, either $|S \setminus \gamma_j| = O((\log\log N)^2)$ or $|\gamma_j \cap S| = O((\log N)^{O(1)})$.

As there are $O((\log N)^{O(1)})$ curves, this concludes the proof.

$\blacksquare$

Theorem (G-Iliopoulou-Peluse, 2024):

Let $S \subset [-N,N]^2$ be an integer distance set. Then there exists a line or circle $C \subseteq \mathbb{R}^2$ such that: $|S \setminus C| = O((\log N)^{O(1)})$.

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In fact, we have that either $|S| = O((\log N)^{O(1)})$, or there is a line/circle C s.t. $|S \setminus C| = O((\log\log N)^2)$.

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Let $S \subset [-N,N]^2$ be an integer distance set. Then there exists a line or circle $C \subseteq \mathbb{R}^2$ such that: $|S \setminus C| = O((\log N)^{O(1)})$.

Corollary: Let $S \subseteq [-N,N]^2$ be an integer distance set with no 3 of its points on a line and no 4 points on a circle. Then

$$|S| = O((\log N)^{O(1)}).$$

Corollary: Let S be a noncollinear integer distance set. If $|S| = N$ then:

$$\operatorname{diam} S \geq N^{c(\log\log N)}.$$

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What is the size & structure of **higher-dimensional** integer distance sets?

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What is the size & structure of **higher-dimensional** integer distance sets?

Can our method be adapted to other long-standing problems?

Thank You!