Generalized birthday problem for October 12

Sumit Mukherjee,
Department of Statistics, Columbia

Joint work with B.B. Bhattacharya (U Penn) and S. Mukherjee (NUS)

Outline

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- In this case the number of student pairs with a common birthday is approximate Poisson with mean $\lambda = \frac{\binom{n}{2}}{c}$.
- Consequently the chance of at least one common birthday is approximately $1 - e^{-\lambda}$.

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- Here $|E(G)|$ is the number of edges in G, i.e. the total number of friendship pairs in the class.
- The classical birthday problem is a special case with $G = K_n$, where everyone knows everyone else.

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- In this case things are more interesting, and the answer depends on the graph in a more delicate way (than just the number of edges).

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- Note that

$$T_n = \sum_{i<j} G_n(i,j) X_i X_j,$$

where $(X_1, \ldots, X_n)$ are i.i.d. Bern(p_n), with $p_n = \frac{1}{c_n}$.

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- We will assume that the sequence $\{G_n, p_n\}_{n \geq 1}$ are chosen such that $\mathbb{E}T_n = \frac{|E(G_n)|}{c_n^2} = O(1)$.

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- Thus $T_n \xrightarrow{D} \binom{S}{2}$, where $S \sim Pois(1)$.

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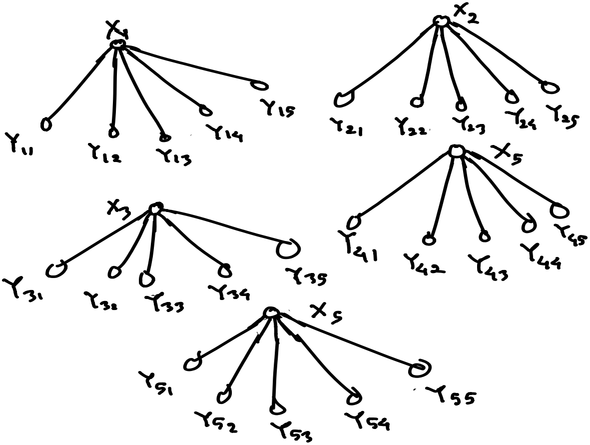

EG : n=25, 5 COPIES OF $K_{1,5}$

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$$T_n = \sum_{i=1}^{\sqrt{n}} X_i \sum_{j=1}^{\sqrt{n}} Y_{ij},$$

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- This gives $T_n \xrightarrow{D} Pois(S)$, where $S \sim Pois(1)$.

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- Also, can we characterize when is this limit exactly a Poisson?

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Towards a general result

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- The disjoint star example captures the contribution of edges between high degree vertices and low degree vertices.
- Finally, edges between low degree vertices gives rise to a Poisson limit.
- Using this philosophy, we partition the edge set into 3 types,

$$\text{High} \leftrightarrow \text{High}, \quad \text{High} \leftrightarrow \text{Low}, \quad \text{Low} \leftrightarrow \text{Low}.$$

Adjacency matrix $\mapsto$ function on positive reals

- Define a function $W_{G_n}(.,.) : [0,\infty)^2 \mapsto [0,1]$ by setting

$$\begin{aligned} W_{G_n}(x,y) =& 1 \text{ if } (\lceil xc_n \rceil, \lceil yc_n \rceil) \in E(G_n) \\ =& 0 \text{ otherwise} \end{aligned}$$

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- Let $d_{G_n}(x) := \int_0^\infty W_{G_n}(x,y) \mathrm{d}x$ be the degree function.

First assumption (A1)

- Given two bounded measurable functions f, g from $[0, K]^2 \mapsto [0, 1]$, define the strong cut distance between f and g by

$$\sup_{A,B \subset [0,1]} \left| \int_{A \times B} f(x,y) \mathrm{d}x \mathrm{d}y - \int_{A \times B} g(x,y) \mathrm{d}x \mathrm{d}y \right|.$$

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- In some sense W captures the limit of the dense part of the graph. Note that this assumption, along with Fatou's lemma automatically implies

$$\int_{[0,\infty)^2} |W(x,y)|\mathrm{d}x\mathrm{d}y < \infty.$$

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- In some sense $\Delta(x)$ counts the edges between the high and low degree vertices.

Third assumption (A3)+Theorem I

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$$\lim_{K\to\infty}\lim_{n\to\infty}\int_{[K,\infty)^2} W_{G_n}(x,y)\mathrm{d}x\mathrm{d}y = 2\lambda_0.$$

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$$T_n \xrightarrow{D} Q_1 + Q_2 + Q_3.$$

- *Here $Q_3 \sim Pois(\lambda_0)$.*
- *(Q_1, Q_2) is jointly independent of Q_3.*
- *The joint Mgf of (Q_1, Q_2) appears on the next slide.*

Mgf of (Q_1, Q_2)

- For $t_1, t_2 > 0$, $\mathbb{E}e^{-t_1 Q_1 - t_2 Q_2}$ equals

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- Q_1 arises from the edges between the high degree vertices, i.e. the dense part of the graph.

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- Suppose $f : [0,\infty)^2 \mapsto \mathbb{R}$ is such that

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- Then $I_2(f)$ is well defined, and

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Example I′: Stochastic Block Model

- Suppose $p_n = \frac{1}{n}$, and G_n is a sequence of random graphs such that

$$\begin{aligned}\mathbb{P}(G_n(i,j)=1) =& a_{11} \text{ if } i,j < \frac{n}{2}\\ =& a_{12} \text{ if } i < \frac{n}{2}, j \geq \frac{n}{2} \text{ or } i \geq \frac{n}{2}, j < \frac{n}{2},\\ =& a_{22} \text{ if } i,j \geq \frac{n}{2}.\end{aligned}$$

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- Similar results apply to unequal blocks, or more than 2 blocks.

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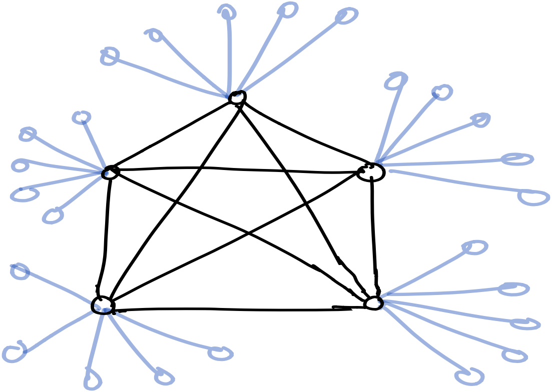
- Put a star graph $K_{1,n}$ centered on every vertex of K_n.
- Then the entire graph G_n has $n + n^2 \sim n^2$ vertices, and $\binom{n}{2} + n^2 \sim \frac{3n^2}{2}$ edges.

Co-existence example



EG: $n = 5$

TOTAL VERTICES $= 5 + 5 \times 5 = 30$

TOTAL EDGES $= \binom{5}{2} + 5 \times 5 = 35$

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- This gives

$$Q_1 + Q_2 \stackrel{D}{\to} \binom{S}{2} + Pois(S), \text{ where } S \sim Pois(1).$$

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- This ensures that G_n has $\Theta(n^2)$ vertices, and $\Theta_P(n^2)$ many edges.
- For the choice $p_n = \frac{1}{n}$, our theorem applies with

$$\begin{aligned} W(x,y) =& c_k \text{ if } x, y \in (r_{k-1}, r_k] \text{ for some } k \geq 1, \\ =& 0 \text{ otherwise }. \end{aligned}$$

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- Using our theorem gives

$$T_n \xrightarrow{D} \sum_{k=1}^{\infty} Bin\left(\binom{S_k}{2}, c_k\right) + Pois\left(\sum_{k=1}^{\infty} a_k S_k\right),$$

where $S_k \sim Pois(b_k)$ are mutually independent.

Outline

Theorem II

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- We claim that the class $\psi(W, d, \lambda)$ is closed under weak topology.

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Theorem (Bhattacharya-Mukherjee-M., AAP-2020)

If $\mathbb{E}T_n = O(1)$, the following are equivalent:

(i) $T_n \xrightarrow{D} Pois(\lambda)$.

(ii)
$$\lim_{M\to\infty}\lim_{n\to\infty}\mathbb{E}(T_{n,M}) = \lambda, \quad \lim_{M\to\infty}\lim_{n\to\infty} Var(T_{n,M}) = \lambda.$$

Theorem III

- Here

$$T_{n,M} := \sum_{i<j} G_n(i,j) X_i X_j 1\{d_i \leq M c_n, d_j \leq M c_n\}$$

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- Compare this with the more well studied fourth moment phenomenon for the Gaussian distribution (see Ivan Nourdin's webpage for a list of papers on this topic).

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- Similar results hold for sparse block models, and random regular graphs.

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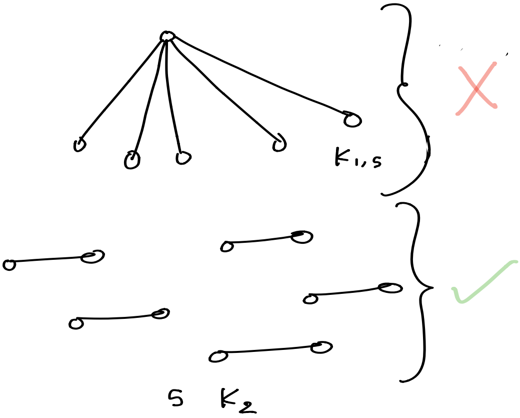

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- The truncated second moment result captures this behavior automatically.

Outline

Proof idea of Theorem I

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- By using a first moment computation using Markov's inequality, we show that the super high degree vertices do not contribute for M large.

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- We argue using method of moments that $T_{3,n}$ is asymptotically independent from $T_{1,n}$ and $T_{2,n}$.

Proof idea of Theorem-I

- Using

  (i) (A1): strong cut metric convergence of W_{G_n} on $[0, K]^2$

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- This gives the joint Mgf of (Q_1, Q_2), thereby proving Theorem I.

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- We characterize the class of all possible limits of Bernoulli quadratic forms.
- As an application, we characterize exactly when is the limit a Poisson random variable.
- We apply our theorem to several examples, which includes both deterministic and random graphs.

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- Finally, our quadratic form is (in terms of) the adjacency matrix of a simple graph. Does a similar analysis apply for general quadratic forms?


The End